An optimal control method for linear systems with time delay

Guo-Ping Cai \textsuperscript{a,*}, Jin-Zhi Huang \textsuperscript{a}, Simon X. Yang \textsuperscript{b}

\textsuperscript{a} Department of Engineering Mechanics, Shanghai Jiaotong University, Shanghai 200030, PR China
\textsuperscript{b} School of Engineering, University of Guelph, Guelph, ON, Canada N1G 2W1

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Abstract

An optimal control method for linear systems with time delay is developed in this paper. In the proposed control method, the differential equation with time delay of the system dynamics is first rewritten into a form without any time delay through a particular transformation. Then, the optimal controller is designed by using the classical optimal control theory. A numerical algorithm for control implementation is presented, since the obtained expression of the optimal controller contains an integral term that is not convenient for online calculation. The time delay is considered at the very beginning of the control design, and no approximation and estimation are made in the control system. Thus the system performance and stability are prone to be guaranteed. Instability in responses might occur only if a system with time delay is controlled by the optimal controller that was designed with no consideration of time delay. The effectiveness of the proposed optimal controller is demonstrated by simulation studies.

Keywords: Linear system; Optimal control; Time delay; Stability; Maximum response

1. Introduction

In recent years, the techniques for active vibration control have been get more and more vigorous, and many successful theoretical findings and applications in engineering have been demonstrated. The studies in both laboratory investigation and realistic applications demonstrate that vibration suppression by active control is emerging as a powerful technique to improve the performance of structures against strong wind, earthquake and many other dynamic loads. Meanwhile, many problems that may affect this technique towards large-scale practical applications have been identified and been under investigation. Time delay in vibration control is one of these problems that need serious attention [1].

Time delay exists inevitably in active control systems, which mainly results from the follows: (1) the time taken in the online data acquisition from sensors at different locations of the system; (2) the time taken in the filtering and processing of the sensory data for the required control force calculation and the transmission of the control force to the actuator; and (3) the time taken by the actuator to produce the required control force. Due to the time delay, when unsynchronized control force is applied to a structure, it may result in degradation in the control efficiency and instability of the control system [2,3]. So there have been many studies on control systems with time delay in various research fields such as aeronautical, astronautical, mechanical, chemical and electrical engineering, and many methodologies have been proposed to deal with the control problem of systems with time delay [4–7]. In vibration control, the technique of time-delay compensation [8–11] is commonly used to eliminate or reduce the effect of time delay. In the time-delay compensation control method, the controller is first designed by assuming that no time delay exists, and then the feedback gain is modified with
time compensation. However, this technique is generally suitable for short time delay only, and the system performance and stability are not guaranteed for a system with long time-delay. Another approach to time delay problem in vibration control is to design the controller directly from the discrete system equation with time delay. For example, Chung et al. [12] proposed a controller for systems with time delay by first discretizing the system motion equation that has no time delay, then adding the time delay into the discretized motion equation, and finally designing a discrete optimal controller according to the discrete optimal control theory. However, the time-delay term in [12] is not considered in the continuous motion equation, which is added into the discrete state equation after the discretization of the system dynamics. Thus the time delay needs to be approximated to integer times of the sampling period.

In this paper, an optimal control method is proposed for linear systems with time delay in vibration control, where the time delay is considered in the continuous system motion equation. In the proposed control approach, through a particular transformation, the motion equation of the time-delay control system is first reformulated into a standard form of a first-order differential equation that contains no time delay, then the optimal controller is designed according to the classical optimal control theory. Since an integral term appears in the expression of the obtained optimal controller, a numerical algorithm for the implementation of the proposed optimal control method is developed. In the proposed controller, because the time delay is incorporated in the motion equation throughout the derivation of the control algorithm, the system performance and stability are prone to be guaranteed. The effectiveness of the proposed control method is demonstrated by simulation studies of two structural models.

