Homework Problems for Math*1160 (W18).

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Note

We use the following abreviations:

d.p. $=$ decimal places, e.g. π to $\mathbf{3}$ d.p. is 3.141
s.f. $=$ significant figures, e.g. π to $\mathbf{3}$ s.f. is 3.14

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Chapter 1

Linear Equations & Matrices

1.1 Formulation of systems of linear equations

1) A bartender makes a cocktail with whisky and rum that contains $\mathbf{10}$ ml of alcohol and $\mathbf{40}$ ml of water. The rum has $\mathbf{14\%}$ alcohol by volume (and hence $\mathbf{86\%}$ water by volume), while the whisky has $\mathbf{43\%}$ alcohol by volume (and hence $\mathbf{57\%}$ water by volume). Write $\mathbf{2}$ equations in $\mathbf{2}$ unknowns for the amount of whisky ($\boldsymbol{x}$) and the amount of rum ($\boldsymbol{y}$) needed. (Hint: see the similar problem in your lecture notes.)

1.2 Systems of Linear Equations

2) Apply back-substitution to solve the following linear system:

$$\begin{cases} 2x_1 - x_2 + 3x_3 - 2x_4 = 1 \\ \qquad\quad x_2 - 2x_3 + 3x_4 = 2 \\ \qquad\qquad\qquad 4x_3 + 3x_4 = 3 \\ \qquad\qquad\qquad\qquad\quad 4x_4 = 4 \end{cases}$$

3) Find the unique solution of the following linear system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 3x_1 - x_2 - 3x_3 = -1 \\ 2x_1 + 3x_2 + x_3 = 4 \end{cases}$$

4) By Attempting to solve the following system show why no solution exists:

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 9 \end{cases}$$

5) Find the infinite solution set corresponding to the following linear system:

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 12 \end{cases}$$

6) (a) Without doing any row operations, explain why the following system of linear equations is consistent:

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 = 0 \\ -5x_1 + 6x_2 - 17x_3 = 0 \\ 7x_1 - 4x_2 + 3x_3 = 0 \end{cases}$$

(b) Without doing any row operations, explain why the following system of linear equations has an infinite number of solutions?

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 + 2x_4 = 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 = 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 = 0 \end{cases}$$

1.3 Matrices

Review your notes and the examples there. Then make your own examples up and check them in Matlab. Alternatively, you can Google many examples for the simple operations of matrix addition, subtraction, scalar multiplication and transpose. Also, the *many* standard texts in Linear Algebra that we have in the library will also have a multitude of examples covering this basic material.

7) Calculate $(A + B^T - 2C)^T$ where

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 1 \\ -1 & 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -3 \end{pmatrix}.$$

1.4 Matrix Multiplication

See the examples in your notes, in standard textbooks, and on the web. This is another type of problem where you can readily construct your own examples and easily check them using Matlab. Keep in mind the comments made regarding the dimensions of the matrices concerned on page 38 of your Workbook.

8) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Find the matrix product AB.

9) Let

$$\boldsymbol{U} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix} \quad \text{and} \quad \boldsymbol{V} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \\ \mathbf{4} \end{pmatrix}.$$

Verify that (a) $\boldsymbol{V^T U} = \boldsymbol{U^T V}$ and (b) $(\boldsymbol{U V^T})^{\boldsymbol{T}} = \boldsymbol{V U^T}$.

10) Find a number $\boldsymbol{k}$ such that $\boldsymbol{A\underline{x}} = \boldsymbol{k\underline{x}}$, where

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\underline{x}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{1} \end{pmatrix}.$$

11) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} \mathbf{7} \\ \mathbf{8} \\ \mathbf{9} \end{pmatrix}.$$

Write the matrix product $\boldsymbol{Ax}$ as a sum of the columns of $\boldsymbol{A}$ multiplied by the entries in $\boldsymbol{x}$. (Hint: use Theorem 1).

1.5 Algebraic Properties of Matrix Operations

Recall that the 'Algebraic Properties of Matrix Operations' that we refer to are listed in Theorem 2 of your notes.

12) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}.$$

(a) Calculate $\boldsymbol{A+B}$ and $\boldsymbol{B+A}$. Is matrix addition commutative? (i.e., is it generally true that $\boldsymbol{A+B = B+A}$?)

