IN-CLASS DISCOVERIES FOR MATH*1160

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Discovery 1 (Section 1.2)

1. Without doing any row operations, explain why the following system of linear equations is consistent:

$$\begin{array}{rcrcrcl} \mathbf{2x_1} & + & \mathbf{3x_2} & + & \mathbf{5x_3} & = & \mathbf{0} \\ \mathbf{-5x_1} & + & \mathbf{6x_2} & - & \mathbf{17x_3} & = & \mathbf{0} \\ \mathbf{7x_1} & - & \mathbf{4x_2} & + & \mathbf{3x_3} & = & \mathbf{0} \end{array}$$

2. The following system has more variables than equations. Why does it have an infinite number of solutions?

$$\begin{array}{rcrcrcrcl} \mathbf{2x_1} & + & \mathbf{3x_2} & + & \mathbf{5x_3} & + & \mathbf{2x_4} & = & \mathbf{0} \\ \mathbf{-5x_1} & + & \mathbf{6x_2} & - & \mathbf{17x_3} & - & \mathbf{3x_4} & = & \mathbf{0} \\ \mathbf{7x_1} & - & \mathbf{4x_2} & + & \mathbf{3x_3} & + & \mathbf{13x_4} & = & \mathbf{0} \end{array}$$

Discovery 2 (Section 1.3 and 1.4)

Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}.$$

1. Calculate $\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{B} + \boldsymbol{A}$. Is matrix addition commutative? (i.e., is it generally true that $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$?)

2. Calculate $\boldsymbol{AB}$ and $\boldsymbol{BA}$. Is matrix multiplication commutative? (i.e., is it generally true that $\boldsymbol{AB} = \boldsymbol{BA}$?)

Discovery 3 (Section 1.5)

Let $\boldsymbol{A}$ be $\boldsymbol{m \times p}$, $\boldsymbol{B}$ be $\boldsymbol{p \times q}$, and $\boldsymbol{C}$ be $\boldsymbol{q \times n}$. Find an expression for

$$\Big((\boldsymbol{AB})\boldsymbol{C}\Big)_{\boldsymbol{ij}}.$$

Hints:

- Recall that $(\boldsymbol{AB})_{\boldsymbol{ij}} = \displaystyle\sum_{\boldsymbol{s=1}}^{\boldsymbol{p}} \boldsymbol{a_{is} b_{sj}}$.
- Let $\boldsymbol{D} = \boldsymbol{AB}$ and consider the expression for $(\boldsymbol{DC})_{\boldsymbol{ij}}$ (using a 'dummy' variable $\boldsymbol{k}$).

Discovery 4 (Section 1.6)

Use the following invertible matrix

$$\begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-4} \end{bmatrix}$$

to encode the message

“ MEET ME MONDAY”

The inverse of the ‘encoding matrix’ above is the following ‘decoding matrix’:

$$\begin{bmatrix} \mathbf{-1} & \mathbf{-10} & \mathbf{-8} \\ \mathbf{-1} & \mathbf{-6} & \mathbf{-5} \\ \mathbf{0} & \mathbf{-1} & \mathbf{-1} \end{bmatrix}$$

Also show how to decode the encoded message.

Discovery 5 (Section 1.6)

Consider an arbitrary square nonsingular matrix $\boldsymbol{A}$ and a nonzero scalar $\boldsymbol{c}$. Prove that

$$(\boldsymbol{cA})^{-1} = \frac{1}{\boldsymbol{c}}\boldsymbol{A}^{-1}$$

Hint:
Recall that

$$\boldsymbol{A}^{-1} = \boldsymbol{B} \iff \boldsymbol{AB} = \boldsymbol{I},$$

where $\boldsymbol{I}$ is the identity matrix.

Discovery 6 (Section 2.2)

Investigate for what values of $a \in \mathbb{R}$ the linear system

$$\begin{cases} x + \quad\quad\quad y &= 3, \\ x + (a^2 - 8)y &= a, \end{cases}$$

has (i) no solution, (ii) an infinite number of solutions, and (iii) a unique solution.

Strategy: apply Gauss-Jordon Elimination to the associated augmented matrix.

Discovery 7 (Section 2.3)

Investigate for what values of $\boldsymbol{a} \in \mathbb{R}$ the homogeneous linear system

$$\begin{cases} (a-1)x + 2y &= 0, \\ 2x + (a-1)y &= 0, \end{cases}$$

has a **non**trivial solution.

Strategy: use the converse of Theorem 6 (either 1., 3. or 5.).

