Chapter 2

Algebra Review

We review some high school arithmetic and algebra.

2.1 Arithmetic Operations

Result 1
Let $\boldsymbol{a, b, c} \in \mathbb{R}$. *Then*

$$\boldsymbol{a+b=b+a} \quad \textit{(Commutative Law of Addition)}$$
$$\boldsymbol{ab=ba} \quad \textit{(Commutative Law of Multiplication)}$$
$$\boldsymbol{(a+b)+c=a+(b+c)} \quad \textit{(Associative Law of Addition)}$$
$$\boldsymbol{(ab)c=a(bc)} \quad \textit{(Associative Law of Multiplication)}$$
$$\boldsymbol{a(b+c)=(b+c)a=ab+ac} \quad \textit{(Distributive Law)}$$

Note:
It is important to realize that these rules only apply to the real numbers. In other number systems these rules may not apply! For example, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of the same size, we have $\boldsymbol{AB \neq BA}$ (except in special cases).

Exercise 3

(a) $(3xy)(-4x) =$ $3(-4)x^2y = -12x^2y.$

(b) $2t(7x + 2tx - 11) =$ $14tx + 4t^2x - 22t.$

(c) $4 - 3(x - 2) =$ $4 - 3x + 6 = 10 - 3x.$

We often need the following formula:

Formula 1

Using the Distributive Law three times yields:

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd.$$

Exercise 4

$(2x + 1)(3x - 5) =$ $6x^2 + 3x - 10x - 5 = 6x^2 - 7x - 5.$

Formula 2

Some special cases of Formula 1 are:

(i) $(a + b)(a - b) = a^2 - b^2$. *('Difference of two squares')*

(ii) $(a + b)^2 = a^2 + 2ab + b^2$. *('A perfect square')*

(iii) $(a - b)^2 = a^2 - 2ab + b^2$. *('A perfect square')*

Exercise 5

(a) $(x + 6)^2 =$ $x^2 + 12x + 36.$

(b) $$\begin{aligned} 3(x-1)(4x+3) - 2(x+6) &= 3(4x^2 - x - 3) - 2x - 12 \\ &= 12x^2 - 3x - 9 - 2x - 12 \\ &= 12x^2 - 5x - 21 \quad . \end{aligned}$$

2.2 Fractions

To add fractions with a common denominator we use the Distributive Law:

$$\frac{a}{b} + \frac{c}{b} = \quad a\left(\frac{1}{b}\right) + c\left(\frac{1}{b}\right) = \frac{1}{b}(a+c) = \frac{a+c}{b}.$$

We have shown the following:

Formula 3

$$\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b}.$$

However, note the following:

Caution !
A common mistake:

$$\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}.$$

To add fractions with a different denominator we make a common denominator:

$$\frac{a}{b} + \frac{c}{d} = \quad \left(\frac{d}{d}\right)\left(\frac{a}{b}\right) + \left(\frac{b}{b}\right)\left(\frac{c}{d}\right) = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

Thus we have shown

Formula 4

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Generally, we seek to find the 'least common denominator'[1] of the fractions, e.g.

$$\frac{a}{xy} + \frac{b}{x^2y} + \frac{c}{xy^2} = \quad \frac{xya}{x^2y^2} + \frac{yb}{x^2y^2} + \frac{xc}{x^2y^2} = \frac{xya+yb+xc}{x^2y^2}.$$

[1]Aka, least common multiple of the denominators.

We multiply and divide fractions as follows:

Formula 5

$$(i)\ \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \quad (ii)\ \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

We can also put the negative sign in a fraction in different places:

Formula 6

$$\frac{-a}{b} = -\frac{a}{b} = -\left(\frac{a}{b}\right) = \frac{a}{-b}.$$

Exercise 6

(a) $\dfrac{(x+3)}{x} = \quad \dfrac{x}{x} + \dfrac{3}{x} = 1 + \dfrac{3}{x}.$

(b) $\dfrac{3}{x-1} + \dfrac{x}{x+2} =$

$$\frac{3(x+2) + x(x-1)}{(x-1)(x+2)} = \frac{3x + 6 + x^2 - x}{x^2 + x - 2} = \frac{x^2 + 2x + 6}{x^2 + x - 2}.$$

