Dynamic games and variational inequalities - II

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Outline

- Evolving (disequilibrium) strategies
  - Definition
  - Mathematical setting
- Hybrid systems
- Dynamic games as hybrid systems
- Network problems and hybrid systems
- Examples
- Bonus: if there is time - ABM market models

Evolving (disequilibrium) states

- Assume a dynamic game or a time dependent network problem involving a time interval [0,T];
- We consider that the problem described by these models is understood to evolve through states that are not equilibrium ones;
- Ideally, if the equilibrium states are stable, then the disequilibrium states should evolve towards the equilibrium ones.
- How can we define disequilibrium in a mathematically rigorous way?

- Assume an equilibrium problem modeled as a VI

$$find\ x \in K\ subset\ X\ s.t. \langle F(x), y - x \rangle \geq 0,\ for\ all\ y \in K$$

$$where\ usually\ K \subset X = L^2([0,T], R^q)$$

- Then x is an equilibrium state
- Consequently:

$$x^o \in K\ \ for\ which\ \exists\ y \in K\ s.t. \langle F(x^o), y - x^o \rangle < 0$$

is a disequilibrium state, or an evolving state.

- Why evolving?

Connection with dynamical systems

- Let us assume that we have a continuous time process described by

$$\frac{dx}{d\tau} = P_{T_K(x(\tau))}(-F(x(\tau))),\ x(0) \in K$$

or

$$\frac{dx(\tau)}{d\tau} = P_K(x(\tau) - \alpha F(x(\tau))) - x(\tau),\ x(0) \in K,\ \alpha > 0$$

or

$$\frac{dx}{d\tau} \in -F(x(\tau)) - N_K(x(\tau)),\ x(0) \in K$$

- Moreover we require two conditions for each of the processes above:
  1. the set of stationary points = solution set of VI(F,K)
  2. all solutions x(t) have to remain in K, for all t.
- With the above, an equilibrium state is defined as:
  - $x(\tau) = constant\ such\ that\ P_{T_K(x)}(-F(x)) = 0,$
  - $x(\tau) = const.\ such\ that\ P_K(x - \alpha F(x)) - x = 0$
  - $x(\tau) = const.\ whenever\ 0 \in -F(x) - N_K(x)$

- Then the disequilibrium or evolving states can be defined as

  - $x^o(\tau) \neq const.\,, x^o(\tau)\, is\ a\ soln\ of\ \ DE1$

  or

  - $x^o(\tau) \neq const\,, x^o(\tau)\, is\ a\ soln.\ of\ \ DE2$

  or

  - $x^o(\tau) \neq const\,, x^o(\tau)\, is\ a\ soln\ of\ \ DI$

- What is τ ? In ODE theory, it is the time of the differential equation, so an abstract time. Thus in general $\tau \in [0, \infty]$

Example

- Can we visualize the dynamic process so far?
- Yes, τ measures the ”deformation” of the initial curve x(0) from K into the equilibrium curve x in K
- Mathematically, great explanation !
- In practice: difficult to implement = we do not have a priori knowledge of x(0) !

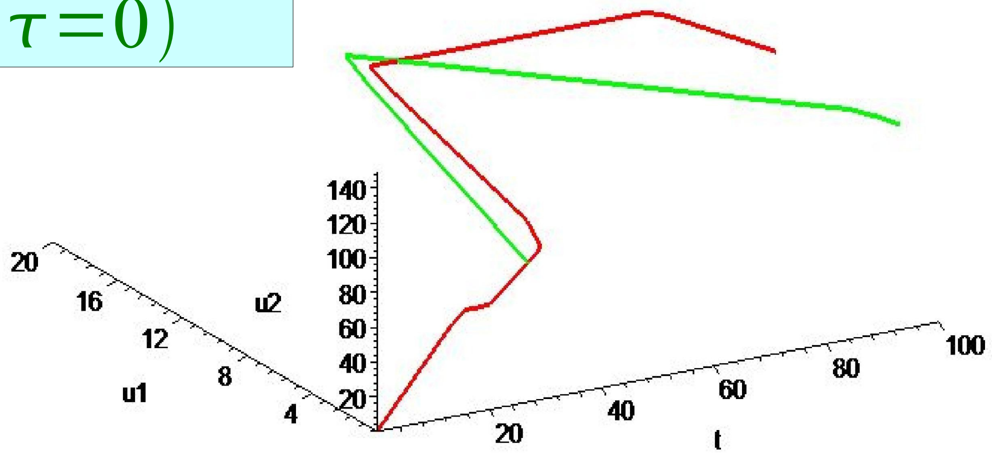
An actual example

In green we have

$x(t, \tau = 0)$

In red, we have

$x(t, \tau = const.)$



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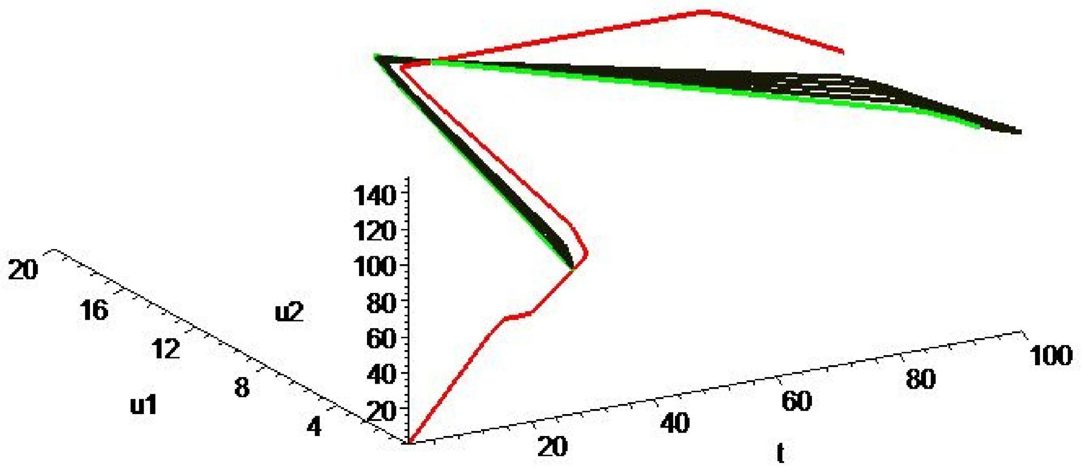
An actual example

