Solutions

UNIVERSITY of GUELPH

Fall 2016 MATH*1160 Test 2

Last name (print): ______________ **First** name (print): ______________

Student #: ______________________

1. This is a 60 minute test. Do NOT start until instructed.
2. Please fill out your personal details above. Once the examination starts fill out your solutions in the spaces provided.
3. You may quote results from lecture notes without proof, unless asked to do otherwise. But notes or books are not permitted during the exam.
4. Give sufficient working as solutions with no justification will receive little or no credit.
5. You may use a calculator, but not a 'graphing calculator' that supports matrix algebra. No other electronic equipment is permitted. **Turn off or 'mute' your cell phone! The use of cell phones during the exam is prohibited.**
6. If you copy the work of your neighbour this is considered **Academic Misconduct** and will be reported to the appropriate university authority.
7. There are a total of **29** points to be awarded on this test.

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1. (6 points)

(a) Without applying a cofactor expansion, or elementary row/column operations, evaluate

$$\begin{vmatrix} 3 & 2 & 3 \\ -6 & 0 & -6 \\ 1 & 2 & 1 \end{vmatrix}.$$

Justify your answer.

The $\boxed{\text{deteterminant} = 0}$ because the 1st and 3rd columns are equal.

1

1

(b) If

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 3,$$

find

$$|\mathbf{B}| = \begin{vmatrix} b & a \\ 2d+b & 2c+a \end{vmatrix}.$$

$$3 = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$$ 1

$$= -\frac{1}{2}\begin{vmatrix} b & a \\ 2d & 2c \end{vmatrix}$$ 1

$$= -\frac{1}{2}\begin{vmatrix} b & a \\ 2d+b & 2c+a \end{vmatrix} = -\frac{1}{2}|B|,$$ 1

$\Rightarrow \boxed{|B| = -6}$ 1

Alternatively: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 3,$ ---(*)

So $\begin{vmatrix} b & a \\ 2d+b & 2c+a \end{vmatrix} = b(2c+a) - a(2d+b) = 2bc + \cancel{ab} - 2ad - \cancel{ab}$

$= 2(bc - ad) = -2(ad - bc) = -2(3) = -6,$

using (*).

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2. (7 points)

(a) Using a cofactor expansion *along the third row*, find the values of x so that

$$|A| = \begin{vmatrix} -1 & 0 & -2x \\ 1 & 1 & 0 \\ x & 0 & 1 \end{vmatrix} = 1.$$

Follow Through

$|A| = +x\begin{vmatrix} 0 & -2x \\ 1 & 0 \end{vmatrix} + 0 + 1\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix}$ (2)

$= x(0+2x) + (-1-0)$ (1)

$= 2x^2 - 1 = 1$ (1)

$\Rightarrow x^2 = 1 \Rightarrow \boxed{x = \pm 1}$ (1)

(b) From our lecture notes we know that

$$\mathbf{A}(\mathbf{adj\,A}) = (\mathbf{adj\,A})\mathbf{A} = |\mathbf{A}|\mathbf{I},$$

where $\mathbf{adj\,A}$ denotes the adjoint of a square matrix $\mathbf{A}$ and $\mathbf{I}$ is the identity matrix. If $|\mathbf{A}| \neq \mathbf{0}$ prove using the above result that

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}(\mathbf{adj\,A}).$$

(State any results you assume from your lecture notes.)

Multiplying both sides of $A(\text{adj}A) = |A| \cdot I$ by $\frac{1}{|A|}$ yields (1)

$A\left(\frac{1}{|A|} \cdot \text{adj}\, A\right) = I$, (1)

which implies the required result.

(other correct answers OK).

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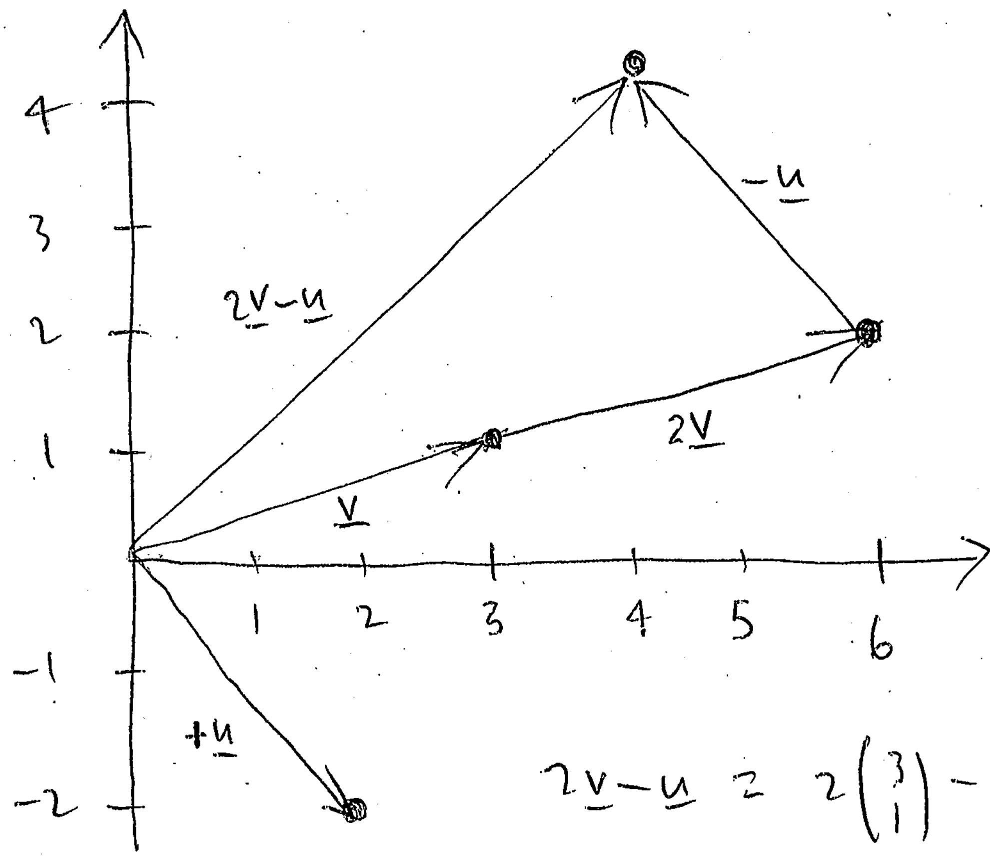
3. (8 points)