(c) $\dfrac{s^2t}{u} \cdot \dfrac{ut}{-2} = \quad \dfrac{s^2t^2u}{-2u} = -\dfrac{s^2t^2}{2}.$

(d) $\dfrac{1}{2 - \frac{x}{2}} = \quad \dfrac{1}{\frac{4}{2} - \frac{x}{2}} = \dfrac{1}{\frac{4-x}{2}} = \dfrac{2}{4-x}.$

2.3 Ratios

Definition 4
A **ratio** *is a comparison of the relative sizes of two (or more) parts of a whole. It is represented using a colon ':', the word 'to', or using a fraction '/'. A* **proportion** *is a comparison of a part considered in relation to the whole.*

Note: The comparison between quantities is 'relative', so for example, the ratios $\mathbf{2:6}$ and $\mathbf{1:3}$ are equivalent (also, notice that $\frac{2}{6} = \frac{1}{3}$).

Exercise 7
Suppose $\mathbf{\$50}$ of a $\mathbf{\$200}$ debt has been repaid. What is the ratio of debt paid to money owed? What is the proportion of money paid back? Proportion owed?

SOLUTION

Ratio of debt paid to money owed = $\boxed{\mathbf{50 : 150}}$, or $\boxed{\mathbf{1 : 3}}$, or $\boxed{\mathbf{1/3}}$.

Proportion of debt paid = $\boxed{\mathbf{50/200 = 1/4}}$.

Proportion of money owed = $\boxed{\mathbf{150/200 = 3/4}}$.

Exercise 8
The ratio of vanilla to chocolate to strawberry ice-cream cones sold is $\mathbf{3:5:2}$. If $\mathbf{380}$ ice-cream cones were sold, how many of each flavour were sold?

SOLUTION

The proportions of vanilla, chocolate, and strawberry ice-cream cones are

$$\boxed{\frac{3}{3+5+2} = \frac{3}{10}}, \quad \boxed{\frac{5}{3+5+2} = \frac{5}{10} = \frac{1}{2}}, \quad \boxed{\frac{2}{3+5+2} = \frac{2}{10} = \frac{1}{5}}.$$

so, # vanilla cones = $\boxed{\frac{3}{10} \times 380 = 114}$

chocolate cones = $\boxed{\frac{1}{2} \times 380 = 190}$

strawberry cones = $\boxed{\frac{1}{5} \times 380 = 76}$

Check: $\boxed{114 + 190 + 76 = 380. \quad \checkmark}$

2.4 Percentages

As everyone knows,

Definition 5
Percent *means "per* 100*", thus* x *percent, written* $x\%$*, is shorthand for the fraction* $\frac{x}{100}$.

Percentages can be negative, or greater than 100, e.g. -10% or 150%, the latter referring to the improper fraction $\frac{150}{100} = \frac{3}{2}$. Finally, when we use the word "of" to express the percentage of a quantity we "multiply", i.e.

Definition 6
The expression "$y\%$ *of* A*" is shorthand for "*$\frac{y}{100} \times A$*".*

E.g., to find 20% of 50 do $\frac{20}{100} \times 50 = \frac{1}{5} \times 50 = 10$.

Exercise 9
The price of a skateboard is reduced by 25% in a sale. If the old price was \$120, what is the sale price?
SOLUTION
The price reduction is $\frac{25}{100} \times 120 = \frac{1}{4} \times 120 = \$30$. Thus the sale price is $\$120 - \$30 = \$90$.

Exercise 10
Gas is selling at $\$1.15$ per litre after a 5% rise in price. What was price of gas before the price increase?

SOLUTION

Method 1: 105% corresponds to $\$1.15$, thus 1% corresponds to $\$\frac{1.15}{105}$, so 100% corresponds to $\$\frac{1.15}{105} \times 100 = \$1.10$ (to 2 decimal places).

Method 2: Let $P =$ the original selling price of gas. Then, $P + 5\%P = 1.15$, so

$$P + \frac{5}{100}P = 1.15 \implies P(1.05) = 1.15 \implies P = \frac{1.15}{1.05} = 1.0952 \quad \text{or } \$1.10.$$

Exercise 11
A dress is selling for $\$100$ after a 20% discount. What was the original selling price?
SOLUTION

Method 1:
80% corresponds to $\$100$, so 1% corresponds to $\$\frac{100}{80} = \$\frac{5}{4}$, so 100% corresponds to $\$\frac{5}{4} \times 100 = \$5 \times 25 = \$125$.