This paper is organized as follows. Section 2 first briefly presents the system motion dynamics with explicit time delay in the differential equations, and then introduces a particular transformation that reformulates system dynamics into a standard form of first-order differential equations without any time delay. The proposed optimal control design is presented in Section 3, including the controller design and a numerical algorithm for its implementation. Section 4 provides simulation and comparison studies of a three-story model and a six-story building model using the proposed optimal control method. Finally, a concluding remark is given in Section 5.

2. System motion equation

To study the vibration control, a linear structure modeled by an n-degree-of-freedom linear system with time delay is considered. The motion equation of the structural system (also called system dynamics) can be written as

\[ M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = H_1E(t) + H_2U(t - \lambda), \]

where \( X = [x_1, x_2, \ldots, x_n]^T \) is the \( n \times 1 \) vector of displacement; \( M, C \) and \( K \) are the \( n \times n \) mass, damping and stiffness matrices, respectively; \( E(t) \) is the \( q \times 1 \) vector of external excitations; \( H_1 \) is an \( n \times q \) matrix denoting the locations of external excitations; \( U(t - \lambda) \) is the \( r \times 1 \) vector of control inputs, where \( \lambda \) is the time delay; and \( H_2 \) is an \( n \times r \) matrix denoting locations of the control inputs.

In the state-space representation, the system dynamics in Eq. (1) can be obtained as

\[ \dot{Z}(t) = AZ(t) + BU(t - \lambda) + \vec{E}(t), \]

where

\[
Z(t) = \begin{bmatrix} X(t) \\ \dot{X}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \]

\[
\vec{E}(t) = \begin{bmatrix} \dot{M}^{-1}H_1E(t) \\ \dot{M}^{-1}H_2 \end{bmatrix}.
\]

By the following transformation [13]

\[ Y(t) = Z(t) + \int_{t-\lambda}^{t} e^{-A(t+\eta)}B \dot{U}(t + \eta) \, d\eta, \]

the system dynamics in Eq. (2) can be rewritten into a standard form of first-order differential equation without any explicit time delay term as

\[ \dot{Y}(t) = AY(t) + B(A)U(t) + \vec{E}(t), \]

where

\[ B(A) = e^{-A\lambda}B. \]

3. Optimal control design

In this section, the optimal controller is first designed using the classical optimal control theory based on the obtained system motion equation with no explicit time-delay term. Then, a numerical algorithm for the online calculation of the integral term in the optimal controller is presented. Thus the implementation of the proposed optimal controller is practically feasible.
3.1. Controller design

According to the classical optimal control theory [14], the time-dependent quadratic objective function for the control system characterized by Eq. (4) can be defined as

\[
J = \frac{1}{2} \int_0^\infty \begin{bmatrix} Y^T(t)QY(t) + U^T(t)RU(t) \end{bmatrix} dt,
\]

where \( Q \) is a \( 2n \times 2n \) positive-semidefinite coefficient matrix, and \( R \) is an \( r \times r \) positive-definite coefficient matrix to represent the relative importance of the system response vector \( Y(t) \) and the control input vector \( U(t) \), respectively. To minimize the objective function \( J \), the optimal controller can be obtained as [14]

\[
U(t) = -R^{-1}[B(A)]^TPY(t),
\]

where the matrix \( P \) is the solution to the following Riccati equation

\[
PA + A^TP - P[B(A)]R^{-1}[B(A)]^TP = -Q.
\]

Substituting Eq. (3) into Eq. (7), the control input becomes

\[
U(t) = -R^{-1}[B(A)]^TP \left[ Z(t) + \int_{-\lambda}^{0} e^{-A(t+\eta)}BU(t + \eta) d\eta \right].
\]

3.2. Control implementation

In the developed optimal controller in Eq. (9), it explicitly contains an integral term that is not practically feasible for online calculation. Thus a numerical algorithm to calculate this integral term is needed for the practical control implementation. For the convenience to derive the numerical algorithm, the integral term in Eq. (9) is denoted by a new variable as

\[
Z_0(t) = \int_{-\lambda}^{0} e^{-A(t+\eta)}BU(t + \eta) d\eta.
\]

