Math Preparedness Booklet
MATH*1160

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1 Introduction

Professors often make the mistake of assuming students know the basics, and forget that students come from various backgrounds with different levels of math preparation. This short booklet covers the sort of notation, terminology and assumptions that you are expected to know *before* starting Math*1160. It's not exhaustive, but hopefully covers some common areas that we will frequently need. The material in Section 6 is particularly relevant to the more formal aspects of a more advanced course in mathematics (with theorems and proofs etc,) and may be new to some of you. Please read through the sections and do the short exercises. Answers are given at the end of the booklet. If there is anything you don't understand please come and see me!

2 Number Systems

1. Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

2. Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

3. Rational Numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$.

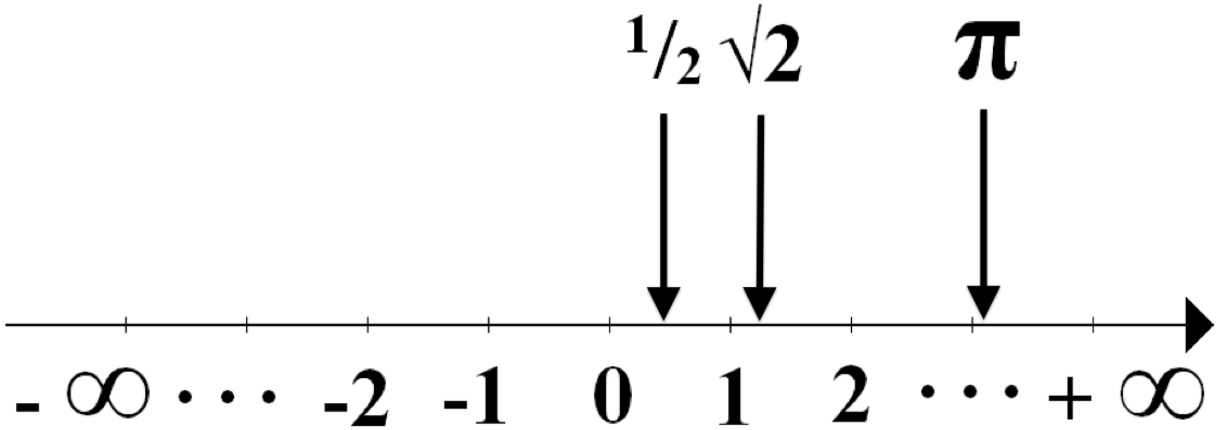
   E.g., $1/4 = 0.25$, $1/3 = 0.333\ldots = 0.\bar{3}$, $3 = 3/1$. Decimal expansions terminate or repeat.

4. Irrational Numbers ($\mathbb{W}$):

   - The set of all numbers that cannot be represented as the ratio of integers p/q (most numbers on the real number line).
   - Decimal expansions do *not* terminate and do *not* repeat.

     E.g., $\sqrt{2} = 1.41421356\ldots$
     Any number that can't be expressed as a perfect square.
     $\pi = 3.14159265\ldots, \quad e = 2.7182818284\ldots$