Method 2:
Let $SP =$ the original selling price. Then

$$SP - \frac{20}{100} \times SP = SP \times \frac{80}{100} = \$100$$
$$\implies SP = 100 \times \frac{100}{80} = \$125.$$

2.5 Factoring

We have used the Distributive Law to expand expressions. We also reverse this process to factor expressions as the product of simpler expressions.

Expanding: $2x(x+5) = 2x^2 + 10x.$
Factoring: $2x^2 + 10x = 2x(x+5).$

To factor a quadratic of the form $x^2 + bx + c$ we note that

$$(x+r)(x+s) = x^2 + (r+s)x + rs.$$

Thus, when factoring a quadratic of the form $x^2 + bx + c$ we look for two numbers that add to give b and multiply to give c.

Exercise 12
Factor $x^2 + 5x - 24$.
SOLUTION

The two numbers that add to give 5 and multiply to give -24 are -3 and 8. Therefore

$$x^2 + 5x - 24 = (x - 3)(x + 8).$$

We can extend this method to problems where the coefficient of x^2 is not 1.

Exercise 13
Factor $2x^2 - 7x - 4$.
SOLUTION

We attempt to fill in the boxes below:

$$2x^2 - 7x - 4 = (2x + \square)(x + \square).$$

After considering the factors of -4 (e.g., 1 and -4, 2 and -2 etc.), experimentation reveals that

$$2x^2 - 7x - 4 = (2x + 1)(x - 4)).$$

Some special cases include the difference of two squares formula we had before: $a^2 - b^2 = (a - b)(a + b)$. We also have the difference and sum of cubes:

Formula 7
(i) $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ *(Difference of two cubes)*

(ii) $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ *(Sum of two cubes)*

Note: We can deduce (ii) from (i) by replacing b with $-b$.

Exercise 14
(a) $4x^2 - 25 =$ $(2x)^2 - 5^2 = (2x - 5)(2x + 5)$

(b) $x^3 + 8 =$ $x^3 + 2^3 = (x + 2)(x^2 - 2x + 4)$

To factor a polynomial of degree 3 or more, sometimes we can use the following result:

Result 2 (The Factor Theorem)
Let $P(x)$ is a polynomial. Then $P(b) = 0$ if and only if $x - b$ is a factor of $P(x)$.

Notes:

- If $x - b$ is a factor of $P(x)$ then there is some polynomial $Q(x)$ s.t. $P(x) = (x - b)Q(x)$.
- So to fully factor $P(x)$ we now need to factor $Q(x)$ (which is hopefully easier to factor than $P(x)$).

Exercise 15
Factor $x^3 - 3x^2 - 10x + 24$

Solution
Let $P(x) = x^3 - 3x^2 - 10x + 24$. By trial and error we try and find b so that $P(b) = 0$. The integer guesses are ± 1, ± 2 etc. It is also true that b is a factor of 24 (think about it!). We find that

$$P(1) = 12, \quad P(-1) = 30, \quad P(2) = 0.$$

Thus by the Factor Theorem, $x - 2$ is a factor. Instead of trying to use the Factor Theorem to find additional factors we use long division as follows:

$$\begin{array}{r|rrrr} & x^2 & -x & -12 & \\ \hline x-2 & x^3 - 3x^2 & -10x & +24 \\ & -x^3 + 2x^2 & & \\ \hline & -x^2 & -10x & \\ & x^2 & -2x & \\ \hline & & -12x & +24 \\ & & 12x & -24 \\ \hline & & & 0 \end{array}$$

Therefore:

$$\begin{aligned} & x^3 - 3x^2 - 10x + 24 \\ &= (x-2)(x^2 - x - 12) \\ &= (x-2)(x+3)(x-4). \end{aligned}$$

Of course if the roots b are not easy integers we need another method for factorizing polynomials.

