# DYNAMIC GAMES AND VARIATIONAL INEQUALITIES ON TIME-DEPENDENT SETS - I

Monica-Gabriela Cojocaru
Associate Professor Mathematics
University of Guelph
Guelph, Ontario, Canada

# Outline

* Quick review of Nash-Cournot games
    1. Nash games - review
    2. Cournot games and oligopolies
    3. Classic methods of solving for a game’s equilibrium strategies

* Variational Inequalities
    1. Classic definitions and examples
    2. Solving games with VI

* Dynamic games & VI

# Quick review of Nash-Cournot games

* Let us recall a classic (linear) non-0 sum game: Battle of the sexes

| | | Woman | |
|---|---|---|---|
| | | $I$ | $II$ |
| $Man$ | $I$ | (1,4) | (0,0) |
| | II | (0,0) | (4,1) |

This game has Nash equilibria in mixed strategies, that is to say $(x^*, y^*) \in [0,1]^2$ so that

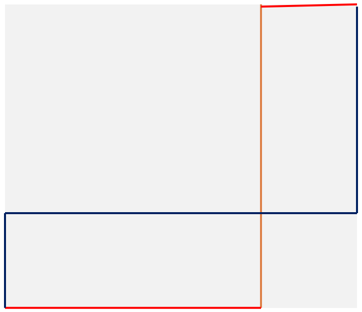
$$\begin{cases} e_1(x^*, y^*) \geq e_1(x, y^*), \forall\, x \in [0,1] \\ e_2(x^*, y^*) \geq e_2(x^*, y), \forall y \in [0,1] \end{cases}$$

where $e_1(x, y) = x^T\, A\, y; e_2(x, y) = y^T B x$ ;

* Here they are:

$$((x^*, 1 - x^*), (y^*, 1 - y^*)) = \begin{cases} ((1,0), (1,0)) \ \rightarrow (\text{I1, II1}) \\ ((0,1), (0,1)) \ \rightarrow (\text{I1, II1}) \\ ((4/5\ \text{I1}, 1/5\text{I2}), (1/5\ \text{II2}, 4/5\ \text{II2})) \end{cases}$$

- This game can be solved relatively easy by the reaction curves method; this is simply saying :
- Solve for
  $x^* \ that\ optimizes\ e_1(x,y) for\ any\ y \in [0,1]$
  $y^* \ that\ optimizes\ e_2(x,y) for\ any\ x \in [0,1]$
- Graphically:



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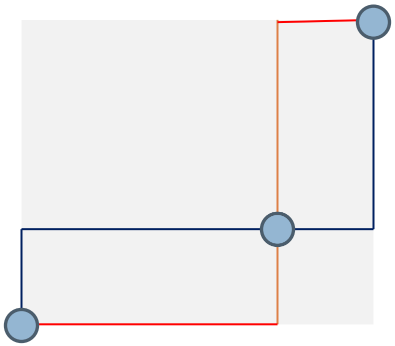
- Graphically:
- The nodes are the

N-equil in this case



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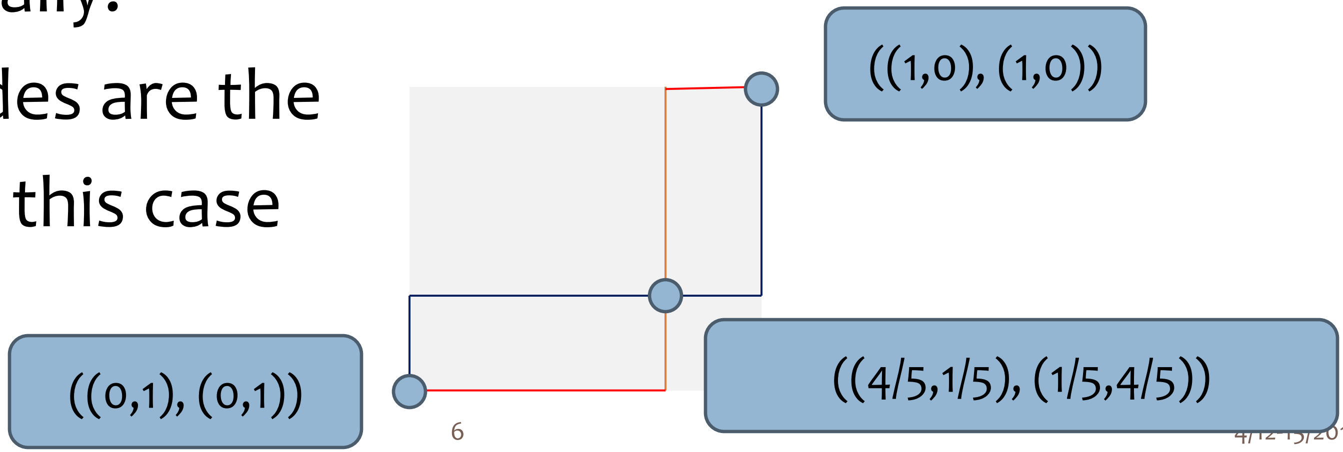