(b) Calculate $\boldsymbol{AB}$ and $\boldsymbol{BA}$. Is matrix multiplication commutative? (i.e., is it generally true that $\boldsymbol{AB = BA}$?)

13) Verify the algebraic property $(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{B^T A^T}$ (Property 11) where

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{-3} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} \end{pmatrix}.$$

14) Determine the values of the number $\boldsymbol{k}$ such that $(\boldsymbol{kA})^{\boldsymbol{T}}(\boldsymbol{kA}) = \mathbf{1}$, where

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}.$$

15) Using the algebraic properties of matrices find an expression for $(\boldsymbol{A - B})^{\boldsymbol{T}}$.

16) Using the algebraic properties of matrices find an expression for $(\boldsymbol{A}-\boldsymbol{B})^2$.

17) Prove that if $\underline{\boldsymbol{x}_1}$ and $\underline{\boldsymbol{x}_2}$ are solutions of the linear system $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$, then $\underline{\boldsymbol{x}_1} - \underline{\boldsymbol{x}_2}$ is a solution of the corresponding homogeneous system $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$.

18) Prove for an arbitrary matrix $\boldsymbol{A}$ that

$$(\boldsymbol{A}^T)^2 = (\boldsymbol{A}^2)^T.$$

(Hint: use Theorem 2, part (11).)

19) Find examples to prove the following statements, where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are $\boldsymbol{2}$-by-$\boldsymbol{2}$ matrices:
(a) $\boldsymbol{AB} = \boldsymbol{AC}$ does not imply $\boldsymbol{B} = \boldsymbol{C}$.
(b) $\boldsymbol{AB} = \boldsymbol{O}$ (where $\boldsymbol{O}$ is the 'zero matrix') does not imply $\boldsymbol{A} = \boldsymbol{O}$ or $\boldsymbol{B} = \boldsymbol{O}$.
(See page 53 of your Workbook.)

20) Let $\boldsymbol{A}$ be $\boldsymbol{m} \times \boldsymbol{p}$, $\boldsymbol{B}$ be $\boldsymbol{p} \times \boldsymbol{q}$, and $\boldsymbol{C}$ be $\boldsymbol{q} \times \boldsymbol{n}$. Find an expression for

$$\Big((\boldsymbol{AB})\boldsymbol{C}\Big)_{ij}.$$

Hints:

- Recall that $(\boldsymbol{AB})_{ij} = \sum_{s=1}^{p} \boldsymbol{a}_{is}\boldsymbol{b}_{sj}$ (see page 37 of your Workbook).
- Let $\boldsymbol{D} = \boldsymbol{AB}$ and consider the expression for $(\boldsymbol{DC})_{ij}$ (using a 'dummy' variable $\boldsymbol{k}$).

Note:
This question isn't typical of the sort of math problems I set. But it gives a taste of what it is like to rigorously prove the algebraic properties of matrices. If you are a math/stats/physics student you should attempt it!

1.6 Special Types of Matrices

21) Consider the matrix equation $\boldsymbol{AC}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ where

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \boldsymbol{2} & \boldsymbol{1} \\ \boldsymbol{-1} & \boldsymbol{1} \end{pmatrix}, \quad \boldsymbol{C}^{-1} = \begin{pmatrix} \boldsymbol{2} & \boldsymbol{1} \\ \boldsymbol{1} & \boldsymbol{2} \end{pmatrix}, \quad \text{and} \quad \underline{\boldsymbol{b}} = \begin{pmatrix} \boldsymbol{2} \\ \boldsymbol{3} \end{pmatrix}.$$

It is easily verified that $\boldsymbol{A}^{-1}$ and $\boldsymbol{C}^{-1}$ are nonsingular. Without first finding $\boldsymbol{A}$ and $\boldsymbol{C}$ determine $\underline{\boldsymbol{x}}$.

22) Let

$$\boldsymbol{A}^{-1} = \begin{pmatrix} \boldsymbol{1} & \boldsymbol{3} \\ \boldsymbol{4} & \boldsymbol{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}^{-1} = \begin{pmatrix} \boldsymbol{-1} & \boldsymbol{1} \\ \boldsymbol{3} & \boldsymbol{7} \end{pmatrix}.$$

Find $(\boldsymbol{AB})^{-1}$.