2.6 Completing the square

This is a technique for factoring a quadratic (polynomial of degree 2). We rewrite a quadratic $ax^2 + bx + c$ in the form $a(x+p)^2 + q$, which is achieved by:

1. Factoring the number a from the terms involving x.
2. Adding and subtracting the square of half the coefficient of x.
In general, we have

$$\begin{aligned} P(x) = ax^2 + bx + c &= a\left[x^2 + \frac{b}{a}x\right] + c \\ &= a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned} \tag{2.1}$$

Now we can solve $P(x) = 0$ for x which leads to the quadratic formula (see next section).

Exercise 16
Rewrite $x^2 + x + 1$ by completing the square.
SOLUTION
The square of half the coefficient of x is $1/4$. Thus

$$x^2 + x + 1 = x^2 + x + \frac{1}{4} - \frac{1}{4} + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Exercise 17
Rewrite $2x^2 - 12x + 11$ by completing the square.
SOLUTION

$$\begin{aligned} 2x^2 - 12x + 11 &= 2[x^2 - 6x] + 11 = 2[x^2 - 6x + 9 - 9] + 11 \\ &= 2[(x-3)^2 - 9] + 11 = 2(x-3)^2 - 7. \end{aligned}$$

2.7 The Quadratic Formula

Completing the square as above leads to the formula for the roots of a quadratic equation.

Formula 8 (The Quadratic Formula)
The roots of the quadratic equation $ax^2 + bx + c = 0$ *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Justification:
Starting from (2.1) we set the quadratic function equal to zero in order to solve for x:

$$\begin{aligned}
& a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) = 0 \\
& \implies a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c = \frac{b^2 - 4ac}{4a} \\
& \implies \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\
& \implies x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
& \implies x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \checkmark.
\end{aligned}$$

Exercise 18
Solve the equation $5x^2 + 3x - 3 = 0$.

SOLUTION

With $a = \boxed{5}$, $b = \boxed{3}$, $c = \boxed{-3}$, the quadratic formula gives the solutions

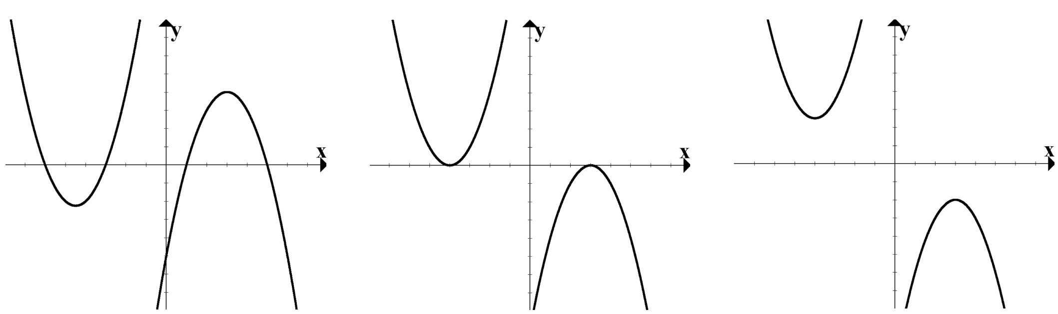
$$x = \frac{-3 \pm \sqrt{3^2 - 4(5)(-3)}}{2(5)} = \frac{-3 \pm \sqrt{69}}{10}.$$

Definition 7
The quantity $b^2 - 4ac$ in the quadratic formula is called the **discriminant**.

There are three possibilities:

Result 3
1. *If $b^2 - 4ac > 0$, the equation has 2 real roots.*
2. *If $b^2 - 4ac = 0$, the roots are equal.*
3. *If $b^2 - 4ac < 0$, the equation has no real roots (the roots are complex).*

These 3 cases correspond to the number of times the parabola $y = ax^2 + bx + c$ crosses the x-axis, namely 2, 1 or 0 respectively. The three cases are illustrated graphically below:



Definition 8
If the quadratic cannot be factored using real numbers (case 3. above) then it is called **irreducible**.

Exercise 19
Is it possible to factor $x^2 + x + 2$ using real numbers?.
SOLUTION The discriminant is

$$b^2 - 4ac = 1^2 - 4(1)(2) = -7 < 0,$$

thus the answer is no. The quadratic is irreducible.