For any time delay \( \lambda \), it can be written as

\[
\lambda = lT - m,
\]

where \( T \) is the data sampling period; \( l > 0 \) is a positive integer; and \( 0 \leq m < T \). When \( m = 0 \), the time delay is integer times of the sampling period. When \( m \neq 0 \), the time delay is non-integer times of the sampling period. Chung et al.’s [12] control method for systems with time delay considers only the special case \( m = 0 \), i.e., the time delay is integer times of the sampling period.

When the regular fourth-order Runge–Kutta method is used to solve numerically the system dynamics in Eq. (4), the integration time step is chosen to be identical to the sampling period. Thus the numerical computation of Eq. (4) is carried out only at every sampling point. Between any two adjoining sampling points, the control forces exerted on the structure can be considered as constants if the sampling period is sufficiently small, i.e.,

\[
U(t) = U(kT), \quad kT \leq t < (k + 1)T.
\]

Thus Eq. (10) can be written as

\[
Z_0(t) = \int_{-(lT - m)}^{0} e^{A(lT-m)}e^{-AT}BU(t + \eta) d\eta
\]

\[
= e^{A(lT-m)} \left[ \int_{-(l-1)T}^{-(l-1)T} e^{-AT}BU(t + \eta) d\eta + \cdots + \int_{-T}^{0} e^{-AT}BU(t + \eta) d\eta \right]
\]

\[
+ e^{A(l-1)T} \int_{0}^{T} e^{-AT}BU(t - lT) + \cdots + e^{AT} \int_{0}^{T} e^{-AT}BU(t - T).
\]

By defining the follow two functions

\[
F(\xi) = e^{\xi T},
\]

\[
G(\xi) = \int_{0}^{\xi} e^{-\eta T} d\eta,
\]

Eq. (13) can be rewritten into a simpler form as

\[
Z_0(t) = I_{2n \times 2n} G(T - m) U(t - T)
\]

\[
+ F(m - T)G(T)U[t - (l - 1)T] + \cdots + F[m - (l - 1)T]G(T)U(t - T).
\]

When \( m = 0 \), Eq. (16) becomes

\[
Z_0(t) = I_{2n \times 2n} G(T)U(t - T)
\]

\[
+ F(-T)G(T)U[t - (l - 1)T] + \cdots + F[-(l - 1)T]G(T)U(t - T).
\]

From Eqs. (9) and (16) or (17), it shows that at every step of numerical computation for the proposed optimal controller, it contains not only the current step of state feedback but also a linear combination of the former \( l \) steps of control inputs.

The integral term \( G(\xi) \) in Eq. (15) can be determined according to the following formula [15]
\[ G(\xi) = \int_0^\infty e^{-\omega t} dt = \sum_{n=1}^{\infty} \frac{(-A)^{n-1} \xi^n}{n!}. \]  

When \( \xi \) is given, \( G(\xi) \) will converge to a constant matrix after limited steps of iterative calculation [15].

4. Simulation studies

To demonstrate the feasibility and effectiveness of the proposed optimal control method, numerical simulations are carried out in this section. Two structure models, three-story and six-story structures, are considered. The structural model is shown in Fig. 1. The Tianjin earthquake (in China) and the El Centro earthquake, scaled to maximum accelerations of 0.2 and 0.4 g, respectively, are used as the external excitations. Earthquake episodes are 10 and 8 s, respectively. Time histories of the two earthquakes are shown in Fig. 2(a) and (b).

4.1. Three-story structure subjected to the Tianjin earthquake

The three-story model studied by Yang [16] is considered herein, where every story unit is identically constructed. The mass, stiffness and damping coefficient of each story unit are \( m_i = 1 \) m ton, \( k_i = 980 \) kN/m, and \( c_i = 1.407 \) kN s/m, \( i = 1, 2, 3 \), respectively. An active control force is applied to the first-story unit as shown in Fig. 1.