(a) Let

$$\mathbf{u} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

(i) Use directed line segments to represent the vectors $\mathbf{u}$ and $\mathbf{v}$.
(ii) Calculate $\mathbf{2v} - \mathbf{u}$ and indicate the resulting vector on your diagram from (i).



-1/2 each error (e.g., axis labels not indicated).

$$2\underline{v} - \underline{u} = 2\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} \text{ ——— } 1$$

$$= \begin{pmatrix} 4 \\ 4 \end{pmatrix} \text{ ——— } 1$$

$\underline{u}$ & $\underline{v}$ correctly drawn ——— 1

$2\underline{v} - \underline{u}$ correctly drawn ——— 2

Question continued on next page

(b) Let V be the set of all 2×2 matrices of the form

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{such that } abcd = 0,$$

where $a, b, c, d \in \mathbb{R}$. Let the operations $\oplus$ and $\odot$ be the standard operations of vector addition and scalar multiplication for matrices. Is V closed under vector addition? Justify your answer carefully.

Any counter-example acceptable (or 'family' of counter-examples).

E.g., Consider

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in V, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in V, \quad \text{(1)}$$

then $$A_1 \oplus A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \notin V, \quad \text{(1)}$$

as the product of the entries $\neq 0$. (1)

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4. (8 marks)

(a) Consider the set of vectors in $\mathbb{R}^n$, with the usual rules for addition and scalar multiplication. For any $\mathbf{u} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$, prove axiom $\boldsymbol{M.3}$ of a vector space, namely

$$(cd) \odot \mathbf{u} = c \odot (d \odot \mathbf{u}).$$

Clearly show *all* the steps in your argument.
(Hint: use the 'coordinate' form to represent vectors, e.g., $\mathbf{u} = (u_1, u_2, \ldots, u_n)$.

$$\begin{aligned}
(cd) \odot \underline{u} &= (cd) \odot (u_1, \ldots, u_n) \\
&= ((cd)u_1, \ldots, (cd)u_n) \quad \text{—— } 1/2 \\
&= (c(du_1), \ldots, c(du_n)) \quad \text{—— } 1/2 \\
&= c \odot (du_1, \ldots, du_n) \quad \text{—— } 1/2 \\
&= c \odot (d \odot (u_1, \ldots, u_n)) \\
&= c \odot (d \odot \underline{u}) \checkmark \quad \text{—— } 1/2
\end{aligned}$$

Note: if you skip steps here you lose marks!

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(b) Let $\mathbf{A}$ be square $(n \times n)$ matrix and $\boldsymbol{W}$ be a subset of $\mathbb{R}^n$ defined by

$$\boldsymbol{W} = \{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{Ax} = 3\mathbf{x}\},$$

with the same rules of addition and scalar multiplication as for $\mathbb{R}^n$. Prove whether or not the set $\boldsymbol{W}$ is a subspace of $\mathbb{R}^n$.

Vector addition:

Let $\underline{x}_1, \underline{x}_2 \in W$, i.e. $A\underline{x}_1 = 3\underline{x}_1$ & $A\underline{x}_2 = 3\underline{x}_2$. ——— 1/2

Consider $A(\underline{x}_1 + \underline{x}_2)$ ——— 1/2

$= A\underline{x}_1 + A\underline{x}_2$ ——— 1/2

$= 3\underline{x}_1 + 3\underline{x}_2$ ——— 1/2

$= 3(\underline{x}_1 + \underline{x}_2)$, ——— 1/2

i.e. $\underline{x}_1 + \underline{x}_2 \in W$ ✓ ——— 1/2

Scalar multiplication:

Let $\underline{x} \in W$ so $A\underline{x} = 3\underline{x}$, & $k \in \mathbb{R}$. ——— 1/2

Consider $A(k\underline{x})$ ——— 1/2

$= kA\underline{x}$ ——— 1/2

$= k(3\underline{x})$ ——— 1/2

$= 3(k\underline{x})$ ——— 1/2

i.e. $k\underline{x} \in W$ ... ✓ ——— 1/2

END OF TEST

TA Initials (print) ______ Total: / **29**