23) Consider the linear system $\boldsymbol{A^T \underline{x} = \underline{b}}$ where

$$\boldsymbol{A^{-1}} = \begin{pmatrix} \mathbf{4} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\underline{b}} = \begin{pmatrix} \mathbf{1} \\ \mathbf{-2} \end{pmatrix}.$$

Find the solution $\boldsymbol{\underline{x}}$.

24) Prove for an arbitrary square matrix $\boldsymbol{A}$ that

$$\boldsymbol{(A^2)^{-1} = (A^{-1})^2}.$$

(Hint: use Theorem 4 (i).)

25) Use the following invertible matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-1} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{-1} & \mathbf{-4} \end{pmatrix}$$

to encode the message

" MEET ME MONDAY"

The inverse of the 'encoding matrix' above is the following 'decoding matrix':

$$\boldsymbol{A^{-1}} = \begin{pmatrix} \mathbf{-1} & \mathbf{-10} & \mathbf{-8} \\ \mathbf{-1} & \mathbf{-6} & \mathbf{-5} \\ \mathbf{0} & \mathbf{-1} & \mathbf{-1} \end{pmatrix}$$

Also show how to decode the encoded message.

26) Consider an arbitrary square nonsingular matrix $\boldsymbol{A}$ and a nonzero scalar $\boldsymbol{c}$. Prove that

$$\boldsymbol{(cA)^{-1} = \frac{1}{c} A^{-1}}$$

Hint:
Recall that

$$\boldsymbol{A^{-1} = B \iff AB = I},$$

where $\boldsymbol{I}$ is the identity matrix. (Note: of course in the hint, $\boldsymbol{A}$ and $\boldsymbol{B}$ are generic matrices, so the A here is different from the A in the statement to be proved.)

Chapter 2

Solving Linear Systems

2.1 Echelon Form of a Matrix

27) Reduce

$$A = \begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix}$$

to REF and then to RREF. (Note: number of rows (m) = number of columns (n).)

28) Reduce

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$$

to REF and then to RREF. (Note: number of rows $(m) >$ number of columns (n).)

29) Reduce

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{pmatrix}$$

to REF and then to RREF. (Note: number of rows $(m) <$ number of columns (n).)

30) Determine the reduced row echelon form of

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \quad \text{for } 0 < x < \tfrac{\pi}{2}.$$

Show all the steps in your argument!

2.2 Solving Linear Systems

31) Solve the system

$$\begin{cases} 2x \quad\quad - 3z = 1 \\ 4x + y - 2z = 2 \\ 3x + y - z = 3 \end{cases}$$

via (a) Gaussian Elimination, and (b) Gauss-Jordon Elimination. (Note: number of equations (m) = number of unknowns (n).)

32) Solve the system

$$\begin{cases} 3x_1 + 4x_2 = 1 \\ x_1 - 2x_2 = 2 \\ -x_1 + 5x_2 = 0 \end{cases}$$

via (a) Gaussian Elimination, and (b) Gauss-Jordon Elimination. (Note: number of equations (m) $>$ number of unknowns (n).)

33) Solve the system

$$\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - x_3 = 1 \end{cases}$$

via (a) Gaussian Elimination, and (b) Gauss-Jordon Elimination. (Note: number of equations (m) $<$ number of unknowns (n).)

34) Investigate for what values of $a \in \mathbb{R}$ the linear system

$$\begin{cases} x + y = 3, \\ x + (a^2 - 8)y = a, \end{cases}$$

has (i) no solution, (ii) an infinite number of solutions, and (iii) a unique solution.

Strategy: This is a harder problem, but just apply Gauss-Jordon Elimination to the associated augmented matrix and see what conditions have to be given to the number a for the three cases.

2.3 Finding A^{-1}

35) Find the inverse of

$$A = \begin{pmatrix} -3 & -3 & -4 \\ 0 & 1 & 1 \\ 4 & 3 & 4 \end{pmatrix}$$

(if it exists).

36) Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{pmatrix}$$

(if it exists).

37) Investigate for what values of $a \in \mathbb{R}$ the homogeneous linear system

$$\begin{cases} (a-1)x + 2y &= 0, \\ 2x + (a-1)y &= 0, \end{cases}$$

has a **non**trivial solution. Do this problem three different ways, using the contrapositive of items 1., 3., and 5. with item 2. in Theorem 6.