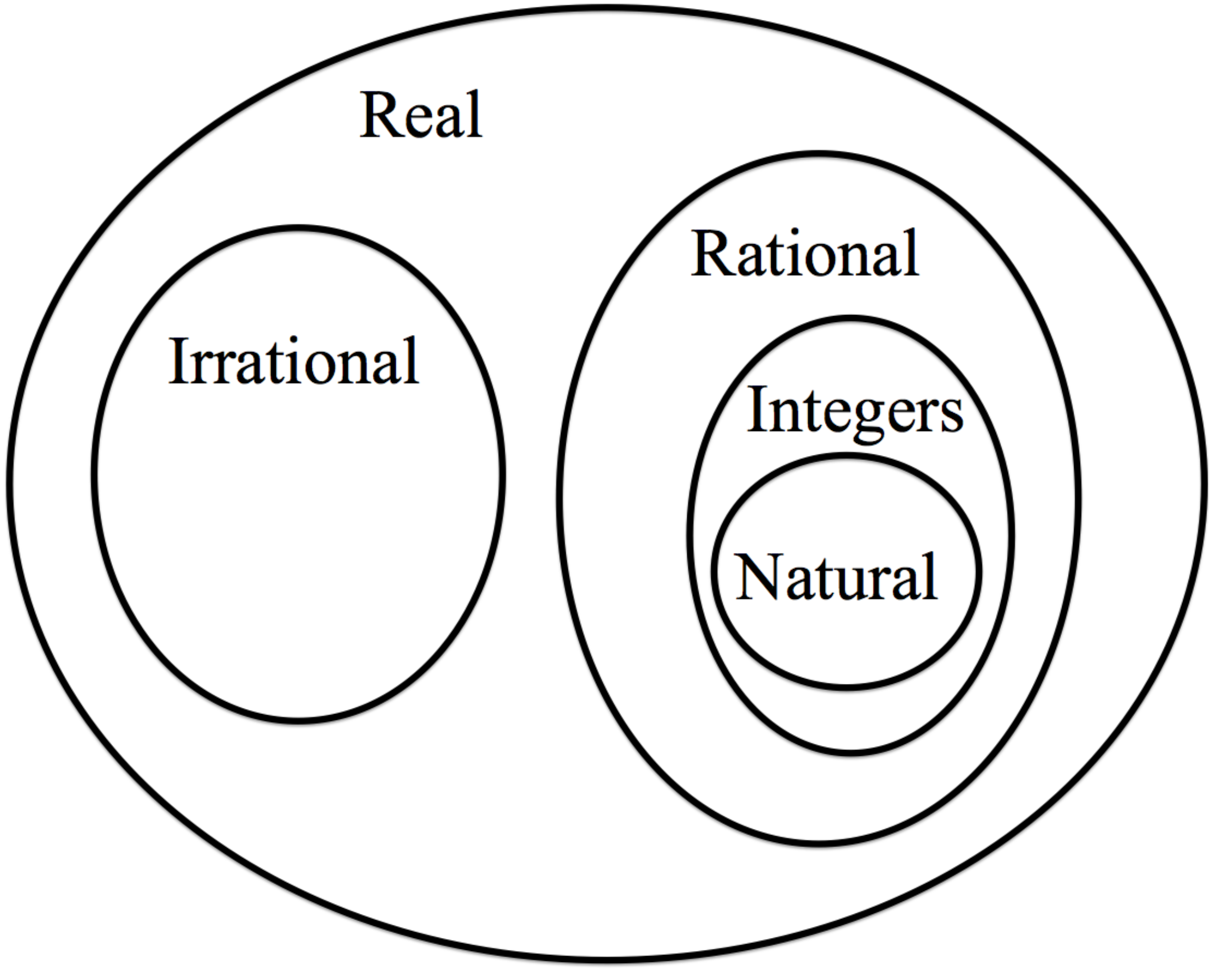
5. Real Numbers: $\mathbb{R} = \mathbb{Q} \cup \mathbb{W}$, i.e. the set of Rationals and Irrationals combined. The real number line is illustrated below:



6. The set of positive and set of negative real numbers are denoted $\mathbb{R}^+$ and $\mathbb{R}^-$ respectively. The set of nonnegative real numbers is thus $\mathbb{R}^+ \cup \{\mathbf{0}\}$.

7. Complex Numbers ($\mathbb{C}$): set of numbers of the form $\boldsymbol{a}+\boldsymbol{bi}$, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}$, $\boldsymbol{i} = \sqrt{-\mathbf{1}}$. These numbers are not needed in this course.

What is the hierarchy of the number systems?



Exercise 1

Classify the following numbers as rational, irrational, or an integer:

(a) $\mathbf{(0.333\ldots)}$ is $\cdots =$ [ ]

(b) $\mathbf{2.5}$ is $\cdots =$ [ ]

(c) $\sqrt{\mathbf{5}}$ is ... [ ]

(d) $\mathbf{0}$ is ... [ ]

(e) $\mathbf{\sin(\pi/2)}$ is $\cdots =$ [ ]

(f) $\mathbf{(0.303003000300003\ldots)}$ is ... [ ]

(g) ∞ is ... [ ]

(h) $\pi + e$ is ... [ ]

3 Sets and subsets

A 'set' is a collection of items. The items (in our case, usually numbers) are called 'elements'. It is usual to represent the set by a capital letter, the elements by lower case letters, and to use curly brackets to enclose the elements in a collection. For example, the set of integers between 3 and 10 is written as:

$$S = \{x \,|\, x \text{ is an integer between } 3 \text{ and } 10\} = \{4, 5, 6, 7, 8, 9\}.$$

