Test 1 Solutions

Math * 1160, F16

Q1. (a) Taking the transpose yields

$$\begin{pmatrix} a-b & 2c+d \\ c-d & a+b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ . } \text{——— } 1$$

So
$$\begin{cases} a-b = 1 & \text{——— } 1. \\ 2c+d = 2 & \text{——— } 2. \\ c-d = 3 & \text{——— } 3. \\ a+b = 4 & \text{——— } 4. \end{cases}$$

follow through

$1.+4.$: $2a = 5 \Rightarrow a = 5/2 \overset{1.}{\Rightarrow} 5/2 - b = 1 \Rightarrow b = 3/2$

$2.+3.$: $3c = 5 \Rightarrow c = 5/3 \overset{3.}{\Rightarrow} 5/3 - d = 3 \Rightarrow d = -4/3$.

$$\boxed{(a, b, c, d) = (5/2, 3/2, 5/3, -4/3)}$$

- 4 correct equations ——— 1
- correct solution ——— 2

(−1/2 each error).

(b) $2A - 3B + C = \begin{pmatrix} 2 & 0 \\ -4 & 6 \end{pmatrix} - \begin{pmatrix} -3 & 3 \\ 6 & 15 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$ ——— 1

$$= \boxed{\begin{pmatrix} 7 & 0 \\ -10 & -10 \end{pmatrix}} \text{ ——— } 1$$

(c)

$$vu^T = \begin{pmatrix}-1\\0\\3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}$$ 1

3×1 1×3

$$= \boxed{\begin{pmatrix}-1 & -2 & -3\\0 & 0 & 0\\3 & 6 & 9\end{pmatrix}}$$ 2

(−1/2 each error).

/9

Q2. (a)

$$(r(sA^T)+B)^T$$

$$= ((rs)A^T+B)^T$$ ($r(sA) = (rs)A$) 1/2

$$= ((rs)A^T)^T + B^T$$ ($(A+B)^T = A^T + B^T$) 1/2

$$= (rs)(A^T)^T + B^T$$ ($(rA)^T = rA^T$) 1/2

$$= \boxed{(rs)A + B^T}$$ ($(A^T)^T = A$) 1/2

3/

(b) We have

$$A\underline{x} = \begin{pmatrix} 3 & 1 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \end{pmatrix} = k^2\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$ ½

so $6 = k^2 \Rightarrow \boxed{k = \pm\sqrt{6}}$ ½

(also $18 = 3k^2 \Rightarrow k^2 = 6$ etc.).

(c) $((AB)C)^T$

$= C^T(AB)^T$ $\quad ((AB)^T = B^TA^T)$ ½

$= \boxed{C^T(B^TA^T)}$ $\quad (\quad '' \quad)$ ½

(or $(C^TB^T)A^T$).

or just $C^TB^TA^T$.

/4

4

Q3. (a) $A^T \underline{x} = \underline{b}$

$\Rightarrow (A^T)^{-1} A^T \underline{x} = (A^T)^{-1} \underline{b}$ (mult. both sides on left by $(A^T)^{-1}$) ½

$\Rightarrow I \underline{x} = (A^T)^{-1} \underline{b}$ $(A^{-1}A = I)$ ½

$\Rightarrow \underline{x} = (A^T)^{-1} \underline{b}$ $(IA = A)$ ½

$\Rightarrow \underline{x} = (A^{-1})^T \underline{b}$ $((A^T)^{-1} = (A^{-1})^T)$ ½

$$= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}^T \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$ ½

$$= \boxed{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$ ½

5/

(b) The no. of cars entering each intersection per hour equals the no. of cars leaving each intersection per hour. Hence

(i)

$$\left.\begin{aligned} A:&\quad 300 + 500 = x_1 + x_2 \\ B:&\quad x_2 + x_4 = x_3 + 300 \\ C:&\quad 100 + 400 = x_4 + x_5 \\ D:&\quad x_1 + x_5 = 600 \end{aligned}\right\} \text{—— } 2$$

$-1/2$ each error

i.e.

$$\begin{cases} x_1 + x_2 = 800 \\ x_2 - x_3 + x_4 = 300 \\ x_4 + x_5 = 500 \\ x_1 + x_5 = 600 \end{cases}$$

—— 2

$-1/2$ each error

(ii)

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 800 \\ 300 \\ 500 \\ 600 \end{pmatrix} \text{ —— } 2$$

4×5 $\quad 5 \times 1$ $\quad 4 \times 1$

$-1/2$ each error.

Line 1 : construct the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 3 & 1 \\ -4 & 1 & -2 \end{pmatrix}$$ ——— 1

Line 2&3 : apply the row operations

$r_2 - 2r_1 \to r_2$ and $r_3 + 4r_1 \to r_3$ ——— 1

yielding as output

$$\boxed{A = \begin{matrix} 1 & 0 & -2 \\ 0 & 3 & 5 \\ 0 & 1 & -10 \end{matrix}}$$ ——— 1

/12

Q4.

$$[A|b] \to \left(\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & 2 & c \\ 0 & 3 & 6 & b \end{array}\right) \quad r_2 \leftrightarrow r_3$$ ——— 1

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & 2 & c \\ 0 & 0 & 0 & b-3c \end{array}\right) \quad r_3 - 3r_2 \to r_3$$ ——— 1

7/ For a consistent solution we need

$b - 3c = 0 \Rightarrow b = 3c$. —— (1)

Proceeding with that assumption we have only 2 equations:

$$\begin{cases} x \quad - 2z = a \\ \quad y + 2z = c \end{cases}$$ —— (1)

$y = -2z + c$, let $z = \alpha \in \mathbb{R}$ ('free')

method (2)

yields $y = -2\alpha + c$, Subst. in 1st equation:

$x = 2\alpha + a$. So the ∞ solution set is:

$$\boxed{\begin{array}{c}\{(2\alpha + a, -2\alpha + c, \alpha) \mid \alpha \in \mathbb{R}\} \\ \text{provided } b = 3c.\end{array}}$$ —— (2)

answer

- Different (correct) row ops acceptable
- Follow through.

/8