Chapter 3

Determinants

3.1 Definition

38) Find $|A|$ where

$$A = \begin{pmatrix} -2 & 3 & 2 \\ 1 & 2 & -1 \\ 4 & 2 & 18 \end{pmatrix}$$

3.2 Properties of Determinants

39) Let

$$A = \begin{pmatrix} -1/2 & 3 & 1/3 \\ 1/4 & 2 & -1/6 \\ 1 & 2 & 3 \end{pmatrix}$$

Use Theorem 9 (b) and Definition 29 in the Workbook to evaluate $|A|$.

40) Let

$$A = \begin{pmatrix} -2 & 3 & 2 \\ 1 & 2 & -1 \\ 4 & 2 & 18 \end{pmatrix}$$

Use row operations to reduce $|A|$ to upper triangular form (using Theorem 9) and then use Theorem 8(c) to evaluate this determinant.

41) Let

$$A = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

(a) Compute $|A|$ and $|A^{-1}|$.

(b) Make a conjecture about the determinant of the inverse of a matrix.

42) Let A and B be square matrices of order 3 such that $|A| = 4$ and $|B| = 5$.
(a) Find $|AB|$
(b) Find $|2A|$
(c) Are A and B singular or nonsingular? Explain.
(d) If A and B are nonsingular find $|A^{-1}|$ and $|B^{-1}|$
(e) Find $|(AB)^T|$

43)

$$\text{If} \qquad |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 2,$$

$$\text{find} \qquad |B| = \begin{vmatrix} (3a_1 - 6a_3) & a_2 & a_3 \\ (3b_1 - 6b_3) & b_2 & b_3 \\ (c_1 - 2c_3) & \frac{1}{3}c_2 & \frac{1}{3}c_3 \end{vmatrix}.$$

Hint: apply row and column operations (see Theorem 9). This is a harder problem.

3.3 Cofactor Expansions

44) Let

$$A = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 2 & 5 \\ 6 & -1 & 7 \end{pmatrix}$$

Find $|M_{23}|$ (the minor of the entry a_{23}) and A_{23} (the cofactor of a_{23}).

45) Let A be the same as in Exercise **44**). Use a cofactor expansion along row 2 to evaluate $|A|$.

46) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Use a cofactor expansion (along row 1 or column 2) to prove the usual formula for the determinant of A, namely

$$|A| = ad - bc.$$

47) Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Use a cofactor expansion to verify

$$|A| = |A^T|.$$

3.4 Inverse of a Matrix (via the Adjoint)

48) Find the adjoint of

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

.

49) Using $\mathrm{adj}(A)$, where A is the matrix in Exercise **48**), find A^{-1}.

50) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a nonsingular matrix. Use the formula

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$$

to verify the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Chapter 4

Real Vector Spaces

4.1 Vectors in the Plane and in 3-Space (generalized to n-Space)

51) Consider the vectors $u = (3, -4)$, $v = (9, 1)$, and $w = (-39, 0)$.

(a) Use directed line segments to represent $\boldsymbol{u}$ and $\boldsymbol{v}$.

(b) Find $\boldsymbol{u}+\boldsymbol{v}$ and represent graphically.

(c) Find $\boldsymbol{2v}-\boldsymbol{u}$ and represent graphically.

(d) Write the vector $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$.

52) (a) Draw the vectors

$$\vec{OP} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \vec{OR} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \vec{OQ} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

(b) How do we get to $\boldsymbol{R}$ using the vectors $\vec{OP}$ and $\vec{OQ}$?

53) Let $\underline{\boldsymbol{x}} = (-1,-2,-2)$, $\underline{\boldsymbol{u}} = (0,1,4)$, $\underline{\boldsymbol{v}} = (-1,1,2)$ and $\underline{\boldsymbol{w}} = (3,1,2)$ be vectors in $\mathbb{R}^3$. Write (if possible) $\underline{\boldsymbol{x}}$ as a linear combination of the vectors $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{w}}$. (In other words, write $\underline{\boldsymbol{x}}$ as a sum of constants times $\underline{\boldsymbol{u}}$, $\underline{\boldsymbol{v}}$ and $\underline{\boldsymbol{w}}$.)

4.2 Vector Spaces

54) Let

$$\boldsymbol{V} = \begin{array}{l}\text{the set of integers with the standard operations} \\ \text{of (vector) addition and (scalar) multiplication.}\end{array}$$

Show that $\boldsymbol{V}$ is NOT a vector space.

