Why pay a penalty for paying off your mortgage? A pedagogical note

J. Atsu Amegashie, December 12, 2006

Consider an individual who has T periods left to pay off his mortgage. Let $r^* > 0$ be the interest rate and let $M > 0$ be his periodic payment. Time goes from $t = 0, 1, 2, 3, \ldots, T-1$. A period could be a week, two weeks, month, year, etc. This is irrelevant to the analysis and so will not affect any of our results. Without any loss of generality but for the sake of exposition, I define a period to be a month and so the interest rate r^* is defined over the same period.

Let $P > 0$ be the remaining mortgage in the current period (i.e., time $t = 0$) and suppose that the individual makes his payments at the beginning of each period. Let P_t be the remaining mortgage in period t. For example, P_0 is the remaining mortgage at the beginning of period 0; P_1 is the mortgage at the beginning of period 1 and so forth.

Then

$$P_0 = P - M$$

$$P_1 = (1 + r^*)P_0 - M = (1 + r^*)(P - M) - M = (1 + r^*)P - (1 + r^*)M - M$$

$$P_2 = (1 + r^*)P_1 - M = (1 + r^*)[(1 + r^*)(P - M) - M] - M$$
$$= (1 + r^*)^2P - (1 + r^*)^2M - (1 + r^*)M - M$$

By inspection, it is easy to show that the remaining mortgage in period t is

$$P_t = (1 + r^*)P_{t-1} - M = (1 + r^*)^tP - (1 + r^*)^tM - (1 + r^*)^{t-1}M - \ldots - (1 + r^*)M - M$$

$$= \delta^tP - M[1 + \delta + \delta^2 + \ldots + \delta^t],$$

$$= \delta^tP - M\frac{\delta^{t+1} - 1}{\delta - 1}, \qquad (1)$$

where $(1 + r^*) \equiv \delta > 1$.

If you have T more periods to pay off your mortgage and the first period is $t = 0$, then the last period is period T-1. So your remaining mortgage should be zero in period T-1. Hence, using equation (1), we require that

$$P_{T-1} = \delta^{T-1}P - M\frac{\delta^{T} - 1}{\delta - 1} = 0, \qquad (2)$$

Therefore, your mortgage per period is obtained by solving for M in equation (2). This gives

$$M^* = \frac{P\delta^{T-1}(\delta - 1)}{\delta^T - 1} \qquad (3)^{1,2}$$

Given an interest rate of $r^* > 0$, the bank's discount factor is $\frac{1}{1+r^*} = \frac{1}{\delta} < 1$. So the present value of your monthly mortgage payments over the T periods is

$$PVM_{T-1} = \sum_{t=0}^{T-1} \frac{1}{\delta^t} M^* = \left(1 + \frac{1}{\delta} + \frac{1}{\delta^2} + \ldots + \frac{1}{\delta^{T-1}}\right) M^* = \frac{\delta^T - 1}{\delta^T} \frac{\delta}{\delta - 1} M^* \qquad (4)$$

Then putting (3) into (4) and simplifying gives

$$PVM_{T-1} = P. \qquad (5)$$

Therefore, the *present value* of your future stream of mortgage payments to the bank is *equal* to your current remaining mortgage before your payment in that period (i.e., P). This is exactly what equation (5) says. It is therefore puzzling that if you want to pay off your mortgage before the end of the amortization period, the bank imposes a penalty on you. Why?

In the above analysis, I assumed that the interest rate was constant over the entire life of the mortgage. The USA has such mortgage contracts (i.e., 25 year fixed rate mortgages). My Scotia Bank mortgage officer tells me that there are no such contracts in Canada. The longest term in Canada is 10 years. However, a variable interest rate does not affect the above analysis because I chose an *arbitrary* time period (i.e., $t = 0$). Hence, so long as the interest rate is fixed over this period (e.g., the last five years of the mortgage), the bank should not penalize me for wanting to pay off my mortgage.

Let me now explicitly analyze the case of a variable interest rate. Suppose that from period $t = 0$ to period N-1, the interest rate is r_1, and from period N to T-1, the interest rate is r_2. And suppose that at the end of period N-1, your remaining mortgage is $P_{N-1} > 0$. Of course, as before $P_{T-1} = 0$. Then using equation (2), we require that

$$P_{N-1} = \delta_1^{N-1} P - M \frac{\delta_1^N - 1}{\delta_1 - 1} > 0, \qquad (6)$$

where $\delta_1 \equiv (1 + r_1)$.

[1] M* is your mortgage before taxes. If the tax rate is t, then your after-tax mortgage is $(1 + t)M^*$. Typically, your mortgage also includes the bank's administrative fees.

[2] If mortgage payments are made at the end of each period, then the mortgage per period is $\delta M^* > M^*$. This is not surprising because making mortgage payments at the end of the period means that you are paying the mortgage at slower rate relative to paying it at the beginning of each period. Hence, your monthly mortgage in the former case must be higher.