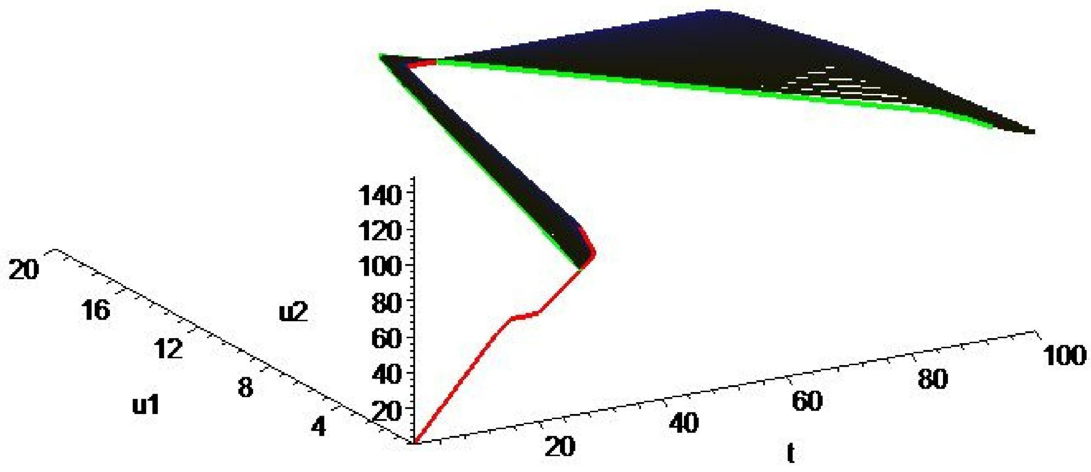
In black, we have

$$x(t, 0<\tau \neq const.)$$



An actual example

This picture is after 100 steps in τ
Can we say something about stability of equilibria?



What is known about disequilibrium states

- Assume we have computed a disequilibrium state

$$x(t, \tau = \tau_0) \in K$$

- Under pseudo-monotonicity conditions on F, as well as L-continuity, when $\tau \to \infty$ the disequilibrium converge in L^2 norm to the equilibrium curve (note that in these cases, the equilibrium curve is unique !)

Different approach to describe disequilibria – Hybrid dynamical systems

- These are systems with two time lines:

  - a continuous time
  - a discrete one (finite or countable)

- The discrete time line consists of a collection of time instances called event times

Different approach to describe disequilibria – Hybrid dynamical systems

- These are systems with two time lines:

  - a continuous time
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- The discrete time line consists of a collection of time instances called event times

- It is considered that we follow a complex system over a time interval: initial → 1st event time

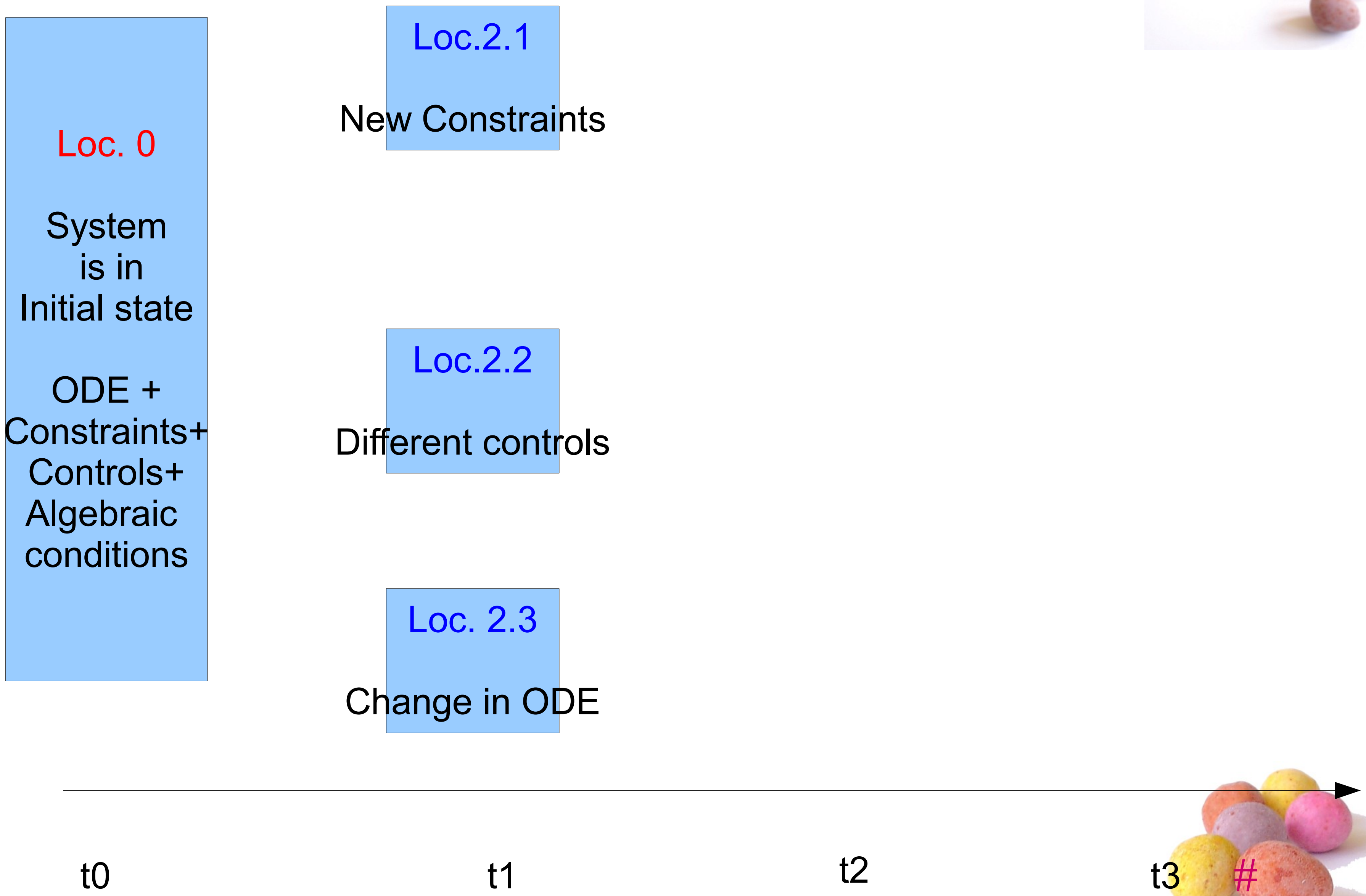
- At each event time the system's behaviour is changing, perhaps substantially, and new rules may apply

Loc. 0

System
is in
Initial state

ODE +
Constraints+
Controls+
Algebraic
conditions

t0 t1 t2 t3 #


Loc. 0
System is in Initial state
ODE + Constraints+ Controls+ Algebraic conditions
Loc.2.1
New Constraints
Loc.2.2
Different controls
Loc. 2.3
Change in ODE
t0
t1
t2
t3
#