We read the vertical line '|' as "such that ". This set has a finite number of elements and so is called a 'finite set'. We also have 'infinite sets', e.g.

$$X = \{x \,|\, x \text{ is an even positive integer}\} = \{2, 4, 6, 8 \ldots\}.$$

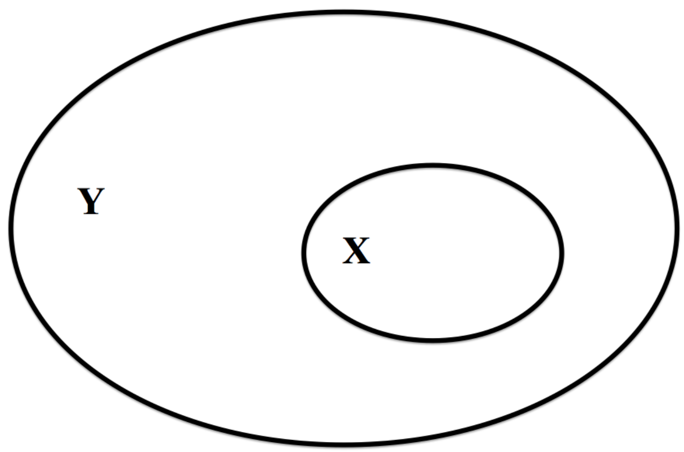
The '$\ldots$' indicates that the sequence of numbers continues indefinitely. To denote that an element x belongs to a set X we write $x \in X$. The empty set is denoted $\emptyset$. As we shall see below, a set does not have to comprise a discrete set of numbers - it can also denote all numbers in a given interval, e.g. $\{x \,|\, 1 < x \leq 3\}$.

Definition 1

Let X and Y be two sets. Then we say that X is a **subset** *of Y, written $X \subseteq Y$ if all the elements of X are in Y. If $X = Y$ then X and Y contain the same elements. If X is a* **proper subset** *of Y, written $X \subset Y$ then all the elements of X are in Y, but some elements of Y are not in X.*

Notes:

- $X \subseteq Y$ means $X \subset Y$ or $X = Y$.
- $X \subset Y$ is illustrated in the following Venn diagram:



Definition 2
Let X and Y be two sets. The **union** *of X and Y, written $X \cup Y$, means the set of elements that are in X, in Y, or in both X and Y:*

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}.$$

Definition 3
Let X and Y be two sets. The **intersection** *of X and Y, written $X \cap Y$, is the set of elements that are in both X and Y (i.e., the elements they share):*

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}.$$

Exercise 2

(a) The proper subsets of $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ are:

(b) $\{\mathbf{1}, \mathbf{2}\} \cup \{\mathbf{2}, \mathbf{3}, \mathbf{4}\} =$

(c) $\{\mathbf{1}, \mathbf{2}\} \cap \{\mathbf{2}, \mathbf{3}, \mathbf{4}\} =$

(d) $\{\mathbf{1}, \mathbf{2}, \mathbf{3}\} \cap \{\mathbf{4}, \mathbf{5}, \mathbf{6}\} =$

(e) $\{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} > \mathbf{2}\} \cap \{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} \leq \mathbf{3}\} =$

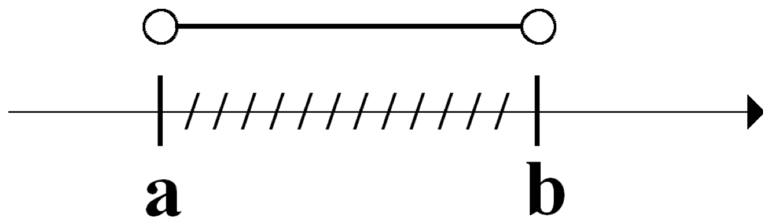
(f) $\{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} \geq \mathbf{0}\} \cup \{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} < \mathbf{0}\} =$

(g) $\{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} \geq \mathbf{5}\} \cap \{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} < \mathbf{4}\} =$