Then your monthly mortgage from period 0 to period N-1 is

$$\hat{M} = \frac{(P\delta_1^{N-1} - P_{N-1})(\delta_1 - 1)}{\delta_1^{N} - 1} \qquad (7)^3$$

We can show that the present value of this stream of payments is

$$PVM_{N-1} = \frac{(P\delta_1^{N-1} - P_{N-1})}{\delta_1^{N-1}} = P - \frac{P_{N-1}}{\delta_1^{N-1}} \qquad (8)$$

Note that $PVM_{N-1} < P$. Also, since P_{N-1} is my remaining mortgage at the end of period N-1, it follows that $\frac{P_{N-1}}{\delta_1^{N-1}}$ is the present value of this mortgage (i.e., its value at period t = 0). Therefore, the bank should allow me to pay off my mortgage by paying them

$$PVM_{N-1} + \frac{P_{N-1}}{\delta_1^{N-1}} = P \qquad (9)$$

Hence, whether the interest rate is fixed or varies over the length of the mortgage, I should *not* pay a penalty if I want to pay off my mortgage, so long as the interest varies across different short terms of the contract but remains constant during a given term. Why then do banks impose this penalty?

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[3] In practice, banks compute the monthly mortgage as follows. For example, for a five-year term mortgage, the monthly mortgage is equal to the monthly mortgage that the customer would have paid if the five-year term interest rate were applied over a 25-year amortization period. Hence, the mortgage over a five-year term will be M* which will then make P_{N-1} endogenous. This is how the bank determines P_{N-1}. Typically, banks compound the interest semi-annually (i.e., every six months). Since a mortgage is paid monthly, the length of a period is a month. Let the annual interest rate be x and the monthly interest rate be y. If the interest is compounded monthly, then after a year (i.e., 12 months) a dollar yields $(1 + y)^{12}$ dollars. If it is compound semi-annually, a dollar yields $(1 + x/2)^2$ dollars. Since the interest rate is compounded semi-annually but the mortgage payments are made monthly, we need to find the monthly interest rate, y, which makes the return on a dollar to the bank under monthly compounding *equal* to the return under semi-annual compounding. That is, we want $(1 + y)^{12} = (1 + x/2)^2$. This gives the effective monthly interest rate as $y = (1 + x/2)^{1/6} – 1$, where x is the annual interest rate. So in our analysis above, $y = r = (1 + x/2)^{1/6} – 1$, where x is the annual interest rate. Given a 25-year amortization period, the number of months is, T = 25(12) = 300.Using these figures and P to compute M* gives the monthly mortgage. As noted in footnote 2, if mortgage payments are made at the end of the month, then the monthly mortgage payment is δM*. I have verified this formula with the results of mortgage calculators on the web (e.g., http://www.canadamortgage,com).

Some Answers

One may argue that the penalty is imposed to cover transaction costs, administrative fees, etc. But this argument is not convincing. If I do not pay off my loan early and follow the usual amortization period, I pay no penalty. I only make my monthly payments for the duration of the contract. So if I want to pay the *present value* of all those future monthly payments, why should the bank impose a penalty on me? Indeed, the bank's transaction costs should be lower when I pay off the mortgage now than when it has to deal with me periodically over a given period of time. I cannot think of any transaction costs which arise when I pay off the loan now but don't arise when the bank deals with me periodically. More importantly, any transaction costs or administrative fees are reflected in one's mortgage payments. So if I pay the present value of these future payments, the imposition of a penalty for doing so remains a puzzle.

A more plausible answer is the following. Suppose that the interest rate fell *after* your mortgage contract took effect. Let r** be the new interest rate. Then using equation (4), the bank's new discount factor $1/\hat{\delta}$ is now higher than what it was before, since $(1 + r^{**}) < (1 + r^*)$. Then given $1/\hat{\delta} > 1/\delta$ and noting the result in equation (5), it follows that the present value of your mortgage payments is

$$PVM_{T-1} = \sum_{t=0}^{T-1} \frac{1}{\hat{\delta}^t} M^* > P \qquad (10)$$

Hence, if you want to pay off your loan, the bank imposes a penalty of $PVM_{T-1} - P > 0$ on you.

The intuition is straightforward.[4] Note that the bank computed your mortgage of M* based on a given interest rate, r*, of financing it. Hence, by continuing to pay a mortgage of M* after the fall in the interest rate, the bank gets a return of r* on each dollar while the cost of financing your mortgage to the bank is now $r^{**} < r^*$. Therefore, the bank makes a profit of $r^* - r^{**} > 0$ on each dollar. By paying off your mortgage, you deny the bank this profit.[5] That is why it imposes the penalty on you. Also, by imposing this penalty the bank discourages you from refinancing your mortgage when interest rates fall.

Conversely, the bank should be more than happy to let you pay off your mortgage, if interest rates rise since they make a loss on your mortgage. Do banks allow their clients to pay off their mortgages in such cases? Well, this is irrelevant because it is not in a client's interest to pay off his mortgage when interest rates rise.

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[4] My thanks are due to Edward Kutsoati for this intuitive answer and for prodding me in this direction.

[5] Note that the present value calculation in equation (10) assumes that the fall in the interest rate will remain during the entire term of the mortgage. This is unlikely to be case. However, this does not affect the argument for why the bank imposes a penalty on you.