2.8 The Binomial Theorem

We briefly cover an extension of the Perfect Square formula we had, namely

$$(a + b)^2 = a^2 + 2ab + b^2.$$

If we multiply both sides by $(a+b)$ and simplify we get another 'binomial expansion':

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

Before we give a general formula for $(a + b)^n$ we define the 'Factorial Notation'.

Definition 9
The **factorial** *of a non-negative integer n, denoted $n!$, is the product of all positive integers less than or equal to n, i.e.*

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

We also define $0! = 1$ (see the note below).

Note: $\ldots, \frac{4!}{4} = 3!, \frac{3!}{3} = 2!, \frac{2!}{2} = 1!$, so it is reasonable that $\frac{1!}{1} = 0! = 1$.

Exercise 20
By definition we have:

$$0! = 1$$

$$1! = 1$$

$$2! = 2 \cdot 1 = 2$$

$$3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Exercise 21
Simplify the following expressions:
(a) $\frac{7!}{5!}$, (b) $\frac{1000!}{998!\,2!}$.

SOLUTION

(a) $$\frac{7!}{5!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 7 \cdot 6 = 42.$$

(b) $$\frac{1000!}{998!\,2!} = \frac{1000 \cdot 999 \cdot 998!}{998!\,2!} = \frac{1000 \cdot 999 \cdot \cancel{998!}}{\cancel{998!}\,2!} = \frac{999000}{2} = 499500.$$

Result 4 (The Binomial Theorem)
If n is a positive integer, then

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + nab^{n-1} + b^n.$$

Notes:

- Observe the pattern in the formula!
- There is a lot more on this topic, for example see http://www.mathsisfun.com/algebra/binomial-theorem.html
- We will need this formula when we study Calculus.

Exercise 22
Expand and simplify $(x-2)^5$.

SOLUTION

We use the Binomial Theorem with $a = x$, $b = -2$, $n = 5$ yielding

$$\begin{aligned}(x-2)^5 &= x^5 + 5x^4(-2)^1 + \frac{5\cdot 4}{2!}x^3(-2)^2 + \frac{5\cdot 4\cdot 3}{3!}x^2(-2)^3 \\ &+ \frac{5\cdot 4\cdot 3\cdot 2}{4!}x^1(-2)^4 + \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{5!}x^0(-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32.\end{aligned}$$

2.9 Radicals

The most commonly used radical is the square root.

Definition 10
The **square root** *of a number* $a \geq 0$*, denoted* $\sqrt{a}$*, is the unique non-negative real number* x *such that* $x^2 = a$*.*

Notes:

- Notice that $\sqrt{a}$ only makes sense if $a \geq 0$ (unless we are considering complex numbers - but that is another story!)
- The solution of the equation $x^2 = a$ is of course $\pm\sqrt{a}$, thus take note that the square root is defined to be the *positive* square root (e.g., $\sqrt{9} = +3$).
- Exponent form: $\sqrt{a} = a^{\frac{1}{2}}$ (see next section).

Here are two rules for working with square roots:

Formula 9
(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$*,* *(ii)* $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$*.*

Caution !

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

For instance, with $a = b = 1$ we would get $\sqrt{2} =$ 2 ?

Exercise 23

1. $\sqrt{18}/\sqrt{2} =$ $\sqrt{18/2} = \sqrt{9} = 3.$
2. $\sqrt{x^2 y} =$ $\sqrt{x^2}\sqrt{y} = |x|\sqrt{y}$ ($\sqrt{x^2} = |x|$, the 'absolute value' of x) .

Let's consider the nth roots of numbers.

Definition 11
The n**th root** $(n \in \mathbb{N})$ *of a number* a*, denoted* $\sqrt[n]{a}$*, is the unique real number* x *such that* $x^n = a$*. If* n *is even then* a *must be non-negative and* $\sqrt[n]{a}$ *is the non-negative solution of* $x^n = a$*.*

Note: Exponent form: $\sqrt[n]{a} = a^{\frac{1}{n}}$ (see next section).

Exercise 24

(a) $\sqrt[4]{16} =$ 2 because $2^4 = 16$.

(b) $\sqrt[4]{-16} =$ undefined, because there is no number x such that $x^4 < 0$.