55)

$$\boldsymbol{V} = \text{ the set of all second-degree polynomials.}$$

Show that $\boldsymbol{V}$ is NOT a vector space.

56) Let

$$\boldsymbol{V} = \{(x, x-2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

Prove that $\boldsymbol{V}$ is NOT a vector space.

57) Verify that the standard examples in our notes (Examples 10 - 13) are vector spaces by checking that all the axioms of Definition 37 hold.

4.3 Subspaces

58) Show that

$$\boldsymbol{W} = \{(x, 0, z) \in \mathbb{R}^3 \mid x, z \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3$ with the standard operations.

59) Which of these two sets is a subspace of $\mathbb{R}^2$?
a) The set of points on the line $x + 2y = 0$.
b) The set of points on the line $x + 2y = 1$.

60) Draw diagrams to illustrate the results we found for (vector) addition in Exercise **59**). For part (a) choose $w_1 = (1, -\frac{1}{2})$, $w_2 = (3, -\frac{3}{2})$, and for part (b) choose $w_1 = (1, 0)$ and $w_2 = (-1, 1)$.

61) Prove that

$$W = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0\}$$

with the standard operations is *not* a subspace of $\mathbb{R}^2$.

62) Prove that the set

$$2. \quad W = \left\{ \begin{pmatrix} x \\ 3x \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$$

with the standard operations is a subspace of $\mathbb{R}^2$.

63) Is the set of all vectors of the form

$$\begin{pmatrix} a \\ b \\ a + 2b \end{pmatrix}$$

where $a, b \in \mathbb{R}$, a subspace of $\mathbb{R}^3$?

64) Let W be the set of all nonsingular 2-by-2 matrices. Prove that W is NOT a subspace of M_{22} (the vector space of all 2-by-2 matrices with the usual operations)?
(**Hint**: all you need is a single counter-example for the closure property of vector addition, i.e., choose two nonsingular matrices whose sum is singular. Recall that we can tell easily if a matrix is singular or not from Theorem 11.)

65) Prove the corresponding result to Exercise **64**) for *singular* matrices.

66) Give a geometrical interpretation of why the set of vectors on the unit circle (centred at the origin) is not a subspace of $\mathbb{R}^2$.

4.4 Span

67) Consider the following set of vector in $\mathbb{R}^3$:

$$S = \{\underbrace{(1, 3, 1)}_{v_1}, \quad \underbrace{(0, 1, 2)}_{v_2}, \quad \underbrace{(1, 0, -5)}_{v_3}\}.$$

Verify that the vector v_1 can be written as a linear combination of v_2 and v_3, in the form $v_1 = 3v_2 + v_3$.

68) Consider the following set of vectors in M_{22}:

$$S = \left\{ \underbrace{\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}}_{v_2}, \quad \underbrace{\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}}_{v_3}, \quad \underbrace{\begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix}}_{v_4} \right\}.$$

Verify that the vector $\boldsymbol{v}_1$ can be written as a linear combination of $\boldsymbol{v}_2$, $\boldsymbol{v}_3$, and $\boldsymbol{v}_4$, in the form $\boldsymbol{v}_1 = \boldsymbol{v}_2 + 2\boldsymbol{v}_3 - \boldsymbol{v}_4$.

69) If possible, write the vector $\boldsymbol{w} = (\mathbf{1}, \mathbf{1}, \mathbf{1})$ as a linear combination of the vectors in the set $\boldsymbol{S}$:

$$\boldsymbol{S} = \{\underbrace{(\mathbf{1}, \mathbf{2}, \mathbf{3})}_{\boldsymbol{v}_1}, \quad \underbrace{(\mathbf{0}, \mathbf{1}, \mathbf{2})}_{\boldsymbol{v}_2}, \quad \underbrace{(-\mathbf{1}, \mathbf{0}, \mathbf{1})}_{\boldsymbol{v}_3}\}.$$

If there is more than one solution find one particular solution.

70) Try and write the vector

$$\boldsymbol{w} := (\mathbf{1}, -\mathbf{2}, \mathbf{2})$$

as a linear combination of the vectors in the set

$$\boldsymbol{S} = \{\underbrace{(\mathbf{1}, \mathbf{2}, \mathbf{3})}_{\boldsymbol{v}_1}, \quad \underbrace{(\mathbf{0}, \mathbf{1}, \mathbf{2})}_{\boldsymbol{v}_2}, \quad \underbrace{(-\mathbf{1}, \mathbf{0}, \mathbf{1})}_{\boldsymbol{v}_3}\}.$$

Why can this not be done? (see Exercise **69**)).