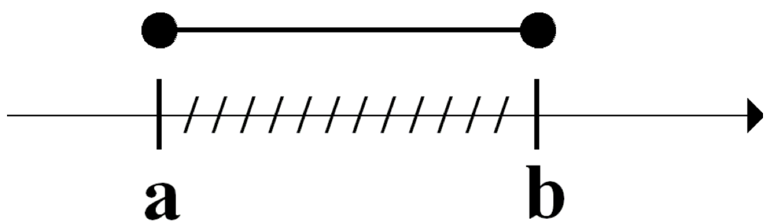
4 Notation for Intervals

Bounded intervals

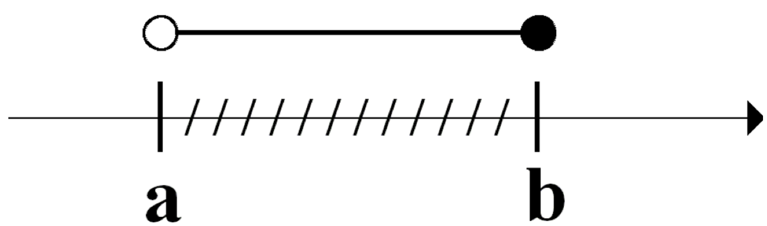
Open interval: $(a,b) = \{x \in \mathbb{R} \,|\, a < x < b\}$



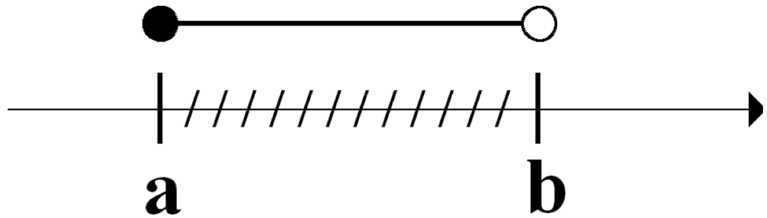
Closed interval: $[a,b] = \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$



Neither Open or Closed: $(a,b] = \{x \in \mathbb{R} \,|\, a < x \leq b\}$



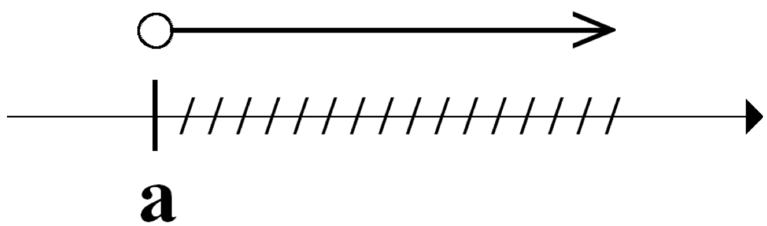
Neither Open or Closed: $[a,b) = \{x \in \mathbb{R} \mid a \le x < b\}$



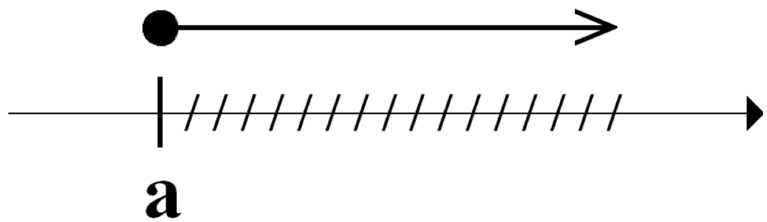
Unbounded intervals

The endpoints a and b can be $-\infty$ or $+\infty$ respectively:

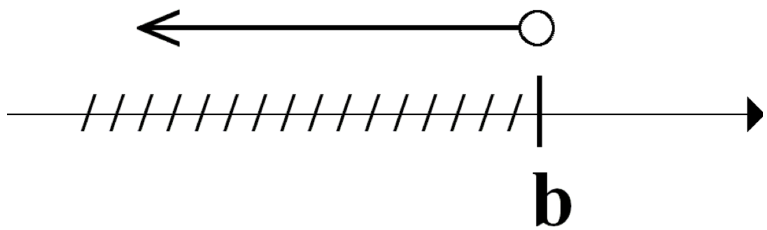
$$(a, +\infty) = \{x \in \mathbb{R} \mid x > a\}$$