(c) $\sqrt[3]{64} =$ 4 because $4^3 = 64$.

(d) $\sqrt[3]{-64} =$ -4 because $(-4)^3 = -64$.

Thus we can see that we must be careful with roots of negative numbers:

Caution !

- *Even roots of negative numbers are undefined*
- *Odd roots of negative numbers are negative*

As with square roots we have the rules:

Formula 10
(i) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$*,* *(ii)* $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Exercise 25
Using Formula 10 we have $\sqrt[3]{x^3y^6} =$ $\sqrt[3]{x^3}\sqrt[3]{y^6} = xy^2$.

Rationalizing a Fraction

To 'rationalize' a fraction where the numerator or denominator contains an an expression like $\sqrt{a}-\sqrt{b}$, we multiply both numerator and denominator by $\sqrt{a}+\sqrt{b}$. Then we take advantage of the formula for a difference of 2 squares:

$$(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b.$$

Exercise 26
Rationalize the expression $\dfrac{\sqrt{x+4}-2}{x}$.
SOLUTION

$$\begin{aligned}\frac{\sqrt{x+4}-2}{x} &= \left(\frac{\sqrt{x+4}-2}{x}\right)\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)\\ &= \frac{(x+4)-4}{x(\sqrt{x+4}+2)}\\ &= \frac{x}{x(\sqrt{x+4}+2)} = \frac{1}{\sqrt{x+4}+2}, \quad x \neq 0.\end{aligned}$$

Note: we had to exclude the possibility that $x = 0$ otherwise we are dividing by 0.

2.10 Exponents

Numbers of the form b^n involve two numbers, the 'base' b and the 'exponent' (or 'power') n. When n is a natural number (i.e. an element of $\mathbb{N}$) then b^n corresponds to repeated multiplication of the base.

Definition 12 (Exponents)
Let a be a positive real number and $n \in \mathbb{N}$. Then

1. $a^n = a \cdot a \cdot \cdots \cdot a$ *(n factors)*
2. $a^0 = 1$
3. $a^{-n} = \frac{1}{a^n}$
4. $a^{1/n} = \sqrt[n]{a}$
5. $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$, *$m$ is an integer*

Note: later we discuss exponential *functions*.

Result 5 (Laws of exponents)
Let a and b be positive numbers and let r and s be any rational numbers (that is, ratios of integers). Then

1. $a^r \times a^s = a^{r+s}$
2. $\frac{a^r}{a^s} = a^{r-s}$
3. $(a^r)^s = a^{rs}$
4. $(ab)^r = a^r b^r$
5. $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$
6. $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$.

Exercise 27
Simply the folllowing:

(a) $2^8 \times 8^2 = \quad 2^8 \times (2^3)^2 = 2^{14}.$

(b) $4^{3/2} = \quad \sqrt{4^3} = \sqrt{64} = 8.$ Alternatively: $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8.$

(c)
$$\frac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} + \frac{1}{y}} = \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y+x}{xy}} = \frac{y^2 - x^2}{x^2y^2} \cdot \frac{xy}{y+x}$$
$$= \frac{(y-x)(y+x)}{xy(y+x)} = \frac{y-x}{xy}, \quad y + x \neq 0.$$

(d) $$\frac{1}{\sqrt[3]{x^4}} = \quad \frac{1}{x^{4/3}} = x^{-4/3}.$$

(e) $$\left(\frac{x}{y}\right)^3 \left(\frac{y^2x}{z}\right)^4 = \quad \frac{x^3}{y^3} \cdot \frac{y^8x^4}{z^4} = x^7y^5z^{-4}, \quad y \neq 0.$$

2.11 Inequalities

When working with inequalities note the following rules:

Result 6 (Rules of inequalities)

1. **Multiplication by a positive constant**:
   If $x > y$, then $ax > ay$ if a is a positive number.
2. **Multiplication by a negative constant**:
   If $x > y$, then $ax < ay$ if a is a negative number.
3. **Addition of inequalities**:
   If $x > y$ and $u \geq v$, then $x + u > y + v$.
   (So if $u = v$ then $x > y$ implies $x + u > y + u$.)