71) Consider the following vectors in $\mathbb{R}^3$:

$$\underbrace{\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{pmatrix}}_{\boldsymbol{v}_1}, \quad \underbrace{\begin{pmatrix}-\mathbf{1}\\\mathbf{0}\\\mathbf{5}\end{pmatrix}}_{\boldsymbol{v}_2}, \quad \underbrace{\begin{pmatrix}\mathbf{6}\\\mathbf{7}\\\mathbf{2}\end{pmatrix}}_{\boldsymbol{v}_3}.$$

Using Theorem 1 find an expression for the span (a noun !) of these vectors involving a matrix-vector product.

72) By considering the span (a noun) of the following vectors in $\mathbb{R}^3$

$$\underbrace{\begin{pmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{pmatrix}}_{\boldsymbol{v}_1}, \quad \underbrace{\begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{pmatrix}}_{\boldsymbol{v}_2}, \quad \underbrace{\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{pmatrix}}_{\boldsymbol{v}_3},$$

show that these vectors span (a verb!) $\mathbb{R}^3$. (If you haven't guessed already, this question is more about terminology and notation than actual math - the math result is actually quite trivial.)

73) Find sets of vectors that span
(a) $\mathbb{R}^2$
(b) $\boldsymbol{P}_2$
(c) $\boldsymbol{M}_{22}$

74) Do the vectors

$$\begin{pmatrix}\mathbf{1}\\\mathbf{2}\\\mathbf{3}\end{pmatrix}, \quad \begin{pmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{2}\end{pmatrix}, \quad \begin{pmatrix}-\mathbf{2}\\\mathbf{0}\\\mathbf{1}\end{pmatrix}$$

span $\mathbb{R}^3$?

4.5 Linear Independence

75) Determine whether the following set of vectors in $\mathbb{R}^3$ is linearly independent or linearly dependent:

$$S = \left\{ \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}_{v_2}, \quad \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{v_2} \right\}.$$

76) Determine whether the following set of vectors in $\mathbb{R}^2$ is linearly independent or linearly dependent:

$$S = \left\{ \underbrace{\begin{pmatrix} 2 \\ 4 \end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix} -3 \\ -6 \end{pmatrix}}_{v_2} \right.$$

77) Determine whether the set of vectors given below is linearly independent or linearly dependent:

$$S = \{\underbrace{1 + x - 2x^2}_{v_1}, \underbrace{2 + 5x - x^2}_{v_2}, \underbrace{x + x^2}_{v_3}\} \subset P_2.$$

78) Determine whether the set of vectors given below is linearly independent or linearly dependent:

$$S = \left\{ \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}}_{v_2}, \quad \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}}_{v_3} \right\} \subset M_{22}$$

4.6 Basis and Dimension

Exercise 62 in the Workbook is a key example showing how to verify that a given set of vectors is a basis for a vector space. Once we know the dimension of the vector space concerned, according to Theorem 19, if the number of vectors is equal to the dimension then all we need to check if we have a basis is that either the set of vectors is linear independence or, the vectors span the space (not both). If the number of vectors is <u>not</u> equal to the dimension of the space then the vectors can't be a basis!

We initially give two simple exercises:

79) Prove that

$$\left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^2$.

80) Show that the set

$$S = \{2, x - 1, x^2 + 1\}$$

is a basis for P_2.

81) Show that the set

$$\boldsymbol{S} = \left\{ \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{-1} & \mathbf{3} \\ \mathbf{2} & \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{1} & \mathbf{-1} \end{pmatrix} \right\}$$

is a basis for $\boldsymbol{M}_{22}$.

82) Let

$$\boldsymbol{S} = \left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix}, \begin{pmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{-4} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{-3} \\ \mathbf{7} \end{pmatrix}, \begin{pmatrix} \mathbf{-4} \\ \mathbf{1} \\ \mathbf{6} \end{pmatrix} \right\}$$

(a) Without doing any calculations why do we know that $\boldsymbol{S}$ is not a basis for $\mathbb{R}^{\mathbf{3}}$?
(b) Show from first principles that $\boldsymbol{S}$ is not a basis for $\mathbb{R}^{\mathbf{3}}$.