# N-players Nash games

* In an N-player non-cooperative, static, finite Nash game, the n-tuple of strategies

$(x_1^*, x_2^*, \dots, x_N^*)$, where player $i$ plays the mixed strategy $x_i^*$ , is an *equilibrium* (in the sense of Nash)if for all other strategies $(y_1, y_2, \dots, y_N)$:

$$e_i(x_1^*, x_2^*, \dots, x_i^*, \dots, x_N^*) \geq e_i(x_1^*, \dots, y_i, \dots x_n^*),$$

$$1 \leq i \leq n$$

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- In general, the reaction curves method (or optimization) is used to solve these games
- They are not necessarily linear anymore!

# Cournot games - oligopolies

- These are non-cooperative, static, infinite games (date back to Cournot in 1838)
- Oligopolies = Also know as [M,∞]-market games

→ M stands for a number of traders (firms) who sell a product

→ ∞ stands for consumers who have money to exchange for the product; however they are so many they can be considered ∞

- Thus we represent them via a single utility function

$$u(p_1, p_2, \dots, p_M, q_1, \dots, q_M)$$

where

$$\begin{cases} p_i = price\ the\ i^{th}\ firm\ sets\ for\ its\ product \\ q_i = amount\ of\ that\ product\ bought\ by\ consumers \end{cases}$$

Assumptions:

1. Consumers are told prices $p$ and they buy $q$ to maximize their utility → *price formulation model*
2. Producers decide quantities $q$ and the consumers' utility function determines the prices $p$ → *quantity formulation model*

* *Let us assume first.* Then consumers are reduced to a set of price-demand equations:

$$q_i = f_i(p_1, p_2, \dots, p_M)$$

* The producers' utility is profit:

$$e_i(p_1, \dots, p_M) = p_i q_i - c_i(q_i)$$

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# Cournot equilibrium

- A Cournot equilibrium is a vector of prices $p^c = (p_1^c, p_2^c, \dots, p_M^c)$ so that for all firms
  i = 1,… ,M we have

$$e_i(p_1^c, \dots, p_i^c, \dots, p_M^c) = \max_{\{p_i\}} e_i(p_1^c\,, \dots, p_i, \dots, p_M^c)$$

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- In general prices are considered bounded

# Oligopoly example

* Let i=1,2 and the price-demand equations:

$$q_1 = \max\left(1 + \frac{1}{3}p_2 - \frac{1}{2}p_1, 0\right), c_1(q_1) = 0$$

$$q_2 = \max\left(1 + \frac{1}{4}p_1 + \frac{1}{2}p_2, 0\right), c_2(q_2) = 0$$

* Since $q_1 \geq 0$ *and* $q_2 \geq 0$ *we can assume*

$$0 \leq p_1 \leq 2 + \frac{2}{3}p_2,$$

$$0 \leq p_2 \leq 2 + \frac{1}{2}p_1$$

* Then the profit functions are:

$$e_1(p_1, p_2) = p_1 + \frac{1}{3}p_1 p_2 - \frac{1}{2}p_1^2$$

$$e_2(p_1, p_2) = p_2 + \frac{1}{4}p_1 p_2 - \frac{1}{2}p_2^2$$

* Although not a Nash game, the reaction curves method comes to the rescue: we find

$$p_1^c = \max e_1(p_1, p_2^c) from \ \frac{de_1}{dp_1}(p_1, p_2^c)$$

$$p_2^c = \max e_2(p_1^c, p_2) from \ \frac{de_2}{dp_2}(p_1^c, p_2)$$

$$which\ give\ p_1^c = \frac{16}{11}, p_2^c = 15/11$$

# Variational Inequalities & games

* Recall a classic VI

$$\text{find x} \in K\ s.t. < F(x), y - x > \geq 0, \forall y \in K,$$

where:

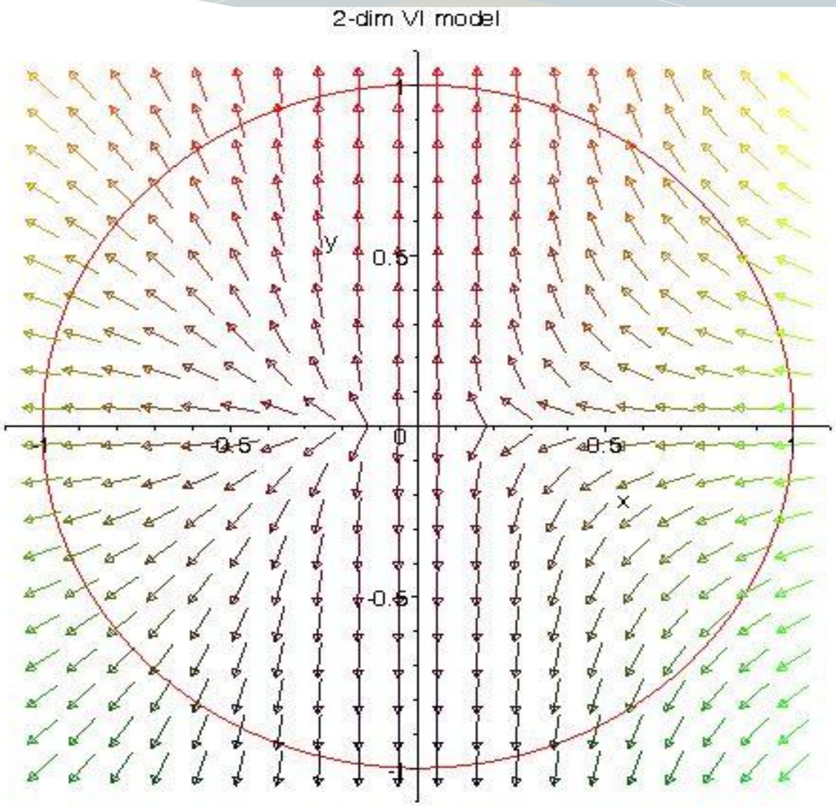
* K is a subset of X – known space, usually closed, convex and non-empty
* F: K → X a mapping with "nice" properties
* <. , .> is an inner product type mapping

What does it mean to solve this problem?

Here is a simple mathematical interpretation in terms of "angles" (inner product):