Discovery 8 (Section 3.2)

$$\text{If} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 2,$$

$$\text{find} \quad \begin{vmatrix} (3a_1 - 6a_3) & a_2 & a_3 \\ (3b_1 - 6b_3) & b_2 & b_3 \\ (c_1 - 2c_3) & \frac{1}{3}c_2 & \frac{1}{3}c_3 \end{vmatrix}.$$

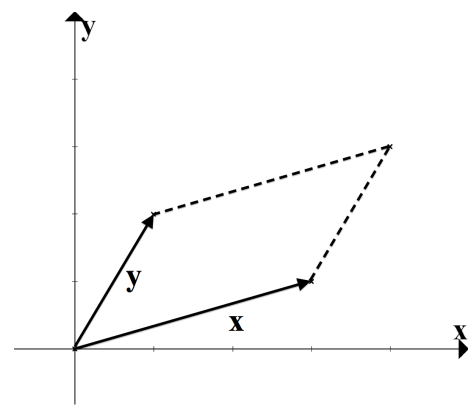
Hint: apply row and column operations.

Discovery 9 (Section 3.2)

Discovery 10a

Section: 3.2

It can be shown from elementary (but tedious) geometry that the ('signed') area of the parallelogram shown below



is given by the determinant

$$\begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} \\ \boldsymbol{y_1} & \boldsymbol{y_2} \end{vmatrix} \quad \text{where} \quad \boldsymbol{x} = \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_2} \end{pmatrix}$$

Give a geometrical interpretation to the rules:

(a)

$$\begin{vmatrix} \boldsymbol{x_1 + y_1} & \boldsymbol{x_2 + y_2} \\ \boldsymbol{y_1} & \boldsymbol{y_2} \end{vmatrix} = \begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} \\ \boldsymbol{y_1} & \boldsymbol{y_2} \end{vmatrix}$$

and

(b)

$$\begin{vmatrix} \boldsymbol{2x_1} & \boldsymbol{2x_2} \\ \boldsymbol{y_1} & \boldsymbol{y_2} \end{vmatrix} = \boldsymbol{2} \begin{vmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} \\ \boldsymbol{y_1} & \boldsymbol{y_2} \end{vmatrix}$$

Hints: In (a) consider the area of the parallelogram formed by the vectors $\boldsymbol{x} + \boldsymbol{y}$ and $\boldsymbol{y}$. In (b) consider the parallelogram formed by $\boldsymbol{2x}$ and $\boldsymbol{y}$.

Discovery 10 (Section 3.2)

Let

$$\boldsymbol{A} = \begin{vmatrix} \mathbf{2} & \mathbf{-3} \\ \mathbf{1} & \mathbf{2} \end{vmatrix}$$

1. Compute $|\boldsymbol{A}|$ and $|\boldsymbol{A}^{-1}|$.
2. Make a conjecture about the determinant of the inverse of a matrix.

Discovery 11 (Section 3.3)

Let

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}.$$

Use a cofactor expansion to prove the usual formula for the determinant of $\boldsymbol{A}$, namely

$$|\boldsymbol{A}| = \boldsymbol{ad - bc}.$$

Discovery 12 (Section 3.3)

Let

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{d} & \boldsymbol{e} & \boldsymbol{f} \\ \boldsymbol{g} & \boldsymbol{h} & \boldsymbol{i} \end{pmatrix}.$$

Use a cofactor expansion to verify

$$|\boldsymbol{A}| = |\boldsymbol{A}^{\boldsymbol{T}}|.$$

Discovery 13 (Section 3.4)

Let

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$$

be a nonsingular matrix. Use the formula

$$\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{|\boldsymbol{A}|}\mathbf{adj}(\boldsymbol{A})$$

to verify that

$$\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\boldsymbol{ad} - \boldsymbol{bc}} \begin{pmatrix} \boldsymbol{d} & -\boldsymbol{b} \\ -\boldsymbol{c} & \boldsymbol{a} \end{pmatrix}$$

Discovery 14 (Section 4.1)

Draw the vectors

$$\vec{OP} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \vec{OR} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \vec{OQ} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

How do we get to R using the vectors $\vec{OP}$ and $\vec{OQ}$?

Discovery 15 (Section 4.1)

Consider the vectors $\boldsymbol{u} = (\mathbf{3}, \mathbf{-4})$, $\boldsymbol{v} = (\mathbf{9}, \mathbf{1})$, and $\boldsymbol{w} = (\mathbf{-39}, \mathbf{0})$.

Using graph paper (see handout) do the following:

(a) Use directed line segments to represent $\boldsymbol{u}$ and $\boldsymbol{v}$.

(b) Find $\boldsymbol{u} + \boldsymbol{v}$ and represent graphically.

(c) Find $\mathbf{2}\boldsymbol{v} - \boldsymbol{u}$ and represent graphically.

(d) Write the vector $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Discovery 16 (Section 4.2)

Let

$$\boldsymbol{V} = \{(\boldsymbol{x}, \boldsymbol{x} - \mathbf{2}) \in \mathbb{R}^2 \,|\, \boldsymbol{x} \in \mathbb{R}\}.$$

Show that $\boldsymbol{V}$ is not closed with respect to vector addition or scalar multiplication (and is therefore not a vector space).