Loc. 0

System
is in
Initial state

ODE +
Constraints+
Controls+
Algebraic
conditions

Loc.2.1

New Constraints

Loc.2.2

Different controls

Loc. 2.3

Change in ODE

Loc. 3.1

Algebraic
conditions
change

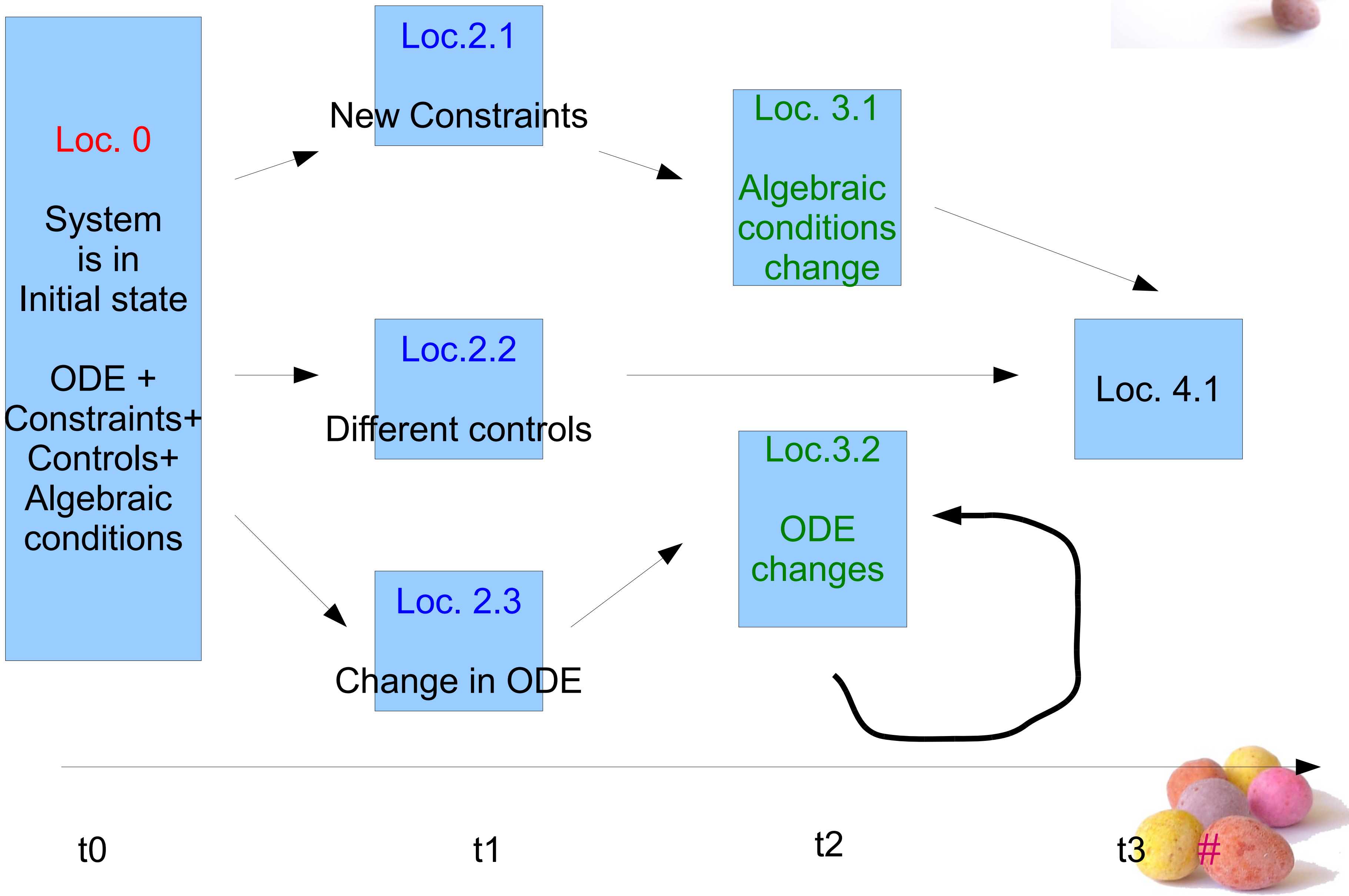
Loc.3.2

ODE
changes

Loc. 4.1

t0 t1 t2 t3

#


Loc. 0
System
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change
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ODE
changes
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#

- For each arrow in the graph, an event time takes place
- In each of the locations, the system we study evolves for a given time interval $[loc\,i, loc\,j]$
- At each location, we can solve the continuous time system and obtain a piece of curve (trajectory):

$$x_{loc} = x_{loc}(t), t \in [loc\,i, loc\,j]$$

- Then a hybrid trajectory is a collection of pieces of curves $x_{loc,\,path}$ where *path* is a path in the graph (a succession of events)

Dynamic games

- Recall our previous definition of a dynamic oligopoly game, as given in Basar and Olsder:
- [0,T] – time interval; we have m players (producers of the commodity)
- Player i's utility function is the profit

$$v_i = v_i(t, x) : [0, T] \, x \, K \rightarrow L^2([0, T], R^n)$$

$$K_i = \{ x \in L^2([0, T], R^n), l_{ij}(t) \leq x_{ij}(t) \leq upp_{ij}(t),$$

$$\forall \; j = 1, n, \; for \, a.a. \, t \}$$

- The producers try to maximize their profits, assuming that their strategies

$$x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))$$

  evolve according to a "differential equation"

$$\frac{dx_i}{dt} = f_i(x_i(t)), x_i(0) \in K_i$$

- Letting $K = K_1 \, x \, K_2 \, x \ldots x \, K_m$ we will build a diff process

$$\frac{dx}{dt} = f(x(t)), x(0) \in K$$

  with the property that

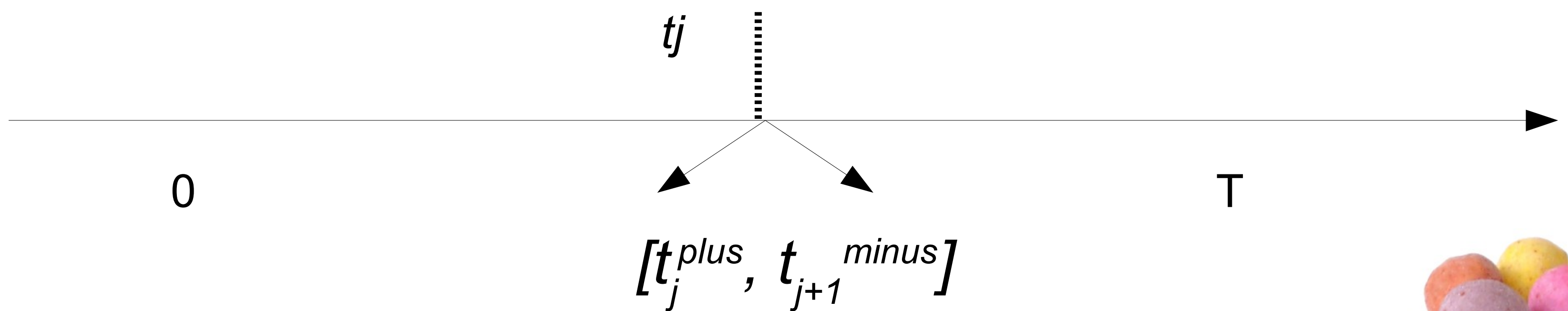
$$x(t) \rightarrow_{L^2-norm} x(t, \tau = const) \, sol \; of \; EVI$$