$F(x, y) = (-x^2, y)$

K = unit disk



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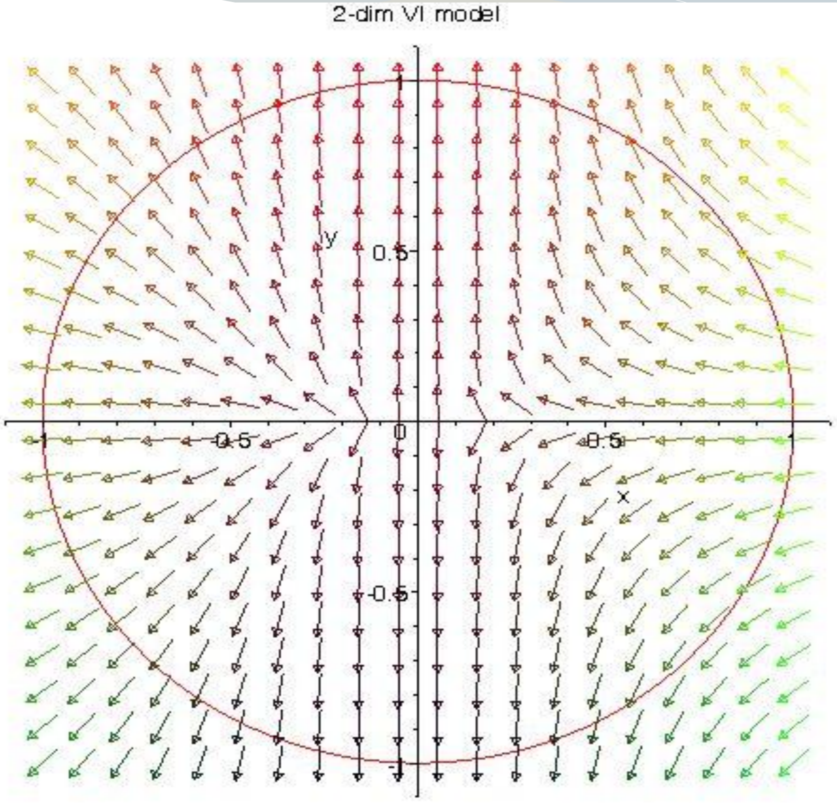
Solutions: (0,0) , (-1,0)



* Another classic example of a VI:
  the characterization of optimal points in differential optimization

$$x \in K \; so \; that \; f(x) = \min_{y \in K} f(y)$$

means find x so that

$$< \nabla f(x), y - x > \geq 0$$
$$K \subset R^N, F: K \to R^N \; diff.$$

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  means find x so that

$$< \nabla f(x), y - x > \geq 0$$
$$K \subset R^N, F: K \rightarrow R^N \; diff.$$

  Of course this is in principle what relates VI and games !

# Nash-Cournot games and VI: finding optimal strategies

* Classic result (L-S ‘67, Gabay-Moulin ‘80 -VI):

1. *Provided the utility(payoff) functions $e_i$ are of class $C^1$ and concave with respect to the variables $x_i$, then x* ∈ K is a Nash equilibrium*

*if and only if it satisfies the VI*

$$< F(x^*, x - x^* > \geq 0,$$
$$where\ F = (-\nabla_{x_1} e_1(x), \dots, -\nabla_{x_n} e_n(x))$$

# Generalized Nash games

* A generalized Nash game is an n-person noncooperative game with non-disjoint strategy sets; other names for this game form include social equilibria and pseudo-Nash games (Harker 1991 – QVI)
* Here players can influence each other's utilities and also each other's strategy sets
* Assume n players

  $X_i \ str.set\ of\ i;$

  $X = \times_{i=1,n} X_i \ - cart.product\ of\ strat\ sets$

  $X^{n\setminus i} = X \setminus X_i$

  $Define\ K^i : X^{n\setminus i} \to X_i\ and\ K^i(x) \subset X_i$

  $u^i : graph(K^i) \to R\ are\ utilities$

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So a GN game is a triple $\left(X^i, K^i, u^i\right)$ and a GNE is a vector $x^* = \left(x_1^*, \dots, {x^*}_n\right) s.t.$

$$x_i^* \in K^i\left({x^*}_{n\backslash \mathrm{i}}\right), \forall i$$

$$u^i(x^*) \geq u^i\left(y^i, {x^*}_{n\backslash \mathrm{i}}\right), \forall y^i \in K^i({x^*}_{n\backslash \mathrm{i}})$$

- This game can be solved via a QVI problem:

$$Find\ x^* \in K(x^*) s.t.$$

$$< F(x^*), x - x^* > \geq 0\ forall\ x \in K(x^*)$$

$$where\ F = (\dots, -\nabla_{x^i}\ u^i\left(x_i^*, {x^*}_{n\backslash \mathrm{i}}\right), \dots.)$$