Strategy:

Let

$$\boldsymbol{v_1} = (\boldsymbol{x_1}, \boldsymbol{x_1} - \mathbf{2}) \in \boldsymbol{V}, \quad \boldsymbol{v_2} = (\boldsymbol{x_2}, \boldsymbol{x_2} - \mathbf{2}) \in \boldsymbol{V},$$

and show that $\boldsymbol{v_1} + \boldsymbol{v_2} \notin \boldsymbol{V}$.

Also, let

$$\boldsymbol{v_1} = (\boldsymbol{x}, \boldsymbol{x} - \mathbf{2}) \in \boldsymbol{V}, \quad \boldsymbol{k} \in \mathbb{R},$$

and show $\boldsymbol{kv} \notin \boldsymbol{V}$ (generally).

Discovery 17 (Section 4.3)

Show the following results:

1. $W = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0\}$
with the standard operations is *not* a subspace of $\mathbb{R}^2$.

2. $W = \left\{ \begin{pmatrix} x \\ 3x \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$
with the standard operations is a subspace of $\mathbb{R}^2$.

Discovery 18 (Section 4.4)

Try and write the vector

$$\boldsymbol{w} := (\mathbf{1}, -\mathbf{2}, \mathbf{2})$$

as a linear combination of the vectors in the set

$$\boldsymbol{S} := \{(\mathbf{1}, \mathbf{2}, \mathbf{3}), \quad (\mathbf{0}, \mathbf{1}, \mathbf{2}), \quad (-\mathbf{1}, \mathbf{0}, \mathbf{1})\}.$$

What goes wrong?

Discovery 19 (Section 4.5)

Determine whether the sets of vectors given below are linearly independent or linearly dependent:

(a)

$$S_1 = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\} \subset P_2$$

(b)

$$S_2 = \left\{ \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \right\} \subset M_{22}$$

Discovery 20 (Section 4.6)

Show that the set

$$\boldsymbol{S = \{(1, 1),\ (1, -1)\}}$$

is a basis for $\mathbb{R}^{\mathbf{2}}$.

Hint:
This is a non-standard basis so you will need to do a little more work than for the standard basis examples.

Discovery 21 (Section 4.6)

Let

$$\boldsymbol{S} = \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{-4} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{-3} \\ \mathbf{7} \end{pmatrix}, \begin{pmatrix} \mathbf{-4} \\ \mathbf{1} \\ \mathbf{6} \end{pmatrix} \right\}$$

Prove that the set of vectors $\boldsymbol{S}$ cannot be a basis for $\mathbb{R}^{\mathbf{3}}$ by

(a) Applying Theorem 19

(b) First Principles. (Hint: prove that the vectors are linearly dependent - *no row operations are required*.)

Discovery 22 (Section 4.6)

Prove that the set of vectors

$$\boldsymbol{S} = \{\boldsymbol{t^2 - t + 1}, \quad \boldsymbol{t^2 - 1}\},$$

cannot be a basis for $\boldsymbol{P_2}$, by

(a) Applying Theorem 19

(b) First Principles. (Hint: prove that the vectors do not span $\boldsymbol{P_2}$ - *no row operations are required*.)

Discovery 23 (Section 4.7)

Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-4} & \mathbf{2} & \mathbf{0} & \mathbf{1} \\ \mathbf{-3} & \mathbf{-9} & \mathbf{12} & \mathbf{-6} & \mathbf{0} & \mathbf{-3} \end{pmatrix}.$$

Find a basis for the null-space of $\boldsymbol{A}$ (i.e., $\boldsymbol{N}(\boldsymbol{A})$).

Discovery 24 (Section 4.8)

Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} & \mathbf{6} & \mathbf{7} & \mathbf{14} \end{pmatrix}$$

(i) Find the dimension of the row space of $\boldsymbol{A}$.

(ii) Find the dimension of the column space of $\boldsymbol{A}$.

(iii) Make a conjecture about the relationship between the row and column space of a matrix.

Discovery 25 (Section 4.8)

Observe that

$$A = \begin{pmatrix} 1 & 3 & -4 & 2 & 0 & 1 \\ -3 & -9 & 12 & -6 & 0 & -3 \end{pmatrix} \overset{E.R.O.'s}{\longrightarrow} \begin{pmatrix} 1 & 3 & -4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(RREF)},$$

(just do $r_2 + 3r_1 \longrightarrow r_2$). Here A is $m \times n$ ($m = 2$ and $n = 6$).
Verify without doing any calculations the result of Theorem 27, namely:

$$\text{rank } A + \text{nullity } A = n.$$

Discovery 26 (Section 5.1 - 5.2)

Let $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ be vectors in an inner product space V. Prove that

$$\|\underline{\boldsymbol{u}} + \underline{\boldsymbol{v}}\| \leq \|\underline{\boldsymbol{u}}\| + \|\underline{\boldsymbol{v}}\|$$

with equality if and only if $\underline{\boldsymbol{u}}$ and $\underline{\boldsymbol{v}}$ are orthogonal.

Hint:
Use the definition of a norm and the Cauchy-Schwartz inequality.