Note rule 2., which says "*if we multiply both sides of an inequality by a negative number we reverse the direction of the inequality*". E.g.: $2 > 1 \implies -2 < -1$

Things NOT to do with inequalities:

- **Subtraction of inequalities**:
  If $x > y$, and $u > v$, then $(x - u) \not> (y - v)$.
- **Multiplication of inequalities**:
  If x, y, u, v are positive, then $x > y$, $u > v$ implies $xu > yv$. The result is not necessarily valid when some of the numbers are negative.
- **Division of inequalities**:
  If $x > y$, and $u > v$, these do *not* imply $x/u > y/v$.

Exercise: you can find your own counter examples here.

Exercise 28

Solve $1 + x < 7x + 5$:

SOLUTION
$$\begin{aligned} 1 + x < 7x + 5 &\iff 1 < 6x + 5 \quad \text{(Rule 3.)} \\ &\iff -4 < 6x \quad \text{(Rule 3.)} \\ &\iff 6x > -4 \quad \text{(same thing)} \\ &\iff x > -\frac{4}{6} = -\frac{2}{3} \quad \text{(Rule 1.)} \end{aligned}$$

In other words, the solution is the interval $(-2/3, \infty)$.

Exercise 29 (Chart Method)
Solve $x^2 - 5x + 6 \le 0$:
SOLUTION First factor the left hand side $(x-2)(x-3) \le 0$. The equation $(x-2)(x-3) = 0$ has the solutions 2 and 3, thus these numbers divide the real number line into three intervals:

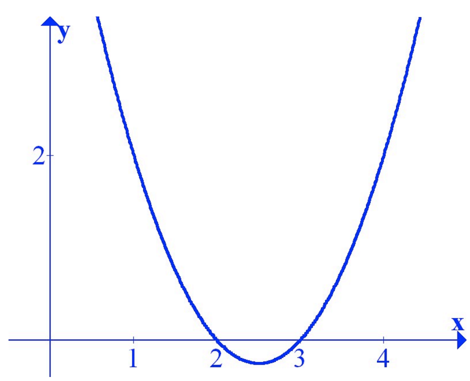
$$(-\infty, 2) \qquad (2, 3) \qquad (3, \infty).$$

On each of these intervals we determine the sign of the factors and record this in a table: Thus from the table we see that $(x-2)(x-3)$ is negative when

Interval	$x-2$	$x-3$	$(x-2)(x-3)$
$x < 2$	$-$	$-$	$+$
$2 < x < 3$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$

$2 < x < 3$, i.e. the solution of the inequality $(x-2)(x-3) \le 0$ is the interval $[2, 3]$.

Another way to get the same information is to use *test values*. For instance, checking $x = 1$ for the interval $(-\infty, 2)$ in $x^2 - 5x + 6$ yields $1^2 - 5(1) + 6 = 2$. The polynomial doesn't change sign inside the three intervals so we conclude that it is positive on $(-\infty, 2)$.
A visual method for solving the inequality is to just graph the parabola $y = x^2 - 5x + 6$ and observe that the curve is on or below the x-axis when $2 \le x \le 3$.



Exercise 30 (Chart Method)
Solve $x^3 + 3x^2 > 4x$:
SOLUTION We can factor this as

$$x(x^2 + 3x - 4) > 0 \quad \text{or} \quad x(x-1)(x+4) > 0.$$

The equation $x(x-1)(x+4) = 0$ has the solutions 0, 1 and -4, thus these numbers divide the real number line into four intervals:

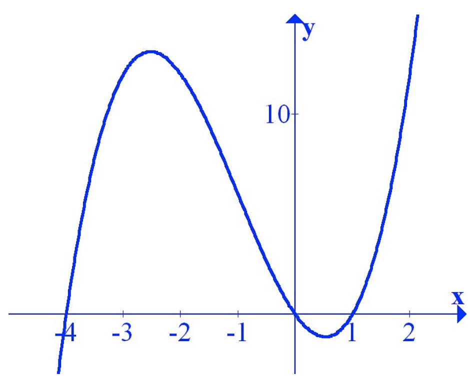
$$(-\infty, -4) \qquad (-4, 0) \qquad (0, 1) \qquad (1, \infty).$$

On each of these intervals we determine the sign of the factors and record this in a table. Thus we see from the table that $x(x-1)(x+4)$ is greater than zero when

Interval	x	$x-1$	$x+4$	$x(x-1)(x+4)$
$x < -4$	$-$	$-$	$-$	$-$
$-4 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$x > 1$	$+$	$+$	$+$	$+$

$-4 < x < 0$ and $x > 1$, thus the required solution set is $(-4, 0) \cup (1, \infty)$.