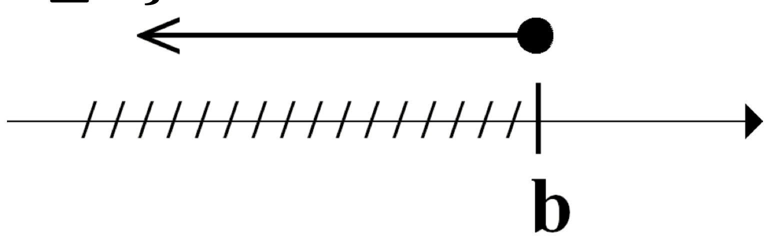
$[\boldsymbol{a}, +\infty) = \{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} \geq \boldsymbol{a}\}$



$(-\infty, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} < \boldsymbol{b}\}$



$(-\infty, \boldsymbol{b}] = \{\boldsymbol{x} \in \mathbb{R} \,|\, \boldsymbol{x} \leq \boldsymbol{b}\}$



Note: $\mathbb{R} = (-\infty, +\infty)$.

5 Some Mathematical Cautions

We often write $x = +\infty$, or $x = -\infty$. This is an abuse of notation. The symbols $\pm\infty$ stand for positive and negative quantities that are unbounded (i.e., no matter how large in magnitude these numbers are they can be made larger). Thus

Caution !
Infinity is not a number.

We can think of $+\infty$ as the limit of dividing a positive number by smaller and smaller positive numbers h, e.g. $\lim_{h\to 0} \frac{1}{h} = +\infty$. In the limit we would be 'dividing by zero' which has no meaning in mathematics. Thus

Caution !
Dividing by zero is not permitted.

Exercise 3

For what value(s) of x is the following equation **in**valid?

$$\frac{3x(x-2)}{(x-2)} = 3x.$$

Answer = [ ] because [ ].

This example also illustrates the following:

Caution !

$\frac{0}{0}$ *is undefined (as division by zero is not allowed). Thus in particular,* $\frac{0}{0} \neq 1$.

In addition to dividing by zero there are two other things we cannot do:

Caution !

We **cannot** *(i) take the square root of a negative number, and (ii) take logarithms of zero or a negative number.*

Exercise 4

For what values of x are the following functions valid?

(a) $\sqrt{x-3}$,

(b) $\log(x^2-1)$

(a) Answer:

(b) Answer:

6 Some Math Notation and Terminology

There are some common notation and terminology used in mathematics that we should clarify[1]. In the following we assume that A, B and C refer to mathematical statements.

<u>$\implies$ **or** $\impliedby$</u> **:**

$A \implies B$ means 'A implies B', in other words if A is true then B is also true. For example, consider the statements

$$A: \; n \text{ is divisible by } 4 \text{ and,}$$
$$B: \; n \text{ is divisible by } 2.$$

[1]Mathematicians use these things frequently and assume that everybody knows what they are talking about!

then clearly $A \implies B$ (e.g. if $n = 16$). The statement $A \impliedby B$ means that 'B implies A', which clearly does NOT follow in this example (e.g., if $n = 6$).

Exercise 5

Let

$$A: \ x \in \mathbb{N}$$
$$B: \ x \in \mathbb{R}.$$

Then

$$\boxed{} \implies \boxed{}$$

$\underline{\iff}$ **:**

$A \iff B$ means that $A \implies B$ AND $A \impliedby B$. In other words A and B are equivalent statements. For example:

A : A triangle has three equal sides

B : A triangle has three equal angles.

Then clearly $A \iff B$.

Exercise 6

Let

A **:** A number x is a rational number

B **:** The decimal expansion of x is terminating or *eventually* repeating

C **:** A number x is an irrational number

Then

$$\boxed{} \iff \boxed{}$$

<u>**iff**</u> **:**

This is shorthand for 'if and only if' and means the same as $\iff$. For example,

> 'A number is divisible by nine if and only if the sum of its digits are divisible by 9'

(E.g., $\mathbf{27}$ is divisible by $\mathbf{9}$ because $\mathbf{2+7=9}$ is divisible by $\mathbf{9}$). Of course like many results this is something that would have to be rigorously justified.

Contrapositive :

Let NA and NB stand for the statements 'not A' and 'not B' respectively, i.e. the statements A and B are false. Then

> **Result 1**
> $A \implies B$ *iff* $NB \implies NA$.

We refer to the statement $NB \implies NA$ as the **contrapositive** of the statement $A \implies B$. For example, let

$$A \text{ : Something is a bat}$$
$$B \text{ : Something is a mammal.}$$

So

$$NA \text{ : Something is NOT a bat}$$
$$NB \text{ : Something is NOT a mammal.}$$

Check that Result 1 makes sense!