Determination of the maximum allowable time delay is first considered. To determine the maximum allowable time delay in this structure system, the sampling period and the integration time step of the fourth-order Runge–Kutta method may be chosen as a small value, both at 0.001 s. The weighting matrix \( Q \) in Eq. (6) is given by \( Q = \text{diag}([10^4, 10^4, 10^4, 1, 1, 1]) \). As there is only one actuator installed in the structure, the weighting matrix \( R \) in Eq. (6) is given by \( R = 7 \times 10^{-9} \) that consists of only one element. When the optimal controller that is designed without considering any time delay is used to control the system with time delay, the maximum interstory drifts with respect to time delay of every story units are shown in Fig. 3(a), where the first, second and third story units are represented by dashed, dot-dashed and solid lines, respectively. Fig. 3(b) gives the maximum absolute accelerations of every story units, and the maximum control force varying with time delay is shown in Fig. 3(c). It shows that instability in responses occurs even when time delay is very small. In addition, the maximum allowable time delay for system stability can be graphically determined as about 0.018 s.

Then the comparison of system performance is considered. For the convenience in numerical calculation, both the sampling period and the integration time step are chosen as 0.01 s. The maximum interstory drifts and maximum absolute accelerations of every story units, with and without controller, are given in Table 1. The results without control to the structure system are shown in columns 2 and 3 of Table 1. When the conventional optimal control method is used and there is no time delay in the control system, i.e., \( \lambda = 0 \), the maximum response quantities and the maximum required control force are shown in columns 4 and 5 of Table 1 (denoted by LQR), respectively. The constants \( Q \) and \( R \) in LQR are chosen as the same as the above. The selection of \( Q \) and \( R \) depends upon the magnitude of the responses of the structure and the magnitude of the external forces that are being exerted upon it. They can be determined by trial and error, until the selected values result in a significant reduction of the response while the control force is also acceptable.

After that, the time delay is considered as \( \lambda = 0.1 \) s. Using the proposed optimal controller in Eq. (9), the maximum response quantities and the maximum required control force are given in columns 6 and 7 of Table 1 (denoted by DLQR), respectively. The parameters \( Q \) and \( R \) in DLQR are chosen as the same as the above cases. The dynamic interstory drifts of the third story unit are shown in Fig. 4(a), where the performance without control is represented by dotted line, with LQR (\( \lambda = 0 \)) is represented by dashed line, and with DLQR (\( \lambda = 0.1 \) s) is represented by solid line. Similarly,
Fig. 3. The maximum response quantities of every story units and maximum control force varying with time delay when the controller designed in case of no time delays is used to control the system with time delay. (a) Maximum interstory drifts: the first story unit (- - -) the second story unit (-----) the third story unit (-); (b) Maximum absolute accelerations: the first story unit (- - -) the second story unit (-----) the third story unit (-); (c) Maximum control force: (- -).

Fig. 4(b) gives the absolute accelerations of the third story unit with and without control. The control forces with LQR and DLQR are shown in Fig. 4(c). By comparing the dynamic system performance in Fig. 4 and the maximum values in Table 1, it is obvious that the system performance with either LQR or DLQR is much better than that without control. In addition, it shows that the DLQR method is more effective in reducing the maximum responses of the structure, although the performance of the LQR is more remarkable than that of the DLQR. On the other hand, if the controller, designed in the case of no time delay, is used for the system with a time delay at $\lambda = 0.1$ s, system instability occurs. This can be also observed from Fig. 3.

To further investigate the system performance with the proposed controller in Eq. (9), a case with a large time delay at $\lambda = 0.4$ s is considered. With the optimal controller, the maximum response quantities of every story unit and the maximum required control force are shown in columns 8 and 9 of Table 1, respectively. It shows that the stability of the structure control system is still guaranteed although there exists a large time delay.