Hybrid systems & dynamic games

- To build the diff process, we use a hybrid system approach;
- We divide [0,T] and analyze the evolution through non-equilibrim states on each subinterval of

$$t_0=0 < t_1 < t_2 < .... < t_{k-1} < t_k=T$$

- In our work, we consider each division point an "event":
  - a finite number of event times t_j -----> $[t_j^{plus}, t_{j+1}^{minus}]$



- The time interval of evolution is now

$$[0,T]' = ([0,T] - \{t_{1,.}\,..,t_{l-1}\}) \cup \{t_{1,}^{minus}\, t_{1,.}^{plus}\,..,t_{l-1}^{plus}\}$$

- On each interval $[t_j^{plus}, t_{j+1}^{minus}]$ we consider the following cont. dynamics:

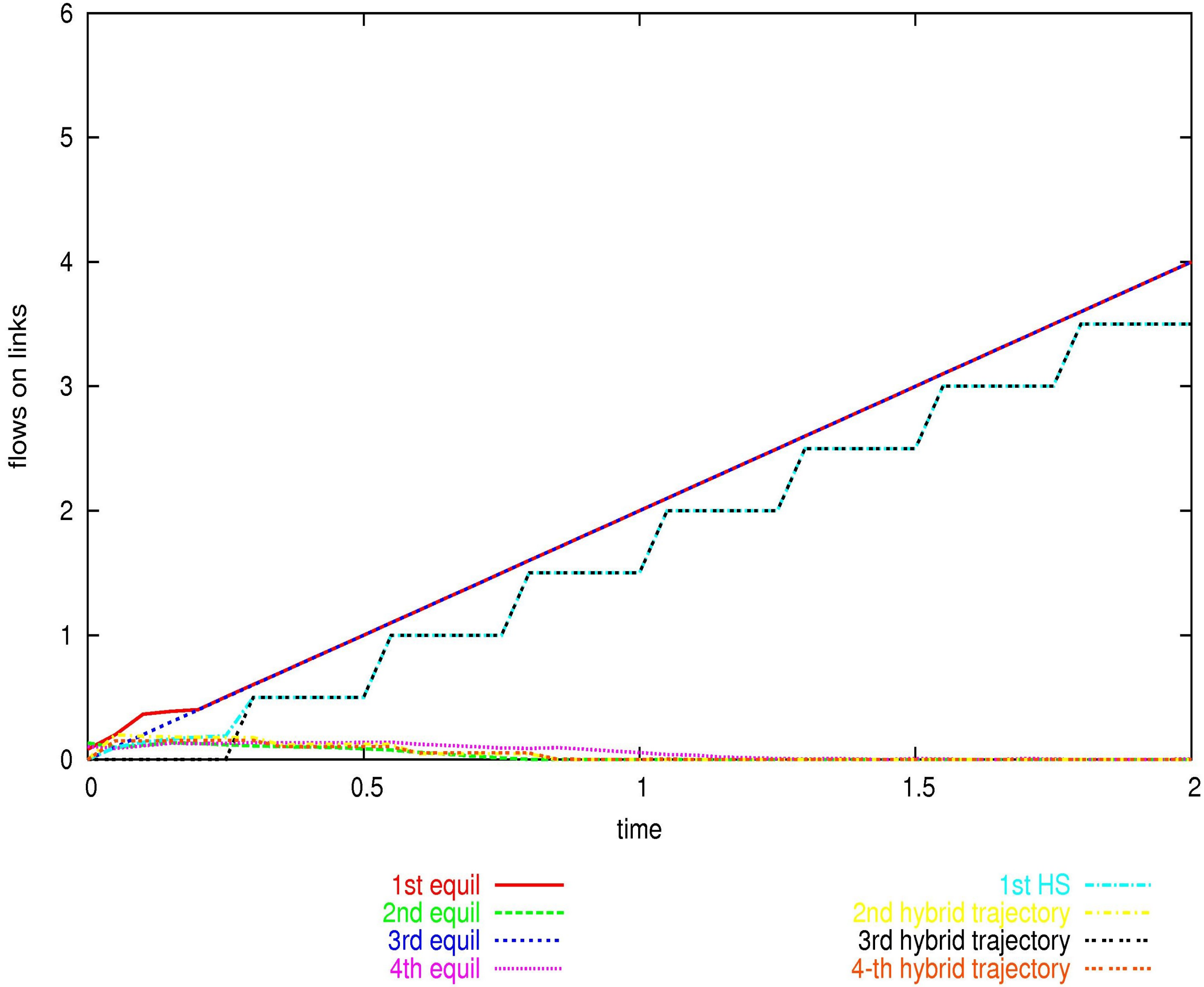
$$dx(t)/dt = P_{T_{K(t_j)}(x(t))}(-F(x(t))),\ x(t_j^{plus}) \in K(t_j),$$

$$F : K(t_j) \to R^q,\ where\ F = (\ldots, -grad_{x_{ij}} v_i(t_j), \ldots)$$

$$P_{T_{K_j(x)}}(-F(x)) = 0\ gives\ the\ equil.\ strats.\ of\ game\ at\ t_j$$

- At t_{j+1}, the dynamics switches to a different constraint set $K(t_{j+1})$ with a jump rule