A much quicker way to see this is just to graph the function $y = x(x-1)(x+4)$, and observe where the curve is above the x-axis:



2.12 Absolute Value

Definition 13
The **Absolute Value** *of a number a, denoted $|a|$, is the distance from a to 0 on the real number line. Thus*

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0. \end{cases}$$

Notes:

- Thus $|a| \geq 0$ for every number a.
- E.g., $|3| = 3$, $|-3| = 3$, $|0| = 0$.
- Sometimes we need to work out the sign of the number before we can evaluate the absolute value, e.g. $|\sqrt{2}-1| = \sqrt{2}-1$, $|3-\pi| = \pi - 3$, $|\ln(0.9)| = -\ln(0.9)$ ($\ln(0.9) \approx -0.1054$).
- $\sqrt{a^2} = |a|$. E.g., $\sqrt{(-1)^2} = \sqrt{+1} = 1 = |-1|$.

Exercise 31
Re-write $|3x-2|$ as a piecewise defined function (without the absolute value symbol).
SOLUTION

$$|3x-2| = \begin{cases} 3x-2 \text{ if } 3x-2 \geq 0 \\ -(3x-2) \text{ if } 3x-2 < 0 \end{cases} = \begin{cases} 3x-2 \text{ if } x \geq 2/3 \\ 2-3x \text{ if } x < 2/3. \end{cases}$$

Note the following basic rules for absolute values:

Result 7
Suppose a and b are real numbers and n is an integer. Then

$$\textbf{1. } |ab| = |a||b|, \qquad \textbf{2. } \left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad (b \neq 0), \qquad \textbf{3. } |a^n| = |a|^n.$$

For solving equations or inequalities involving absolute values, the following statements are helpful:

Result 8
Suppose $a > 0$. *Then*

4. $|x| = a$ *if and only if* $x = \pm a$,
5. $|x| < a$ *if and only if* $-a < x < a$,
6. $|x| > a$ *if and only if* $x > a$ *or* $x < -a$.

Notes:

- For instance, the statement $|x| < a$ says that the distance from x to the origin is less than a, which is true if x lies between $-a$ and a.
- The expression $|a - b|$ $(= |b - a|)$ represents the distance between a and b.

Exercise 32
Solve $|2x - 5| = 3$.
SOLUTION By Property 4., $|2x - 5| = 3$ is equivalent to

$$2x - 5 = 3 \quad \text{or} \quad 2x - 5 = -3.$$

So $2x = 8$ or $2x = 2$. Thus, $x = 4$ or $x = 1$.

Exercise 33
Solve $|x - 5| < 2$.
SOLUTION 1 By Property 5., $|x - 5| < 2$ is equivalent to

$$-2 < x - 5 < 2.$$

Therefore, adding 5 to each side, we have

$$3 < x < 7,$$

and the solution set is the open interval $(3, 7)$.

Exercise 34
Solve $|3x+2| \geq 4$.
Solution By Property 4. and 6., $|3x+2| \geq 4$ is equivalent to

$$3x+2 \geq 4 \quad \text{or} \quad 3x+2 \leq -4.$$

So $3x \geq 2$, which gives $x \geq 2/3$. In the second case $3x \leq -6$, which gives $x \leq -2$. So the solution set is $(-\infty, -2] \cup [2/3, \infty)$.

Chapter 3

Minimal Trigonometry and Radians

3.1 Some Trig Definitions & Identities

Although we *briefly* review some basic trigonometry, we assume students have studied trigonometry before. Trigonometry is useful for more than just calculating angles in a triangle. The 'trig functions' are among the list of 'elementary functions' that are used to model the real world, for example, periodicity in the stock market, or economic trends using 'Fourier Series' [1].

[1]A representation of a function in terms of a sum of sine and cosine functions.