Simulations of the same three-story building model are carried with various time-delay value using the control technique of time-delay compensation given in [11]. The simulation results indicate that instability in responses occurs when a time delay is at $\lambda = 0.4$ s. In the time-delay compensation control, the control algorithm is first derived without considering the presence of time delay, and then the state feedback gain is modified to compensate the time delay. Thus its system stability is not guaranteed where there exists a longer time delay.

Additional simulation studies of system performance and stability under various time delays are carried out. Using the proposed control method, the maximum response quantities of every story unit and the maximum required control force with respect to time delay are shown in Fig. 5(a), (b) and (c), respectively. It shows that the system stability is guaranteed with all time delays up to 1.0 s.

4.2. Six-story structure subjected to the El Centro earthquake

The structural model studied by Rofooei and Tadjbarsh [17] is considered herein. It is a six-story, based-isolated structure with the following parameters: $m_1 = 6.8$ m ton; $k_1 = 1.200$ kN/m; $c_1 = 2.4$ kN/s; $m_2 = 5.897$ m ton, $i = 2, 3, 4, 5$: $k_2 = 33,732$ kN/m; $k_3 = 29,093$ kN/m; $k_4 = 28,621$ kN/m; $k_5 = 24,954$ kN/m; $k_6 = 19,054$ kN/m; $c_1 = 0.002k_1$, $i = 2, 3, 4, 5$. The first story unit is the base-isolated system. Both the sampling period and the integration time step are chosen as 0.01 s in the following numerical simulations.

<table>
<thead>
<tr>
<th>Story</th>
<th>No control</th>
<th>LQR ($\lambda = 0$)</th>
<th>DLQR ($\lambda = 0.1$ s)</th>
<th>DLQR ($\lambda = 0.4$ s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$\dot{x}$</td>
<td>$x$</td>
<td>$\dot{x}$</td>
</tr>
<tr>
<td>(1)</td>
<td>1.26</td>
<td>307</td>
<td>0.58</td>
<td>198</td>
</tr>
<tr>
<td>(2)</td>
<td>1.02</td>
<td>463</td>
<td>0.56</td>
<td>240</td>
</tr>
<tr>
<td>(3)</td>
<td>0.57</td>
<td>559</td>
<td>0.34</td>
<td>332</td>
</tr>
</tbody>
</table>

Table 1
The maximum response quantities and maximum control force of the three-story structure ($x$: cm, $\dot{x}$: cm/s$^2$)
Under the El Centro earthquake excitation shown in Fig. 2(b), the maximum response quantities of all the story units without control to the base-isolated structure are shown in columns 2 and 3 of Table 2, respectively. As observed from Table 2, the advantage of using a base isolation system to protect the upper structure is clearly shown. The interstory deformation of each story unit on the base is very small compared to that of the base isolation system. However, the deformation of the base isolation system shown in row 1 of Table 2 may be excessive.

To protect the safety and integrity of the base isolation system, an actuator is connected to the base isolation system, namely an active control force is applied to