$$x(t_{j+1}^{plus}) = g(x(t_{j+1}^{minus})),\ g : K(t_j) \mapsto K(t_{j+1})$$


flows on links
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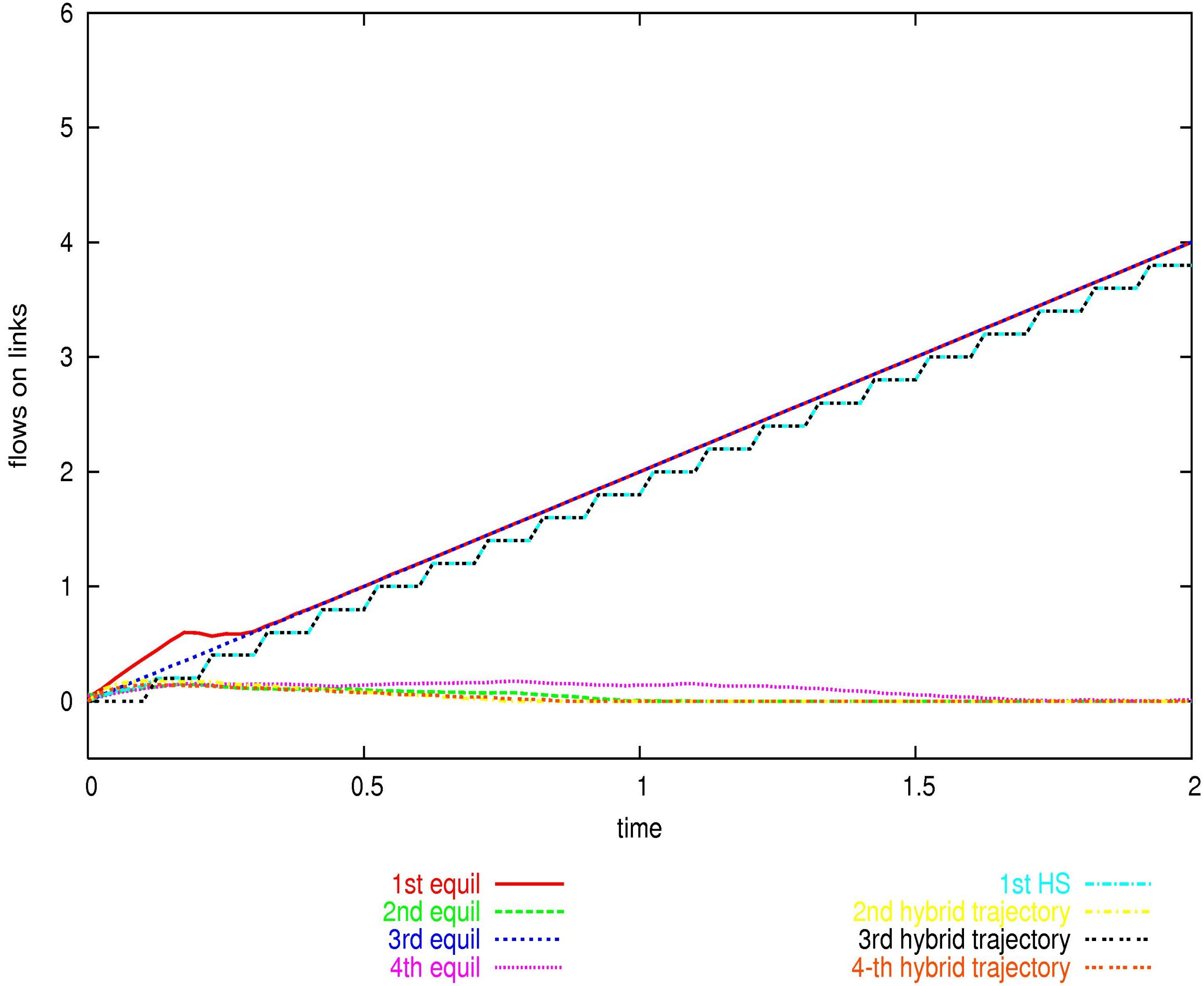
1st HS
2nd hybrid trajectory
3rd hybrid trajectory
4-th hybrid trajectory


flows on links
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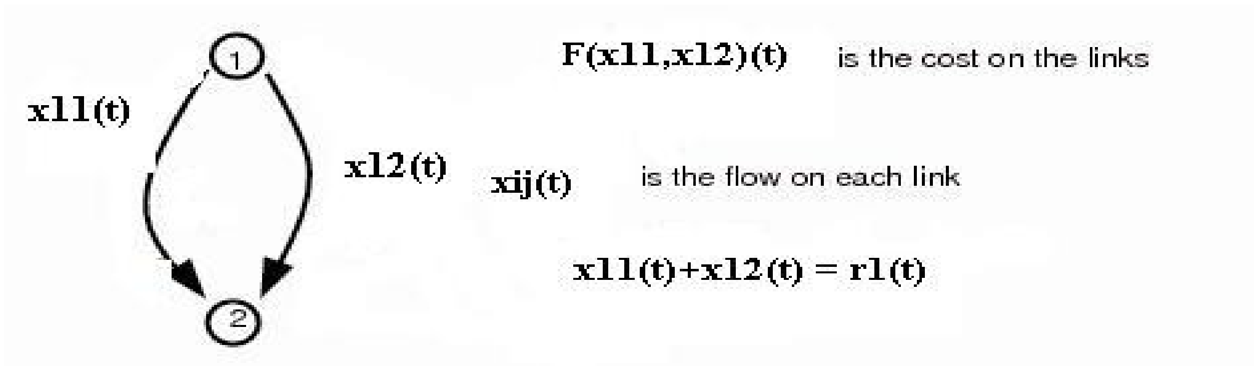
4th equil
1st HS
2nd hybrid trajectory
3rd hybrid trajectory
4-th hybrid trajectory

Dynamics of traffic networks

- 

- Here the flows are bounded in time; both bounds can depend on time
- Constraint set:

$$K = \{ x \in L^2([0,T], R^2), (0,0) \leq x(t) \leq \mu(t) \; a.e. ,$$

$$\Sigma_{j=1}^{2} x_j^{\,1}(t) = r^1(t) \; a.e. \}$$

- We assume now we have a time-dependent network with $i \in \{1,2,\ldots,l\}$ origin/destination pairs with a generic closed, convex nonempty constraint set given by

$$K = \{ x \in L^2([0,T], R^q), \lambda(t) \leq x(t) \leq \mu(t) \; a.e.,$$

$$\Sigma_{j=1}^{q} a_{ij}(t) x_j^i(t) = r^i(t), i = 1,\ldots, l \; a.e. \}$$

- In this case: what happens to the user equilibrium defined before, over the course of a finite time interval [0,T]

- Definition is maintained, but the mathematical context is different
- A curve $x^0(.) \in L^2([0,T], R^q)$ is a succession of equilibrium states if, for a.a t in [0,T]:
  - routes with zero flows have higher cost than those with positive flows.
  - all routes with positive flow should have the same cost
- Using a hybrid systems approach, we can describe the evolution of non-equilibrium states of this problem

Hybrid systems & dynamic equil. pbs.