Hint: for (b) consider linear independence, but do not apply any row operations to the resulting matrix equation.

4.7 Homogeneous Systems

83) Consider the homogeneous linear system $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\mathbf{0}}$ where

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-2} & \mathbf{1} \\ \mathbf{3} & \mathbf{6} & \mathbf{-5} & \mathbf{4} \\ \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{3} \end{pmatrix}.$$

Find a basis for the null space of $\boldsymbol{A}$ (i.e., $\boldsymbol{N}(\boldsymbol{A})$). What is the dimension of $\boldsymbol{N}(\boldsymbol{A})$?

4.8 Rank of a Matrix

84) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

(a) Find expressions for the row space and the column space of $\boldsymbol{A}$.
(b) What are the dimensions of the row space and the column space of $\boldsymbol{A}$?

85) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} & \mathbf{6} & \mathbf{7} & \mathbf{14} \end{pmatrix}$$

(a) Find the dimension of the row space of $\boldsymbol{A}$ (i.e., the row rank of $\boldsymbol{A}$).

(b) Find the dimension of the column space of $\boldsymbol{A}$ (i.e., the column rank of $\boldsymbol{A}$).

86) Let

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix},$$

for arbitrary $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}$. The matrix

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{a}+\boldsymbol{c} & \boldsymbol{b}+\boldsymbol{d} \\ \boldsymbol{c} & \boldsymbol{d} \end{pmatrix},$$

is obtained by applying the row operation $\boldsymbol{r_1} + \boldsymbol{r_2} \to \boldsymbol{r_1}$ to $\boldsymbol{A}$. Show that

row space of $\boldsymbol{A}$ = row space of $\boldsymbol{B}$.

87) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{2} & \mathbf{-5} & \mathbf{1} \\ \mathbf{1} & \mathbf{-4} & \mathbf{-7} \end{pmatrix}$$

Find a basis for the row space of $\boldsymbol{A}$.

88) Find a basis for the subspace $\boldsymbol{V}$ of $\mathbb{R}^3$ spanned by $\boldsymbol{S} = \{\underline{\boldsymbol{u}}_1, \underline{\boldsymbol{u}}_2, \underline{\boldsymbol{u}}_3\}$, where

$$\underline{\boldsymbol{u}}_1 = \begin{pmatrix} \mathbf{1} \\ \mathbf{-2} \\ \mathbf{3} \end{pmatrix}, \quad \underline{\boldsymbol{u}}_2 = \begin{pmatrix} \mathbf{2} \\ \mathbf{-5} \\ \mathbf{1} \end{pmatrix}, \quad \underline{\boldsymbol{u}}_3 = \begin{pmatrix} \mathbf{1} \\ \mathbf{-4} \\ \mathbf{-7} \end{pmatrix} \quad (\boldsymbol{V} = \textbf{span}\,\boldsymbol{S}),$$

(Hint: see Exercise 68 in the notes.)

89) Find the rank of the following matrices:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{3} & \mathbf{-2} \\ \mathbf{3} & \mathbf{5} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-3} \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} \end{pmatrix}.$$

90) Consider the linear system $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ where

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{3} & \mathbf{6} & \mathbf{1} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-2} & \mathbf{-5} \end{pmatrix}, \quad \underline{\boldsymbol{b}} = \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{-1} \end{pmatrix}.$$

Consider the solvability of this system using the concept of rank.

Chapter 5

Inner Product Spaces

5.1 Length and Direction in $\mathbb{R}^n$ $(n \geq 2)$

91) Consider the following vectors in $\mathbb{R}^4$:

$$\underline{u} = (1, -1, 0, 4), \quad \underline{v} = (3, 2, 1, 0).$$

(a) Find the norms of $\underline{u}$ and $\underline{v}$
(b) Find the distance between $\underline{u}$ and $\underline{v}$
(c) Find the standard inner product of $\underline{u}$ and $\underline{v}$
(d) Find $\cos(\theta)$ where θ is the angle between $\underline{u}$ and $\underline{v}$
(e) Find the normalized vector $\widehat{\underline{u}}$ corresponding to $\underline{u}$
(f) What value do we need to change the last component of $\underline{v}$ to in order for $\underline{u}$ and $\underline{v}$ to be orthogonal?