<table>
<thead>
<tr>
<th>Story</th>
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<th>LQR ($\lambda = 0$)</th>
<th>DLQR ($\lambda = 0.1$ s)</th>
<th>DLQR ($\lambda = 0.3$ s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
<td>$\ddot{x}$</td>
<td>$x$</td>
<td>$\ddot{x}$</td>
</tr>
<tr>
<td>(cm)</td>
<td>(cm)</td>
<td>(cm)</td>
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<td>594</td>
<td>10.38</td>
<td>332</td>
</tr>
<tr>
<td>2</td>
<td>0.56</td>
<td>611</td>
<td>0.31</td>
<td>350</td>
</tr>
<tr>
<td>3</td>
<td>0.52</td>
<td>626</td>
<td>0.29</td>
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<td>4</td>
<td>0.40</td>
<td>638</td>
<td>0.22</td>
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<td>0.31</td>
<td>648</td>
<td>0.17</td>
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<tr>
<td>6</td>
<td>0.20</td>
<td>658</td>
<td>0.12</td>
<td>377</td>
</tr>
</tbody>
</table>
the base floor. From the results in columns 2 and 3 of Table 2, it is observed that the base-isolated structure tends to behave like a rigid body under the El Centro earthquake. As a result, sensors are not needed for the upper story units. In other words, displacement and velocity sensors are installed on the base isolation system only. In the controller design, the weighting matrix $Q$ is chosen to be the same as that in [17], given by $Q(1, 1) = 170,000$ and $Q(7, 7) = 17 \times 10^6$ with other elements being zero. Since there is one actuator applied to the structure, the weighting matrix $R$ consists of only one element and is given by $R = 0.015$. These values of $Q$ and $R$ will be used in all numerical calculations below. Using the conventional optimal control method for the structure without any time delay, the maximum response quantities of each story unit are listed in columns 4 and 5 of Table 2, respectively. It is observed from these results that the deformation of the base isolation system is greatly reduced and the maximum response quantities of the upper structure are further reduced as well.

Then the case with time delay in the control system is considered. Using the proposed control method, the maximum response quantities with the existence of the time delays, $\tilde{\lambda} = 0.1$ s and $\tilde{\lambda} = 0.3$ s, are displayed in columns 6 and 7, and columns 8 and 9 of Table 2, respectively. It shows that these maximum values are acceptable.

Using the proposed control method, the maximum response quantities of the base isolation system and the maximum required control force with respect to time delay are shown in Fig. 6(a), (b) and (c), respectively, denoted by solid line. For comparison, the results using the controller designed by neglecting the time delay to control the system with time delay are also shown in Fig. 6, denoted by dashed line. It is observed from Fig. 6 that the dashed line appears a rapid rise as time delay increases, which means that the control performance becomes worst and worst along with the increasing of the time delay. The maximum interstory drifts of the upper five-story units on the base with respect to time delay are shown in Fig. 7(a), where the solid line represents the results using the proposed control method and the dashed line represents the results using the controller designed by neglecting the time delay to control the system with time delay. Fig. 7(b) gives the results of maximum absolute acceleration. From Fig. 7, it can be observed that the proposed control method is effective to deal with the time delay in the control system. If neglecting the time delay in the controller design, the structural responses quickly become unacceptable as the time delay increases.

![Fig. 6. The maximum response quantities of the base-isolated system and the maximum required control force varying with the time delay $\tilde{\lambda}$. (a) Maximum deformation: using the proposed controller (—) using the controller without considering time delay (- - -); (b) Maximum absolute acceleration: using the proposed (—) controller using the controller without considering time delay (- - -); (c) Maximum control force: using the proposed controller (—) using the controller without considering time delay (- - -).](image)

![Fig. 7. The maximum response quantities of the upper five story units on the base varying with the time delay $\tilde{\lambda}$. (a) Maximum deformation: using the proposed controller (—) using the controller without considering time delay (- - -); (b) Maximum absolute acceleration: using the proposed controller (—) using the controller without considering time delay (- - -).](image)
5. Conclusion

In this paper, a novel optimal control method for linear systems with time delay in vibration control is developed. A numerical algorithm for practical implementation of the proposed optimal controller is presented. Since the optimal controller is obtained directly from the differential equation with time-delay and no approximation is made in the derivation of the control algorithm, the system stability is prone to be guaranteed. Theoretical development and numerical verification show that the proposed control method is feasible for realistic applications. Simulation results indicate that the proposed control method is effective in reducing maximum responses of vibration system. If the existence of time delay is neglected in control design, the control system may suffer from system instability. In addition, the proposed optimal control algorithm for time-delay problem is compared to the time-delay compensation control algorithm by numerical simulations. It shows that the proposed control algorithm is superior to the time-delay compensation control algorithm.

References