- on each interval $[t_j^{plus}, t_{j+1}^{minus}]$ we follow a continuous dynamics constrained to the set

$$\frac{dx}{dt} = F_j(x(t)),\ x \in K(t_j) \subset R^{mn},\ where$$

$$F_j(x) = 0 \text{ gives the equilibrium strategies of the game at } t_j$$

- At the end of the interval we "jump" to the next continuous dynamics on the next constraint set via

$$x(t^{plus}{}_j) = g(x(t^{minus}{}_j)) \in K(t_{j+1})$$

- The constraint sets arise from the applied problem; the # of event times, and the jump rule are defined by the modeller

Example

- We have

$$K = \{ x \in L^2([0,T], R^q), \lambda(t) \le x(t) \le \mu(t) \; a.e.,$$

where $$\Sigma_{j=1}^{q} a_{ij}(t) x_j^{\,i}(t) = r^i(t), i = 1,\dots, l \; a.e. \}$$

q=5, $$\lambda(t) = 0, \mu(t) = 200, \; for \; all \; t \wedge l = 2$$

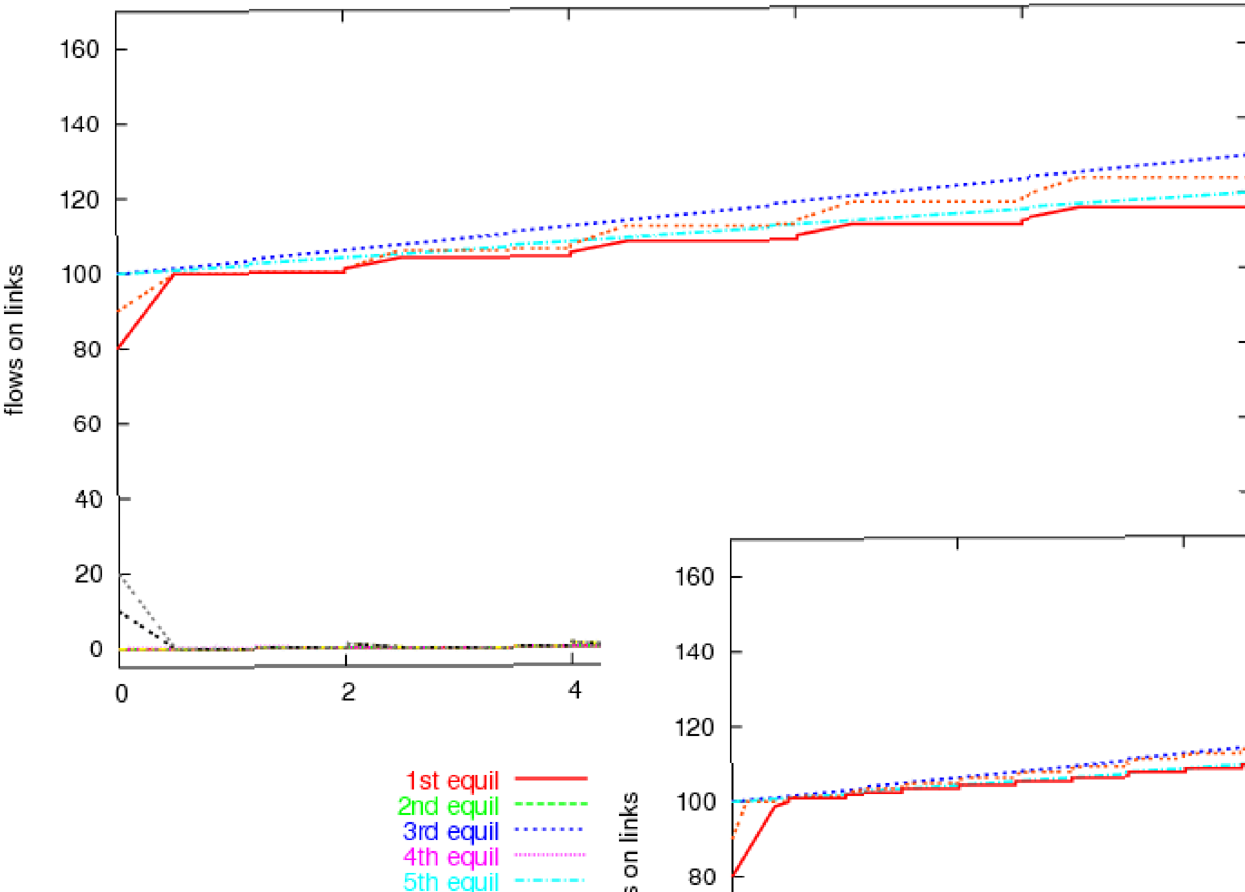
$$F_1(t) = 3\mathrm{x}_1^1(t) - x_1^2(t) + 5\mathrm{x}_1^3(t) - \frac{t}{2}, \; with \; \rho_1(t) = 3\mathrm{t} + 100$$

$$F_2(t) = 4\mathrm{x}_1^1(t) + 6\, x_1^2(t), \; with \; \rho_2(t) = 2\mathrm{t} + 100$$