5.2 Inner Product Spaces

92) Let u and v belong to an inner product space V. Given that $\|u\| = 2$, $\|v\| = 3$ and $(u, v) = 4$ calculate $\|u + 2v\|^2$.

93) Consider the standard inner product $(u, v) := u \cdot v$, where $u, v \in \mathbb{R}^3$.
(a) Show that $(u + v) \cdot (u - v) = \|u\|^2 - \|v\|^2$.
(b) Verify the result in (a) for

$$u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

94) Let $V = C[0, 1]$* with the inner product

$$(f, g) := \int_0^1 f(x)g(x)\, dx,$$

*Set of continuous function on the interval $[0, 1]$.

and $f(t) = 1$ and $g(x) = x$.
(a) Evaluate (f, g)
(b) Without performing any calculations explain why $(g, g) > 0$
(c) Define an induced norm for the inner product space V and hence calculate $\|f\|$ and $\|g\|$
(d) Verify the Cauchy-Schwartz inequality $|(f, g)| \leq \|f\|\|g\|$

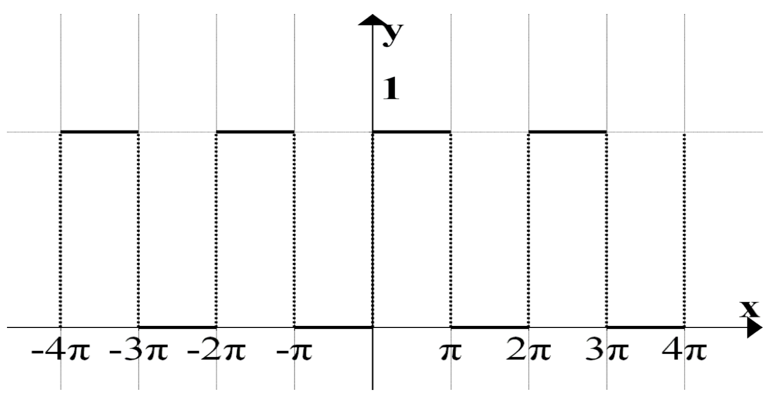
95) Complete Exercise 76 (page 243) in the Workbook by deriving formulae for a_0 and a_m corresponding to Definition 57 for Fourier Series.
(Hint: to derive the formulae for a_0 and a_m, multiply $(*)$ (page 240 of the Workbook) by 1 and $\cos(mx)$ respectively, integrate from $-\pi$ to π, and then use the Calculus results on page 241 of the Workbook.)

96) (a) Using Definition 57 in your Workbook find the Fourier coefficients and Fourier series of the 'square wave' function

$$f(x) = \begin{cases} 0 & \text{if } -\pi \leq x < 0 \\ 1 & \text{if } 0 \leq x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x),$$

i.e. f is periodic with period 2π and has the shape shown below (only shown on $[-4\pi, 4\pi]$):

(b) Graph several terms of your Fourier series against the square wave to see how well successive numbers of terms approximates the given function.



97) Let u and v be vectors in an inner product space V. Prove that

$$\|u + v\| \leq \|u\| + \|v\|$$

with equality if and only if u and v are orthogonal.

(**Hint**: start by expanding $\|u+v\|^2$ with the aid of the rules in Definition 55, use the definition of an induced norm (see page 239 in the Workbook), and then use the Cauchy-Schwartz inequality.)

Chapter 6

Eigenvalues and Eigenvectors

6.1 Eigenvalues and Eigenvectors

98) Prove Theorem 35 in the Workbook which states: if $\boldsymbol{A}$ is a square $\boldsymbol{n} \times \boldsymbol{n}$ matrix and $\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{A}$, then the eigenspace of $\boldsymbol{\lambda}$ is a subspace of $\mathbb{R}^{\boldsymbol{n}}$.

99) Find the eigenvalues and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{-12} \\ \mathbf{1} & \mathbf{-5} \end{pmatrix}.$$

6.2 Diagonalization and Similar Matrices

100) Diagonalize the matrix $\boldsymbol{A}$ from Exercise **99**). Justify this process.

101) Use diagonalization of the matrix $\boldsymbol{A}$ in Exercise **99**) to calculate $\boldsymbol{A}^{\mathbf{10}}$.