Exercise 7
Let

$$\boldsymbol{A}: \boldsymbol{x} \text{ is an integer}$$
$$\boldsymbol{B}: \boldsymbol{x} \text{ is rational.}$$

The contrapositive of the statement $\boldsymbol{A} \implies \boldsymbol{B}$ is

If $\boldsymbol{x}$ is not [ ] then it is not [ ].

Necessary and Sufficient conditions:

The statement that '$\boldsymbol{A}$ is a sufficient condition for $\boldsymbol{B}$' is equivalent to '$\boldsymbol{A} \implies \boldsymbol{B}$', while the statement '$\boldsymbol{A}$ is a necessary condition for $\boldsymbol{B}$' is equivalent to '$\boldsymbol{A} \impliedby \boldsymbol{B}$'. For example,

$$\boldsymbol{A}: \text{An object has } 4 \text{ sides}$$
$$\boldsymbol{B}: \text{An object is a square.}$$

Then $\boldsymbol{A}$ is a necessary condition for $\boldsymbol{B}$ ($\boldsymbol{A} \impliedby \boldsymbol{B}$). But clearly, $\boldsymbol{A}$ is NOT a sufficient condition for $\boldsymbol{B}$ (e.g., rectangles are not squares unless all the sides are the same length). However, $\boldsymbol{B}$ IS a sufficient condition for $\boldsymbol{A}$. Finally, the statement that 'A is necessary and sufficient for B' is just another way of saying $A \iff B$.

Exercise 8

Answer TRUE or FALSE to the following questions:

(a) Being a mammal is a sufficient condition for being human ☐

(b) Being human is a sufficient condition for being a mammal ☐

(c) Being brave is a necessary condition for being an effective firefighter ☐

(d) Being rich is necessary for being happy ☐

(e) Being an unmarried man is both necessary and sufficient for being a bachelor ☐

$\underline{\forall}$ **:**

The symbol '$\forall$' means 'for all'. For example

$$\frac{4x+2}{2} = 2x+1 \quad \forall x \in \mathbb{R}.$$

In other words, the equation is valid for all real numbers.

<u>s.t.</u> :

The letters 's.t.' stands for 'such that'. For example,

$$\text{Solve } n^2 > 4 \text{ s.t. } n \in \mathbb{N},$$

i.e., find the natural numbers that solve $n^2 > 4$ (clearly, $\{3, 4, 5 \ldots\}$).

<u>$\exists$ or $\nexists$</u> :

The symbol '$\exists$' is shorthand for 'there exists'. For example,

$$\exists x \in \mathbb{R} \text{ s.t. } x^2 - 4 = 0,$$

i.e. there exists a real solution to the given equation. We also use the symbol '$\nexists$' to mean 'does not exist'. For example,

$$\nexists x \in \mathbb{R} \text{ s.t. } x^2 + 1 = 0,$$

i.e., there are no real solutions of the given equation.

! :

We use the exclamation mark '**!**' in mathematics to denote 'unique' or 'only one'. For example,

$$\exists\,!\,x \in \mathbb{R} \text{ s.t. } x^2 - 2x + 1 = 0,$$

i.e., there is only one real solution of the given equation, because $x^2 - 2x + 1 = (x-1)^2$.

Answers

Exercise 1: (a) $1/3$. Rational, (b) $5/2$. Rational, (c) Irrational, (d) An integer, (e) 1, thus an integer, (f) Irrational, (g) Not a number, (h) An open question

Exercise 2: (a) $\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}$, (b) $\{1,2,3,4\}$, (c) $\{2\}$, (d) $\emptyset$, (e) $\{x \in \mathbb{R} \,|\, 2 < x \leq 3\}$, (f) $\mathbb{R}$, (g) $\emptyset$

Exercise 3: $x = 2$, because this would cause us to divide by zero

Exercise 4: (a) $\{x \in \mathbb{R} \mid x \geq 3\}$, (b) $\{x \in \mathbb{R} \mid |x| > 1\}$

Exercise 5: $\boldsymbol{A}$, $\boldsymbol{B}$

Exercise 6: $\boldsymbol{A}$, $\boldsymbol{B}$

Exercise 7: rational, an integer

Exercise 8: (a) false, (b) true, (c) true, (d) false, (e) true