$$F_3(t) = 2\mathrm{x}_1^1(t) - 2\, x_1^2(t) + 4\, x_1^3(t) - t$$

$$F_4(t) = 2\mathrm{x}_2^1(t) - 0.5\, x_2^2(t) + 1.5\mathrm{t} + t^2$$

$$F_5(t) = 1\mathrm{x}_2^1(t) - 1.5\, x_2^2(t) + t^2$$


flows on links
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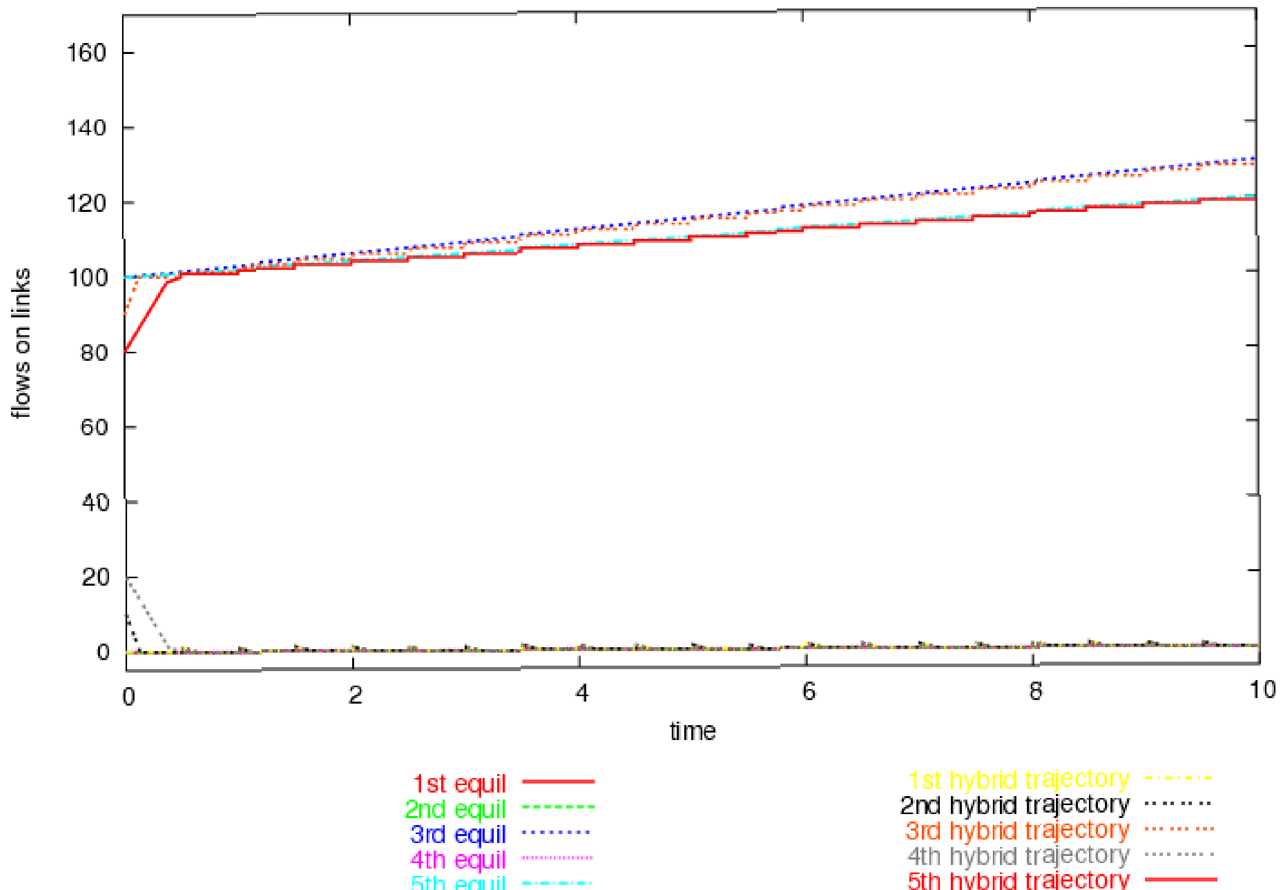
flows on links
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4th equil
5th equil
1st hybrid trajectory
2nd hybrid trajectory
3rd hybrid trajectory
4th hybrid trajectory
5th hybrid trajectory

Stability results with hybrid trajectories

- The fact that hybrid trajectories converge to the equilibrium strategies can be proven if the ODE's used have strongly pseudo-monotone righthand sides with a degree < 2
- This result holds for projected ODE's and for a jump rule that is at least non-expansive
- It is possible the result can be relaxed or modified, depending on the cont. time process chosen in this construction

- Numerical implementation: Matlab;
- On each subinterval, one can choose the appropriate num method for the cont process chosen
- NOTE: to construct the hybrid trajectories, we need to know/prove that the EVI model has a unique solution
- To construct the hybrid trajectories we piece together the trajectories on each subinterval

Vaccination games

- **Vaccine coverage (VC)** is the collective result of the **vaccination decisions of other individuals**
- Hence, the individuals in a given population are effectively engaged in a strategic interaction (a "game") with one another, mediated by transmission dynamics.
- We study a population of size N with k distinct population groups, each of proportion ε_i where

$$\varepsilon_i \neq \{0, 1\} \wedge \varepsilon_i \neq \varepsilon_j, \forall i, j \in \{1,.., k\}.$$

- Each group has a strategy set $\{P_i | P_i \in [0,1]\}$

where $p = \Sigma_{i=1}^{k} \epsilon_i P_i$ is the VC (vacc. coverage).

A game that does not have stability

- The payoff function in a group where the perceived relative risk is r_i , and where the vaccine coverage in the population as a whole is *p*, is given by

$$u_i(P) = -r_i P_i - \pi_p^i (1 - P_i)$$

$$\pi_p^i = \frac{b}{a+p}, \; a = 0.1, b = 0.08$$

- Imagine a game like above but where the strategies depend on time ($P_i = P_i(t)$), and the risks and group sizes may vary; the VC, $p = p(t)$, is computed from the model;

Example

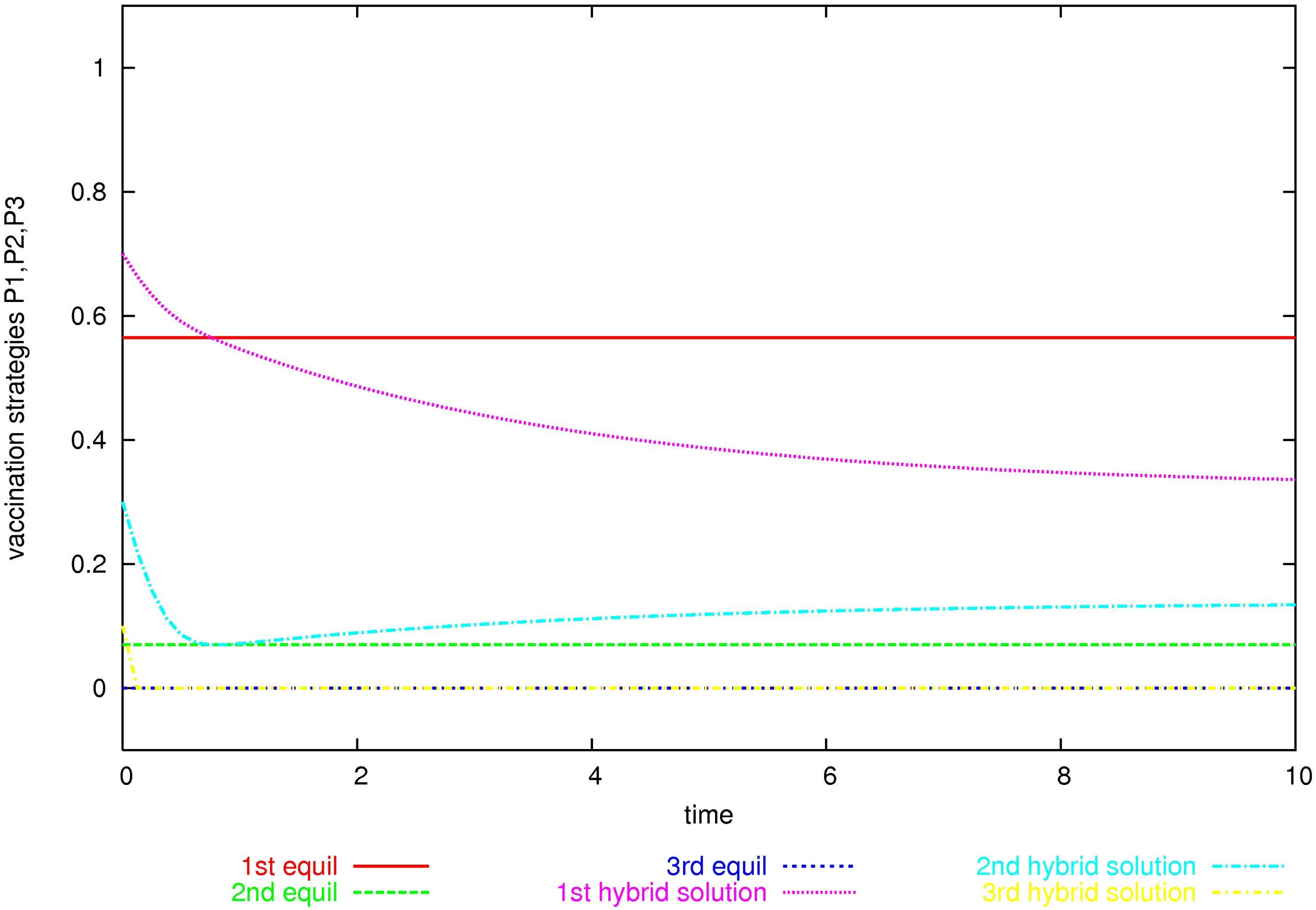
- The interval is [0, 10 yrs] and group sizes are:

$$\varepsilon_1(0)=0.2, \varepsilon_2(0)=0.6, \varepsilon_3(0)=0.2$$

- The risks: $r_1\ (0)=0.5,\ r_2\ (0)= 1,\ r_3\ (0)= 1.5$

$$\pi_p(t)=\frac{0.08}{0.1+\Sigma_{i=1}^{3}\varepsilon(t)P_i(t)}$$

- We then consider the risks decreasing suddenly by 0.1, resp. 0.2, 0.2 at t=4;
- Size of group 2 decreses by 0.2 where groups 1 and 3 gain 0.1 lineary over [0,10];


vaccination strategies P1,P2,P3
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time
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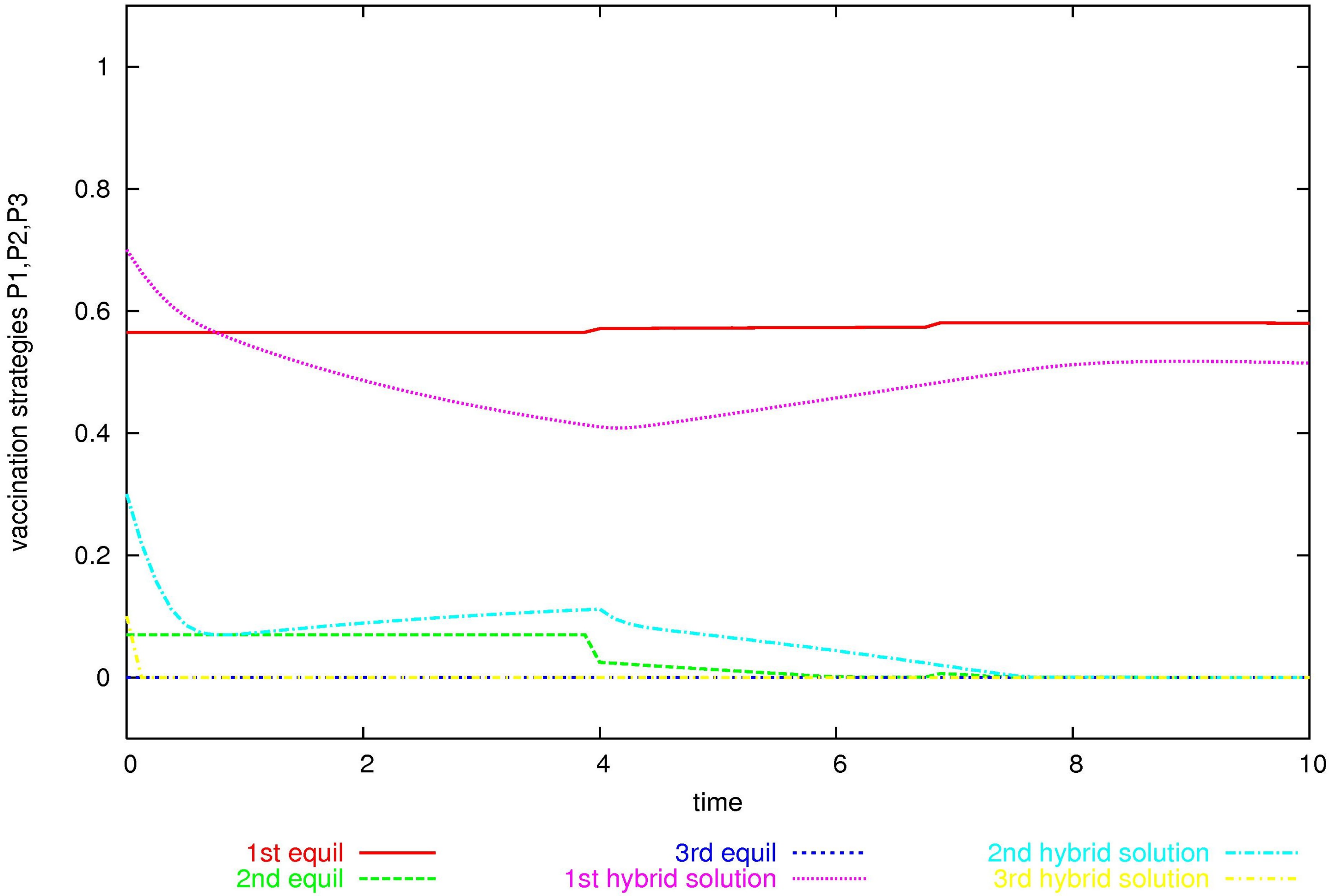
2nd equil
3rd equil
1st hybrid solution
2nd hybrid solution
3rd hybrid solution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time
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1st hybrid solution
2nd hybrid solution
3rd hybrid solution

References

- Basar & Olsder, Noncooperative Dynamic Game Theory
- Van der Schaft & Schumacher, Introduction to Hybrid Dynamical Systems

Other:

- www.uoguelph.ca/~mcojocar