- Original result has only one implication, not the other

# Games and EVI – dynamic games

- (Basar & Olsder – 2002ed. ) These are games whose strategy sets are time dependent:

i ∈ 1,…,k is the number of players and [0,T] is the pre-specified time interval the game is played over;

$E$ach player's payoff functional is given by $U_i(t, x(t))$

- x(t) are the state trajectories x(t) ∈ K, and

- The trajectory x ∈ K is the solution of the differential equation:

$$\frac{dx(t)}{dt} = f\big(t, x(t)\big), and \;\; x(0) \in K(0)$$

# Remarks

- Notes: if we can solve the equation describing the strategies, then we know all strategies of the game;
- Do we need to find them all however?
- No: if all we want is to compute the optimal strategies;
- Yes: if we want to study stability of strategies, their evolution (disequilibrium)
- Normally: the DE is where hard mathematics is needed

* However, since most times we do not have the DE in question, or we have it but it is very difficult to solve, we want a different approach
* This approach is given by EVI:

$$Find\ x^* \in K \subset L^p\big([0,T], R^k\big) s.t.$$
$$\ll F(x^*), y - x^*) \gg\ \geq 0, \forall y \in K, where$$
$$\ll .,. \gg\ is\ a\ duality\ mapping$$

Let us give an example of an oligopoly game that can be solved via an EVI, together with a theorem.

# Dynamic oligopoly games

- Here we present a [m,n] –market game, where m=producers, n= demand markets
- A commodity is produced and consumed in this market and we have

$$p_i(t) - vector\ of\ production\ output\ of\ i$$
$$i = 1,..,m, t \in [0,T]$$

$$q_j(t) - vector\ of\ demand\ at\ j, j = 1,..,n$$

$$x_{ij}\ - vector\ of\ shipments$$

$$The\ following\ market\ equil.conditions\ hold:$$

$$p_i(t) = \sum_j x_{ij}(t) \text{ and } \mathrm{q_j(t)} = \sum_i x_{ij}\ (t)$$

$$Also: l_{ij}(t) \leq x_{ij}(t) \leq upp_{ij}(t)$$

$$Then$$

$$K = \{x \in L^2([0,T], R^{mn}) | l_{ij}(t) \leq x_{ij}(t)$$
$$\leq upp_{ij}\,(t), a.a.t\}$$

$$Assume\ prod.cost\ at\ i\ depends\ on\ all\ production$$

$$f_i = f_i\big(p(t)\big)$$

$Assume$ demand price depends on all consumption

$$d_j = d_j(q(t))$$

Also we assume

$$c_{ij} = c_{ij}\,(\,x_{ij}\,(t))$$

* Then each producer's profit is expressed as $v_i\big(t, x(t)\big) = \sum d_j\big(p(t)\big)x_{ij}(t) - f_i(q(t) - \sum c_{ij}(x(t)x_{ij}(t)$
* Then a dynamic oligopolistic market equil is a vector $x^* \in K\ where$ $v_i(t, x_i{}^*) \geq v_i(t, x_i, x_i{}^{\sim *}), \forall i, and\ a.a.t$
* Next we give an equivalent EVI reformulation of this game.

- Theorem (Barb-Coj '09):
- Assume for each firm i , the profit $v_i$ is pseudo-concave (defn. below) with resp. to variables $(x_{i1}, \ldots, x_{in})$ and cont. diff for a.a. t. If $\nabla v$ is Caratheodory with $\exists\, h \in L^2([0,T], R^{mn})\, st.\, ||\nabla v(t, x(t)|| \leq h(t)$

a.a., then the market equilibrium is the solution of the EVI

$$\ll -\nabla v(x^*), x - x^* \gg \geq 0, \forall x \in K$$

* Reminder: a function v is pseudo-concave w.r.t. $x_i$ if:

$$< \frac{dv(t,x)}{dx_i}, x_i - y_i > \geq 0 \rightarrow$$

$$v(t, x_1, \dots, x_i, \dots, x_n) \geq v(t, , x_1, \dots, y_i, \dots, x_n)$$

* Here is an example: