Homework Solutions for Math*1160 (W18).

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STOP!

Before looking at the answers the best study strategy is to:

1. First read your lecture notes for the relevant section.
2. Then attempt the questions *without* first looking at the answers.
3. Finally, if you are stuck, look at the general approach in the relevant answer and try again.

Remember, struggle is usually necessary for effective learning.

Note
We use the following abreviations:

d.p. = decimal places, e.g. π to 3 d.p. is 3.141
s.f. = significant figures, e.g. π to 3 s.f. is 3.14

Contents

Chapter 1

Linear Equations & Matrices

1.1 Formulation of systems of linear equations

1) A bartender makes a cocktail with whisky and rum that contains $\mathbf{10}$ ml of alcohol and $\mathbf{40}$ ml of water. The rum has $\mathbf{14\%}$ alcohol by volume (and hence $\mathbf{86\%}$ water by volume), while the whisky has $\mathbf{43\%}$ alcohol by volume (and hence $\mathbf{57\%}$ water by volume). Write $\mathbf{2}$ equations in $\mathbf{2}$ unknowns for the amount of whisky (x) and the amount of rum (y) needed. (Hint: see the similar problem in your lecture notes.)

We formulate two equation for the total alcohol and the total water content of the drink:

$$\begin{aligned} \text{Alcohol:} \quad & \frac{43}{100}x + \frac{14}{100}y = 10 \\ \text{Water:} \quad & \frac{57}{100}x + \frac{86}{100}y = 40 \end{aligned}$$

I.e.,

$$\begin{cases} 0.43x + 0.14y = 10 \\ 0.57x + 0.86y = 40 \end{cases}$$

($x = 10.3$ ml (1d.p.), $y = 39.7$ ml (1 d.p.))

1.2 Systems of Linear Equations

2) Apply back-substitution to solve the following linear system:

$$\begin{cases} 2x_1 - x_2 + 3x_3 - 2x_4 = 1 \\ \quad\quad\;\; x_2 - 2x_3 + 3x_4 = 2 \\ \quad\quad\quad\quad\;\; 4x_3 + 3x_4 = 3 \\ \quad\quad\quad\quad\quad\quad\;\; 4x_4 = 4 \end{cases}$$

Solution:
Starting from the last equation:

$$\begin{aligned} 4x_4 = 4 &\implies x_4 = 1. \\ 4x_3 + 3(1) = 3 &\implies x_3 = 0. \\ x_2 - 2(0) + 3(1) = 2 &\implies x_2 = -1. \\ 2x_1 - (-1) + 3(0) - 2(1) = 1 &\implies x_1 = 1. \end{aligned}$$

Thus the solution is $(1, -1, 0, 1)$

3) Find the unique solution of the following linear system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ 3x_1 - x_2 - 3x_3 = -1 \\ 2x_1 + 3x_2 + x_3 = 4 \end{cases}$$

Solution:
As in the class examples we denote the ith equation by E_i. Do the following:

replace E_2 with $E_2 - 3E_1$, then replace E_3 with $E_3 - 2E_1$
yielding:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ \quad - 7x_2 - 6x_3 = -10 \\ \quad - x_2 - x_3 = -2 \end{cases}$$

Then:
replace E_3 with $E_3 - (1/7)E_2$, yielding:

$$\begin{cases} x_1 + 2x_2 + x_3 = 3 \\ \quad - 7x_2 - 6x_3 = -10 \\ \quad\quad - \frac{1}{7}x_3 = -\frac{4}{7} \end{cases}$$

Back-substitution yields the solution
$x_3 = 4$, $x_2 = -2$, $x_1 = 3$.

4) By Attempting to solve the following system show why no solution exists:

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 9 \end{cases}$$

Solution:
Replace E_2 with $E_2 - 2E_1$ yielding

$$\begin{cases} 2x + 3y = 6 \\ 0 = -3 \end{cases}$$

The 2nd equation is impossible, thus the system is inconsistent.

5) Find the infinite solution set corresponding to the following linear system:

$$\begin{cases} 2x + 3y = 6 \\ 4x + 6y = 12 \end{cases}$$

Solution:
Replace E_2 with $E_2 - 2E_1$ yielding

$$\begin{cases} 2x + 3y & = 6 \\ 0 + 0 & = 0 \end{cases}$$

The 2nd equation is redundant (yields no information). Thus from the first equation we have

$$2x = 6 - 3y$$
$$\text{or} \quad x = 3 - \frac{3}{2}y,$$

where y is free to be chosen (a 'free variable'). E.g., if $y = 2/3$ then $x = 3 - 1 = 2$. In general, if $y = \alpha \in \mathbb{R}$ then $x = 3 - (3/2)\alpha$. Thus there are an infinite number of solutions belonging to the set:

$$\left\{ \left(3 - \frac{3}{2}\alpha, \alpha \right) \mid \alpha \in \mathbb{R} \right\}.$$

6) (a) Without doing any row operations, explain why the following system of linear equations is consistent:

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 = 0 \\ -5x_1 + 6x_2 - 17x_3 = 0 \\ 7x_1 - 4x_2 + 3x_3 = 0 \end{cases}$$

(b) Without doing any row operations, explain why the following system of linear equations has an infinite number of solutions?

$$\begin{cases} 2x_1 + 3x_2 + 5x_3 + 2x_4 = 0 \\ -5x_1 + 6x_2 - 17x_3 - 3x_4 = 0 \\ 7x_1 - 4x_2 + 3x_3 + 13x_4 = 0 \end{cases}$$

Solution:
(a) Homogeneous linear systems of the form $A\underline{x} = \underline{0}$ always possess the zero solution, i.e. here it is obvious that a solution is $x_1 = x_2 = x_3 = 0$.

(b) Homogeneous linear systems either have only the unique zero ('trivial') solution, or have an infinite number of solutions. Here we have three equations in four unknowns. This means after applying row operations to get the associated augmented matrix in row echelon form we will have one free variable, leading to an infinite number of solutions.

Note:
For those of you who want more details we can use techniques from Section 2.2* to solve this system.

*Of course if you haven't covered that section yet you can still apply the 'method of elimination' (page 21) to get the same answer, but it's going to be a lengthy calculation.

I reduced the coefficient matrix to RREF using elementary row operations to get[†]

$$\begin{pmatrix} 1 & 0 & 0 & 533/261 \\ 0 & 1 & 0 & 2/261 \\ 0 & 0 & 1 & -110/261 \end{pmatrix}$$

So the associated linear system is

$$\begin{cases} x_1 \quad\quad + \frac{533}{261}x_4 = 0 \\ \quad x_2 \quad + \frac{2}{261}x_4 = 0 \\ \quad\quad x_3 - \frac{110}{261}x_4 = 0 \end{cases}$$

So we see that x_4 is a free variable, leading to the infinite solution set (exercise)

$$\left\{(-\frac{533}{261}\alpha, -\frac{2}{261}\alpha, \frac{110}{261}\alpha, \alpha) \mid \alpha \in \mathbb{R}\right\}.$$

E.g., taking $\alpha = 261$ yields the particular solution $(-533, -2, 110, 261)$.

1.3 Matrices

Review your notes and the examples there. Then make your own examples up and check them in Matlab. Alternatively, you can Google many examples for the simple operations of matrix addition, subtraction, scalar multiplication and transpose. Also, the *many* standard texts in Linear Algebra that we have in the library will also have a multitude of examples covering this basic material.

7) Calculate $(A + B^T - 2C)^T$ where

$$A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \\ 2 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 1 \\ -1 & 3 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -3 \end{pmatrix}.$$

Solution:
First calculate

$$\begin{aligned} A + B^T - 2C &= \begin{pmatrix} -1 & 2 \\ 1 & 0 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 1 \\ -1 & 3 & 4 \end{pmatrix}^T - 2\begin{pmatrix} 2 & -1 \\ 1 & 2 \\ 3 & -3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 2 \\ 1 & 0 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 4 & -1 \\ -2 & 3 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 2 & 4 \\ 6 & -6 \end{pmatrix} \\ &= \begin{pmatrix} -1+4-4 & 2-1+2 \\ 1-2-2 & 0+3-4 \\ 2+1-6 & 5+4+6 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 3 \\ -3 & -1 \\ -3 & 15 \end{pmatrix}. \end{aligned}$$

Thus

$$(A + B^T - 2C)^T = \begin{pmatrix} -1 & 3 \\ -3 & -1 \\ -3 & 15 \end{pmatrix}^T = \begin{pmatrix} -1 & -3 & -3 \\ 3 & -1 & 15 \end{pmatrix}.$$

[†] I used the 'rref' command in Matlab and typed 'format rat' beforehand.

1.4 Matrix Multiplication

See the examples in your notes, in standard textbooks, and on the web. This is another type of problem where you can readily construct your own examples and easily check them using Matlab. Keep in mind the comments made regarding the dimensions of the matrices concerned on page 38 of your Workbook.

8) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Find the matrix product AB.

Solution:

$$\begin{aligned} AB &= \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix}}_{2\times 3} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 4 \end{pmatrix}}_{3\times 2} \\ &= \begin{pmatrix} 1(1)+2(0)+3(1) & 1(2)+2(-1)+3(4) \\ 0(1)+(-1)(0)+2(1) & 0(2)+(-1)(-1)+2(4) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 4 & 12 \\ 2 & 9 \end{pmatrix}}_{2\times 2} \end{aligned}$$

9) Let

$$U = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}.$$

Verify that (a) $V^T U = U^T V$ and (b) $(UV^T)^T = VU^T$.

Solution:

(a)

$$V^T U = \underbrace{\begin{pmatrix} -2 & 1 & 4 \end{pmatrix}}_{1\times 3} \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{3\times 1} = -2(1)+1(2)+4(3) = 12.$$

And

$$U^T V = \underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{1\times 3} \underbrace{\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}}_{3\times 1} = 1(-2)+2(1)+3(4) = 12 = V^T U \quad \checkmark.$$

(b)

$$UV^T = \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{3\times 1} \underbrace{\begin{pmatrix} -2 & 1 & 4 \end{pmatrix}}_{1\times 3} = \begin{pmatrix} 1(-2) & 1(1) & 1(4) \\ 2(-2) & 2(1) & 2(4) \\ 3(-2) & 3(1) & 3(4) \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & 1 & 4 \\ -4 & 2 & 8 \\ -6 & 3 & 12 \end{pmatrix}}_{3\times 3}.$$

Thus

$$(UV^T)^T = \begin{pmatrix} -2 & 1 & 4 \\ -4 & 2 & 8 \\ -6 & 3 & 12 \end{pmatrix}^T = \begin{pmatrix} -2 & -4 & -6 \\ 1 & 2 & 3 \\ 4 & 8 & 12 \end{pmatrix}.$$

And

$$VU^T = \underbrace{\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}}_{3\times 1} \underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{1\times 3} = \begin{pmatrix} -2(1) & -2(2) & -2(3) \\ 1(1) & 1(2) & 1(3) \\ 4(1) & 4(2) & 4(3) \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & -4 & -6 \\ 1 & 2 & 3 \\ 4 & 8 & 12 \end{pmatrix}}_{3\times 3} = (UV^T)^T \quad \checkmark.$$

10) Find a number k such that $A\underline{x} = k\underline{x}$, where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \underline{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solution:

$$\begin{aligned} A\underline{x} &= k\underline{x} \\ \implies \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= k \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \implies \begin{pmatrix} 3 \\ 3 \end{pmatrix} &= \begin{pmatrix} k \\ k \end{pmatrix}, \end{aligned}$$

i.e., $k = 3$.

11) Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad x = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}.$$

Write the matrix product Ax as a sum of the columns of A multiplied by the entries in x.
(Hint: use Theorem 1).

Solution:

$$\begin{aligned} Ax = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} &= \begin{pmatrix} 1(7) + 2(8) + 3(9) \\ 4(7) + 5(8) + 6(9) \end{pmatrix} \\ &= 7 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 8 \begin{pmatrix} 2 \\ 5 \end{pmatrix} + 9 \begin{pmatrix} 3 \\ 6 \end{pmatrix}. \end{aligned}$$

1.5 Algebraic Properties of Matrix Operations

Recall that the 'Algebraic Properties of Matrix Operations' that we refer to are listed in Theorem 2 of your notes.

12) Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Calculate $A + B$ and $B + A$. Is matrix addition commutative? (i.e., is it generally true that $A + B = B + A$?)

(b) Calculate AB and BA. Is matrix multiplication commutative? (i.e., is it generally true that $AB = BA$?)

Solution:
(a) We find that

$$A + B = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} = B + A.$$

Yes, this is generally true.

(b) We find that

$$AB = \begin{pmatrix} 2 & 5 \\ 4 & 11 \end{pmatrix},$$

but

$$BA = \begin{pmatrix} 3 & 4 \\ 7 & 10 \end{pmatrix} \neq AB,$$

thus, no, it is NOT generally true that $AB = BA$.

Note:
See properties (1) and the comment on page 61 of our Workbook.

13) Verify the algebraic property $(AB)^T = B^T A^T$ (Property 11) where

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

Solution:
Now

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1(3) + 3(2) + 2(1) & 1(-1) + 3(4) + 2(2) \\ 2(3) + 1(2) + (-3)(1) & 2(-1) + 1(4) + (-3)(2) \end{pmatrix} \\ &= \begin{pmatrix} 11 & 15 \\ 5 & -4 \end{pmatrix} \end{aligned}$$

Thus

$$(AB)^T = \begin{pmatrix} 11 & 15 \\ 5 & -4 \end{pmatrix}^T = \begin{pmatrix} 11 & 5 \\ 15 & -4 \end{pmatrix}.$$

But

$$
\begin{aligned}
B^T A^T &= \begin{pmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{pmatrix}^T \begin{pmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{pmatrix}^T \\
&= \begin{pmatrix} 3 & 2 & 1 \\ -1 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -3 \end{pmatrix} \\
&= \begin{pmatrix} 3(1) + 2(3) + 1(2) & 3(2) + 2(1) + 1(-3) \\ -1(1) + 4(3) + 2(2) & -1(2) + 4(1) + 2(-3) \end{pmatrix} \\
&= \begin{pmatrix} 11 & 5 \\ 15 & -4 \end{pmatrix},
\end{aligned}
$$

which is the same as before ✓.

14) Determine the values of the number k such that $(kA)^T(kA) = 1$, where

$$
A = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}.
$$

Solution:
We use the algebraic properties of matrices (see your notes):

$$
\begin{aligned}
(kA)^T(kA) &= (kA^T)(kA) \quad \text{(Property 12)} \\
&= k^2(A^T A) \quad \text{(Property 5 \& 8)} \\
&= k^2 \begin{pmatrix} -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \\
&= k^2[(-2)^2 + 1^2 + (-1)^2] \\
&= k^2 6 \\
&= 1.
\end{aligned}
$$

I.e., $6k^2 = 1 \implies k^2 = 1/6 \implies k = \pm 1/\sqrt{6}$.

15) Using the algebraic properties of matrices find an expression for $(A - B)^T$.

Solution:

$$
\begin{aligned}
(A - B)^T &= (A + (-1)B)^T \quad \text{(Definition 12)} \\
&= A^T + ((-1)B)^T \quad \text{(Property 10)} \\
&= A^T + (-1)B^T \quad \text{(Property 12)} \\
&= A^T - B^T \quad \text{(Definition 12)}.
\end{aligned}
$$

16) Using the algebraic properties of matrices find an expression for $(A - B)^2$.

Solution:

$$\begin{aligned}(\boldsymbol{A}-\boldsymbol{B})^2 &= (\boldsymbol{A}-\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B}) \quad \text{(Definition 16)}\\ &= (\boldsymbol{A}+(-1)\boldsymbol{B})(\boldsymbol{A}+(-1)\boldsymbol{B}) \quad \text{(Definition 12)}\\ &= \boldsymbol{A}(\boldsymbol{A}+(-1)\boldsymbol{B})+(-1)\boldsymbol{B}(\boldsymbol{A}+(-1)\boldsymbol{B}) \quad \text{(Property 4)}\\ &= \boldsymbol{A}^2+(-1)\boldsymbol{AB}+(-1)\boldsymbol{BA}+(-1)^2\boldsymbol{B}^2 \quad \text{(Property 4)}\\ &= \boldsymbol{A}^2-\boldsymbol{AB}-\boldsymbol{BA}+\boldsymbol{B}^2 \quad \text{(Definition 12)}.\end{aligned}$$

17) Prove that if $\underline{x_1}$ and $\underline{x_2}$ are solutions of the linear system $\boldsymbol{A}\underline{x}=\underline{b}$, then $\underline{x_1}-\underline{x_2}$ is a solution of the corresponding homogeneous system $\boldsymbol{A}\underline{x}=\underline{0}$.

Solution:
We are told that

$$\boldsymbol{A}\underline{x_1}=\underline{b} \quad \text{and} \quad \boldsymbol{A}\underline{x_2}=\underline{b} \qquad (*)$$

So we have

$$\begin{aligned}\boldsymbol{A}(\underline{x_1}-\underline{x_2}) &= \boldsymbol{A}\underline{x_1}-\boldsymbol{A}\underline{x_2} \quad \text{(Property 4)}\\ &= \underline{b}-\underline{b} \quad \text{(using } (*))\\ &= \underline{0} \quad \checkmark.\end{aligned}$$

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18) Prove for an arbitrary matrix $\boldsymbol{A}$ that

$$(\boldsymbol{A}^T)^2=(\boldsymbol{A}^2)^T.$$

(Hint: use Theorem 2, part (11).)

Solution:
Using the hint with $\boldsymbol{A}=\boldsymbol{B}$ yields

$$(\boldsymbol{AA})^T=\boldsymbol{A}^T\boldsymbol{A}^T \quad \text{i.e.} \quad (\boldsymbol{A}^2)^T=(\boldsymbol{A}^T)^2,$$

as required.

19) Find examples to prove the following statements, where $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are $\mathbf{2}$-by-$\mathbf{2}$ matrices:
(a) $\boldsymbol{AB}=\boldsymbol{AC}$ does not imply $\boldsymbol{B}=\boldsymbol{C}$.
(b) $\boldsymbol{AB}=\boldsymbol{O}$ (where $\boldsymbol{O}$ is the 'zero matrix') does not imply $\boldsymbol{A}=\boldsymbol{O}$ or $\boldsymbol{B}=\boldsymbol{O}$.
(See page 53 of your Workbook.)

Solution:
(a) Choose $\boldsymbol{A}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}2&1\\3&3\end{pmatrix}$, and $\boldsymbol{C}=\begin{pmatrix}3&3\\2&1\end{pmatrix}$, and observe that $\boldsymbol{AB}=\boldsymbol{AC}=\begin{pmatrix}5&4\\5&4\end{pmatrix}$.

(b) Choose $\boldsymbol{A}=\begin{pmatrix}-1&1\\-1&1\end{pmatrix}$, $\boldsymbol{B}=\begin{pmatrix}1&1\\1&1\end{pmatrix}$, and observe that $\boldsymbol{AB}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$.

Note:
Now you may be wondering how I got the answers I did. Well apart from 'guessing' one can apply some

results from the next section (Special Types of Matrices) to help us[‡]. In case (a) we know that A must be singular (no inverse) because if it was nonsingular, then $AB = AC$ implies $A^{-1}(AB) = A^{-1}(AC)$ implies $(A^{-1}A)B = (A^{-1}A)C$ implies $IB = IC$ or $B = C$, which contradicts what we are trying to prove. You can easily check in our example above that $|A| = 0$, i.e. A is singular. A similar argument applies to (b). For example, if A is nonsingular then $AB = O$ implies $A^{-1}(AB) = A^{-1}O$ implies $(A^{-1}A)B = O$ implies $IB = O$ i.e. $B = O$, which contradicts what we are trying to prove. Thus A must be singular, which is easily checked for our example. A similar argument shows that B must be singular, which it is for our example. The rest is just trial and error.

20) Let A be $m \times p$, B be $p \times q$, and C be $q \times n$. Find an expression for

$$\Big((AB)C\Big)_{ij}.$$

Hints:

- Recall that $(AB)_{ij} = \sum_{s=1}^{p} a_{is}b_{sj}$ (see page 37 of your Workbook).
- Let $D = AB$ and consider the expression for $(DC)_{ij}$ (using a 'dummy' variable k).

Note:
This question isn't typical of the sort of math problems I set. But it gives a taste of what it is like to rigorously prove the algebraic properties of matrices. If you are a math/stats/physics student you should attempt it!

Solution:
Following the hints we have

$$(AB)_{ij} = \sum_{s=1}^{p} a_{is}b_{sj} = (D)_{ij} \equiv d_{ij}, \qquad (*)$$

where $D = AB$. Now consider

$$\begin{aligned}((AB)C)_{ij} = (DC)_{ij} &= \sum_{k=1}^{q} d_{ik}c_{kj} \qquad \text{(By definition)}\\ &= \sum_{k=1}^{q}\left(\sum_{s=1}^{p} a_{is}b_{sk}\right)c_{kj} \qquad \text{(Use } (*)\text{, but replace } j \text{ with } k\text{)}\\ &= \sum_{k=1}^{q}\sum_{s=1}^{p} a_{is}b_{sk}c_{kj}.\end{aligned}$$

1.6 Special Types of Matrices

21) Consider the matrix equation $AC\underline{x} = \underline{b}$ where

$$A^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \text{and} \quad \underline{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

It is easily verified that A^{-1} and C^{-1} are nonsingular. Without first finding A and C determine $\underline{x}$.

[‡]Of course this doesn't help you if you haven't done that section yet, but I'm assuming you do re-read your notes.

Solution:
Like any equation involving '$\boldsymbol{x}$' we need to isolate the unknown. We use the algebraic properties of matrices to do so as follows:

$$\begin{aligned}
&\boldsymbol{AC}\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}} \\
\implies &\boldsymbol{A}^{-1}(\boldsymbol{AC}\underline{\boldsymbol{x}}) = \boldsymbol{A}^{-1}\underline{\boldsymbol{b}} \quad \text{(multiply both sides on the left by } \boldsymbol{A}^{-1}) \\
\implies &\underbrace{(\boldsymbol{A}^{-1}\boldsymbol{A})}_{\boldsymbol{I}}\boldsymbol{C}\underline{\boldsymbol{x}} = \boldsymbol{A}^{-1}\underline{\boldsymbol{b}} \quad \text{(using the algebraic properties of matrices)} \\
\implies &\boldsymbol{C}\underline{\boldsymbol{x}} = \boldsymbol{A}^{-1}\underline{\boldsymbol{b}} \\
\implies &\boldsymbol{C}^{-1}(\boldsymbol{C}\underline{\boldsymbol{x}}) = \boldsymbol{C}^{-1}(\boldsymbol{A}^{-1}\underline{\boldsymbol{b}}) \quad \text{(multiply both sides on the left by } \boldsymbol{C}^{-1}) \\
\implies &\underbrace{(\boldsymbol{C}^{-1}\boldsymbol{C})}_{\boldsymbol{I}}\underline{\boldsymbol{x}} = \boldsymbol{C}^{-1}\boldsymbol{A}^{-1}\underline{\boldsymbol{b}} \quad \text{(using the algebraic properties of matrices)} \\
\implies &\underline{\boldsymbol{x}} = \boldsymbol{C}^{-1}\boldsymbol{A}^{-1}\underline{\boldsymbol{b}} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{(using the given information)} \\
&= \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{(matrix multiplication)} \\
&= \begin{pmatrix} 15 \\ 9 \end{pmatrix} \quad \text{(matrix multiplication)}
\end{aligned}$$

Note:
From now on when doing problems of this type we don't always show every step of forming an identity matrix $\boldsymbol{I} = \boldsymbol{AA}^{-1}$. We also assume that you know that multiplication of a matrix (on the left or the right) by an identity matrix leaves the matrix unchanged.

22) Let

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}^{-1} = \begin{pmatrix} -1 & 1 \\ 3 & 7 \end{pmatrix}.$$

Find $(\boldsymbol{AB})^{-1}$.

Solution:
We could find the inverses of the given matrices ($\boldsymbol{A}$ and $\boldsymbol{B}$), and then find the inverse of the matrix product $\boldsymbol{AB}$ to get $(\boldsymbol{AB})^{-1}$. However, there is a much easier way to calculate this. Recall

$$\begin{aligned}
(\boldsymbol{AB})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1} \quad \text{(Theorem 4 (i))} \\
&= \begin{pmatrix} -1 & 1 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 4 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 3 & -3 \\ 31 & 9 \end{pmatrix}.
\end{aligned}$$

23) Consider the linear system $\boldsymbol{A}^T\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}$ where

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \underline{\boldsymbol{b}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Find the solution $\underline{\boldsymbol{x}}$.

Solution:

$$\begin{aligned}
& A^T \underline{x} = \underline{b} \\
\Longrightarrow\ & \underline{x} = (A^T)^{-1}\underline{b} \quad \text{(multiply both sides by } (A^T)^{-1} \text{ on the left)} \\
& \quad = (A^{-1})^T \underline{b} \quad \text{(Theorem 4(ii))} \\
& \quad = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix}^T \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
& \quad = \begin{pmatrix} 4 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\
& \quad = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\end{aligned}$$

24) Prove for an arbitrary square matrix A that

$$(A^2)^{-1} = (A^{-1})^2.$$

(Hint: use Theorem 4 (i).)

Solution:
Using the hint with $A = B$ yields

$$(AA)^{-1} = A^{-1}A^{-1} \quad \text{i.e.} \quad (A^2)^{-1} = (A^{-1})^2,$$

as required.

25) Use the following invertible matrix

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix}$$

to encode the message

" MEET ME MONDAY"

The inverse of the 'encoding matrix' above is the following 'decoding matrix':

$$A^{-1} = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix}$$

Also show how to decode the encoded message.

Solution:
As in Application 5, assign a number for each letter of the alphabet and assign 27 to spaces between (and for a space at the end of a word - you will see why in a minute):

$$\begin{matrix} M & E & E & T & * & M & E & * & M & O & N & D & A & Y & * \\ 13 & 5 & 5 & 20 & 27 & 13 & 5 & 27 & 13 & 15 & 14 & 4 & 1 & 25 & 27 \end{matrix}$$

As A is 3×3 break the enumerated message up into sequences of 3×1 vectors and use these to make the columns of a new matrix B:

$$B = \begin{pmatrix} 15 & 20 & 5 & 15 & 1 \\ 5 & 27 & 27 & 14 & 25 \\ 5 & 13 & 13 & 4 & 27 \end{pmatrix}$$

Now form the matrix product

$$C = AB = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 3 \\ 1 & -1 & -4 \end{pmatrix} \begin{pmatrix} 15 & 20 & 5 & 15 & 1 \\ 5 & 27 & 27 & 14 & 25 \\ 5 & 13 & 13 & 4 & 27 \end{pmatrix}$$
$$= \begin{pmatrix} 15 & -8 & -23 & -5 & 5 \\ 5 & 46 & 61 & 11 & 105 \\ -10 & -59 & -74 & -15 & -132 \end{pmatrix}.$$

The columns of this matrix give the encoded message in the following linear form:

$$\begin{matrix} 15, & 5, & -10, & -8, & 46, & -59, & -23, & 61, & -74 \\ -5, & 11, & -15, & 5, & 105, & -132 \end{matrix}$$

To decode the message using the inverse matrix (sent to the receiver of the message separately), we write this string as a sequence of 3×1 vectors (yielding C), and observe that

$$C = AB \implies B = A^{-1}C = \begin{pmatrix} -1 & -10 & -8 \\ -1 & -6 & -5 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 15 & -8 & -23 & -5 & 5 \\ 5 & 46 & 61 & 11 & 105 \\ -10 & -59 & -74 & -15 & -132 \end{pmatrix}$$
$$= \begin{pmatrix} 15 & 20 & 5 & 15 & 1 \\ 5 & 27 & 27 & 14 & 25 \\ 5 & 13 & 13 & 4 & 27 \end{pmatrix} \quad \checkmark$$

The numbers written column-wise give the original enumerated message.

26) Consider an arbitrary square nonsingular matrix A and a nonzero scalar c. Prove that

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

Hint:
Recall that

$$A^{-1} = B \iff AB = I,$$

where I is the identity matrix. (Note: of course in the hint, A and B are generic matrices, so the A here is different from the A in the statement to be proved.)

Solution:
Using the hint observe that

$$\begin{aligned} (cA)\left(\frac{1}{c}A^{-1}\right) &= \left(c \cdot \frac{1}{c}\right)(AA^{-1}) \quad \text{(Using algebraic matrix properties)} \\ &= 1 \cdot I \quad (c/c = 1 \text{ and } AA^{-1} = I) \\ &= I \quad \checkmark \end{aligned}$$

If we want to be pedantic, we could spell it out: in the hint take A to be cA and B to be $\frac{1}{c}A^{-1}$, etc.

Chapter 2

Solving Linear Systems

2.1 Echelon Form of a Matrix

27) Reduce

$$A = \begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix}$$

to REF and then to RREF. (Note: number of rows (m) = number of columns (n).)

Solution:
Note that the math is simpler if a_{11} is equal to 1.

$$\begin{aligned}
A &\longrightarrow \begin{pmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{pmatrix} r_1 \leftrightarrow r_3 \\
&\longrightarrow \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{pmatrix} \begin{matrix} \\ r_2 + 4r_1 \to r_2 \\ r_3 - 6r_1 \to r_3 \end{matrix} \\
&\longrightarrow \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ r_3 + r_2 \to r_3 \end{matrix} \\
&\longrightarrow \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{pmatrix} r_2 \cdot (1/9) \to r_2 \qquad \text{(REF)} \\
&\longrightarrow \begin{pmatrix} 1 & 0 & 7/9 \\ 0 & 1 & -26/9 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} r_1 - 2r_2 \to r_1 \\ \\ \\ \end{matrix} \qquad \text{(RREF)}.
\end{aligned}$$

28) Reduce

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$$

to REF and then to RREF. (Note: number of rows (m) > number of columns (n).)

Solution:

$$A \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & -2 \\ 0 & 3 \end{pmatrix} \begin{matrix} \\ r_2 - r_1 \to r_2 \\ r_3 + r_1 \to r_3 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{matrix} \\ r_2 \cdot (-1/2) \to r_2 \\ r_3 \cdot (1/3) \to r_3 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{matrix} \\ \\ r_3 - r_2 \to r_3 \end{matrix} \qquad \text{(REF)}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{matrix} r_1 - r_2 \to r_1 \\ \\ \\ \end{matrix} \qquad \text{(RREF).}$$

29) Reduce

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{pmatrix}$$

to REF and then to RREF. (Note: number of rows (m) $<$ number of columns (n).)

Solution:

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} r_2 - 2r_1 \to r_2 \qquad \text{(REF)}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} r_1 - r_2 \to r_1 \qquad \text{(RREF)}$$

30) Determine the reduced row echelon form of

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \quad \text{for } 0 < x < \tfrac{\pi}{2}.$$

Show all the steps in your argument!

Solution:

Before we start we provide two reminders:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \sin^2 x + \cos^2 x = 1.$$

We use elementary row operations as usual.

Method 1:

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \tan x \\ -\sin x & \cos x \end{pmatrix} r_1 \cdot \frac{1}{\cos x} \to r_1$$

$$\longrightarrow \begin{pmatrix} 1 & \tan x \\ 0 & \cos x + \frac{\sin^2 x}{\cos x} \end{pmatrix} r_2 + \sin x \cdot r_1 \to r_2$$

After observing that

$$\cos x + \frac{\sin^2 x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos x},$$

we have

$$\begin{pmatrix} 1 & \tan x \\ 0 & \cos x + \frac{\sin^2 x}{\cos x} \end{pmatrix} = \begin{pmatrix} 1 & \tan x \\ 0 & \frac{1}{\cos x} \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} 1 & \tan x \\ 0 & 1 \end{pmatrix} \; r_2 \cdot \cos x \to r_2$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; r_1 - \tan x \cdot r_2 \to r_1$$

In the past though I have seen students take a more non-standard approach to solving this problem, as the next method shows.

Method 2:

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \longrightarrow \begin{pmatrix} \cos^2 x & \cos x \sin x \\ \sin^2 x & -\sin x \cos x \end{pmatrix} \; \begin{matrix} r_1 \cdot \cos x \to r_1 \\ r_2 \cdot (-\sin x) \to r_2 \end{matrix}$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ \sin^2 x & -\sin x \cos x \end{pmatrix} \; r_1 + r_2 \to r_1$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -\sin x \cos x \end{pmatrix} \; r_2 - \sin^2 x \cdot r_1 \to r_2$$
$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; r_2 \cdot \frac{1}{(-\sin x \cos x)} \to r_2$$

2.2 Solving Linear Systems

31) Solve the system

$$\begin{cases} 2x \qquad - 3z = 1 \\ 4x + y - 2z = 2 \\ 3x + y - \; z = 3 \end{cases}$$

via (a) Gaussian Elimination, and (b) Gauss-Jordon Elimination. (Note: number of equations (m) = number of unknowns (n).)

Solution:
(a) First write down the associated matrix equation:

$$\begin{pmatrix} 2 & 0 & -3 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

so the augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 0 & -3 & 1 \\ 4 & 1 & -2 & 2 \\ 3 & 1 & -1 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 2 & 0 & -3 & 1 \\ 1 & 0 & -1 & -1 \\ 3 & 1 & -1 & 3 \end{array}\right) r_2 - r_3 \to r_2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 1 & -1 & 3 \end{array}\right) r_1 \leftrightarrow r_2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 2 & 6 \end{array}\right) \begin{array}{l} r_2 - 2r_1 \to r_2 \\ r_3 - 3r_1 \to r_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & -1 & 3 \end{array}\right) r_2 \leftrightarrow r_3$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & -3 \end{array}\right) r_3 \cdot (-1) \to r_3 \qquad \text{(REF)} \qquad (*)$$

The associated (equivalent) linear system is:

$$\begin{cases} x \quad - \quad z = -1 \\ \quad y + 2z = 6 \\ \qquad\quad z = -3 \end{cases}$$

Back-substitution yields

$$z = -3, \quad y = 12, \quad x = -4.$$

(b) Alternatively, from $(*)$

$$(*) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & -3 \end{array}\right) \begin{array}{l} r_1 + r_3 \to r_1 \\ r_2 - 2r_3 \to r_2 \end{array} \quad \text{(RREF)}.$$

The associated (equivalent) linear system is:

$$\begin{cases} x \qquad = -4 \\ \quad y \quad = 12 \\ \qquad z = -3 \end{cases}$$

as before.

32) Solve the system

$$\begin{cases} 3x_1 + 4x_2 = 1 \\ x_1 - 2x_2 = 2 \\ -x_1 + 5x_2 = 0 \end{cases}$$

via (a) Gaussian Elimination, and (b) Gauss-Jordon Elimination. (Note: number of equations $(m) >$ number of unknowns (n).)

Solution:
(a) The augmented matrix is

$$
\left(\begin{array}{cc|c} 3 & 4 & 1 \\ 1 & -2 & 2 \\ -1 & 5 & 0 \end{array}\right) \qquad \text{leading to}
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & 4 & 1 \\ -1 & 5 & 0 \end{array}\right) r_1 \leftrightarrow r_2
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 10 & -5 \\ 0 & 3 & 2 \end{array}\right) \begin{array}{l} r_2 - 3r_1 \to r_2 \\ r_3 + r_1 \to r_3 \end{array}
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 2 & -1 \\ 0 & 3 & 2 \end{array}\right) r_2 \cdot (1/5) \to r_2
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 3 \end{array}\right) r_3 - r_2 \to r_3
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & -1 \end{array}\right) r_2 \leftrightarrow r_3
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{array}\right) r_3 - 2r_2 \to r_3
$$

$$
\longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right) r_3 \cdot (-1/7) \to r_3 \qquad \text{(REF)} \qquad (*)
$$

The associated system of equations is:

$$
\begin{cases} x_1 - 2x_2 &= 2 \\ \qquad\; x_2 &= 3 \\ \qquad\;\; 0 &= 1 \end{cases}
$$

The last equation is impossible, so the system is inconsistent.

(b) We could continue to get $(*)$ into RREF, i.e.

$$
\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)
$$

(check!), but clearly there is no point.

33) Solve the system

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - \ \ x_3 = 1 \end{cases}$$

via (a) Gaussian Elimination, and (b) Gauss-Jordon Elimination. (Note: number of equations $(m) <$ number of unknowns (n).)

Solution:

(a) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 3 & -1 & 1 \end{array}\right) \quad \text{leading to}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -5 & -7 \end{array}\right) r_2 - 2r_1 \to r_2 \quad \text{(REF)} \qquad (*)$$

The associated system of equations is:

$$\begin{cases} x_1 + x_2 + 2x_3 = 4 \\ \qquad x_2 - 5x_3 = -7 \end{cases}$$

The last equation yields $x_2 = 5x_3 - 7$, where x_3 is free to be chosen. Let $\underline{x_3 = \alpha} \in \mathbb{R}$, then $\underline{x_2 = 5\alpha - 7}$. From the first equation

$$\begin{aligned} x_1 &= -x_2 - 2x_3 + 4 \\ &= -(5\alpha - 7) - 2\alpha + 4 \\ &= \underline{-7\alpha + 11} \end{aligned}$$

Thus the infinite solution set is

$$\{(-7\alpha + 11, 5\alpha - 7, \alpha) \mid \alpha \in \mathbb{R}\}$$

(b) Alternatively, using Gauss-Jordon Elimination we continue the elimination via:

$$(*) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 7 & 11 \\ 0 & 1 & -5 & -7 \end{array}\right) r_1 - r_2 \to r_1 \quad \text{(RREF).}$$

The associated linear system is

$$\begin{cases} x_1 \qquad + 7x_3 = 11 \\ \quad x_2 - 5x_3 = -7 \end{cases}$$

From the 2nd equation $x_2 = 5x_3 - 7$. Set $\underline{x_3 = \alpha} \in \mathbb{R}$, so $\underline{x_2 = 5\alpha - 7}$, and from the 1st equation $x_1 = -7x_3 + 11 = \underline{-7\alpha + 11}$, etc. as before.

34) Investigate for what values of $a \in \mathbb{R}$ the linear system

$$\begin{cases} x + \qquad\quad y \ = 3, \\ x + (a^2 - 8)y \ = a, \end{cases}$$

has (i) no solution, (ii) an infinite number of solutions, and (iii) a unique solution.

Strategy: This is a harder problem, but just apply Gauss-Jordon Elimination to the associated augmented matrix and see what conditions have to be given to the number a for the three cases.

Solution:

The eliminations process applied to the augmented matrix initially yields:

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & a^2-8 & a \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & a^2-9 & a-3 \end{array}\right) r_2 - r_1 \to r_2 \qquad (*)$$

Now in order to proceed with the GJE algorithm we would divide the 2nd row by $a^2 - 9$. However, this can only be done if $a^2 - 9 \neq 0$, i.e. $a \neq \pm 3$. Thus (for now) we exclude these two possibilities of a. Continuing

$$\begin{aligned} (*) &\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & \frac{1}{a+3} \end{array}\right) r_2 \cdot \left(\frac{1}{a^2-9}\right) \to r_2 \\ &\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 3 - \frac{1}{a+3} \\ 0 & 1 & \frac{1}{a+3} \end{array}\right) r_1 - r_2 \to r_1, \end{aligned}$$

where in the 1st step above we used the fact that

$$\frac{a-3}{a^2-9} = \frac{\cancel{(a-3)}}{\cancel{(a-3)}(a+3)} = \frac{1}{a+3} \quad (\text{as } a \neq 3).$$

Thus we have the unique solution for each $a \neq \pm 3$ given by

$$x = 3 - \frac{1}{a+3}, \quad y = \frac{1}{a+3}.$$

Now we go back and deal with the two cases $a = +3$ and $a = -3$:

$a = +3$:

From $(*)$ we get the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right)$$

i.e., $x + y = 3$ so $x = 3 - y$, where y is free to be chosen, etc. Thus in this case we get an infinite number of solutions.

$a = -3$:

From $(*)$ we get the augmented matrix

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 0 & -6 \end{array}\right).$$

From the last equation we have $0 = -6$, an impossibility, thus the system is inconsistent. I.e., in this case we have no solutions.

2.3 Finding A^{-1}

35) Find the inverse of

$$A = \begin{pmatrix} -3 & -3 & -4 \\ 0 & 1 & 1 \\ 4 & 3 & 4 \end{pmatrix}$$

(if it exists).

Solution:

$$[A|I] = \left(\begin{array}{ccc|ccc} -3 & -3 & -4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 3 & 4 & 0 & 0 & 1 \end{array}\right) r_1 + r_3 \to r_1$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 3 & 4 & -4 & 0 & -3 \end{array}\right) r_3 - 4r_1 \to r_3$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -4 & -3 & -3 \end{array}\right) r_3 - 3r_2 \to r_3$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 4 & 4 & 3 \\ 0 & 0 & 1 & -4 & -3 & -3 \end{array}\right) r_2 - r_3 \to r_2$$

Thus

$$A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 4 & 3 \\ -4 & -3 & -3 \end{pmatrix}$$

36) Find the inverse of

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{pmatrix}$$

(if it exists).

Solution:

$$[A|I] = \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 5 & -2 & -3 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & -12 & 12 & -5 & 0 & 1 \end{array}\right) \begin{array}{l} r_2 - r_1 \to r_2 \\ r_3 - 5r_1 \to r_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & -4 & 4 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -3 & 1 \end{array}\right) r_3 - 3r_2 \to r_3 .$$

Thus we see that the original matrix A is *not* equivalent to the identity matrix (the last row is all zeros) and thus from Theorem 6 the matrix A must be singular (i.e., it does not have an inverse).

37) Investigate for what values of $a \in \mathbb{R}$ the homogeneous linear system

$$\begin{cases} (a-1)x + 2y &= 0, \\ 2x + (a-1)y &= 0, \end{cases}$$

has a **non**trivial solution. Do this problem three different ways, using the contrapositive of items 1., 3., and 5. with item 2. in Theorem 6.

Solution:

The contrapositive of item 5 with item 2 in Theorem 6 yields:

$$\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}} \text{ has a non-trivial solution } \iff |\boldsymbol{A}| = \boldsymbol{0}.$$

Writing the given system in matrix form

$$\begin{pmatrix} a-1 & 2 \\ 2 & a-1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or $\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$. So to show that we have a non-trivial solution (i.e., an infinite number of solutions) we must have that $|\boldsymbol{A}| = \boldsymbol{0}$. We solve

$$\begin{aligned} |\boldsymbol{A}| = \begin{vmatrix} a-1 & 2 \\ 2 & a-1 \end{vmatrix} &= (a-1)^2 - 4 = 0 \\ &\implies (a-1)^2 = 4 \\ &\implies a - 1 = \pm 2 \\ &\implies a = 3 \quad \text{or} \quad -1. \end{aligned}$$

The contrapositive of item 3 with item 2 in Theorem 6 yields:

$$\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}} \text{ has a non-trivial solution } \iff \text{ the RREF of } \boldsymbol{A} \text{ is NOT } \boldsymbol{I_n}.$$

Remember, to get a non-trivial solution, during the row elimination procedure we would get a row of zeros, leading to a free variable etc. Writing the given system in matrix form and applying row operations:

$$\begin{aligned} \begin{pmatrix} a-1 & 2 \\ 2 & a-1 \end{pmatrix} &\longrightarrow \begin{pmatrix} 2 & a-1 \\ a-1 & 2 \end{pmatrix} r_1 \leftrightarrow r_2 \\ &\longrightarrow \begin{pmatrix} 1 & \frac{1}{2}(a-1) \\ a-1 & 2 \end{pmatrix} r_1 \cdot (1/2) \to r_1 \\ &\longrightarrow \begin{pmatrix} 1 & \frac{1}{2}(a-1) \\ 0 & 2 - \frac{1}{2}(a-1)^2 \end{pmatrix} r_2 - (a-1) \cdot r_1 \to r_2. \end{aligned}$$

We thus have a non-trivial solution provided (for a row of zeros)

$$\begin{aligned} 2 - \frac{1}{2}(a-1)^2 = 0 &\implies 4 - (a-1)^2 = 0 \\ &\implies (a-1)^2 = 4, \quad \text{etc., as before.} \end{aligned}$$

The contrapositive of item 1 with item 2 in Theorem 6 yields:

$$\boldsymbol{A}\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}} \text{ has a non-trivial solution } \iff \boldsymbol{A} \text{ is singular.}$$

So we try and find the inverse of $\boldsymbol{A}$ and in the process find conditions telling us that the inverse does not exist:

$$
\begin{aligned}
[A|I] &= \left(\begin{array}{cc|cc} a-1 & 2 & 1 & 0 \\ 2 & a-1 & 0 & 1 \end{array}\right) \\
&\longrightarrow \left(\begin{array}{cc|cc} 2 & a-1 & 0 & 1 \\ a-1 & 2 & 1 & 0 \end{array}\right) r_1 \leftrightarrow r_2 \\
&\longrightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2}(a-1) & 0 & \frac{1}{2} \\ a-1 & 2 & 1 & 0 \end{array}\right) r_1 \cdot (1/2) \to r_1 \\
&\longrightarrow \left(\begin{array}{cc|cc} 1 & \frac{1}{2}(a-1) & 0 & \frac{1}{2} \\ 0 & 2-\frac{1}{2}(a-1)^2 & 1 & -\frac{1}{2} \end{array}\right).
\end{aligned}
$$

So A^{-1} does not exist if (see Exercise **36**))

$$2 - \frac{1}{2}(a-1)^2 = 0 \quad \text{etc.} \quad \checkmark$$

Of course we already saw from the previous argument that A cannot be row reduced to I so we didn't really need to do this calculation.

Chapter 3

Determinants

3.1 Definition

38) Find $|A|$ where

$$A = \begin{pmatrix} -2 & 3 & 2 \\ 1 & 2 & -1 \\ 4 & 2 & 18 \end{pmatrix}$$

Solution:
We use Definition 29 in the Workbook and use a diagram as explained on page 106 (arrows omitted):

$$\begin{matrix} -2 & 3 & 2 & -2 & 3 \\ 1 & 2 & -1 & 1 & 2 \\ 4 & 2 & 18 & 4 & 2 \end{matrix}$$

Then

$$
\begin{aligned}
|A| &= (-2)(2)(18) + (3)(-1)(4) + (2)(1)(2) \\
&\quad - (1)(3)(18) - (-2)(-1)(2) - (2)(2)(4) \\
&= -72 - 12 + 4 - 54 - 4 - 16 \\
&= -154.
\end{aligned}
$$

3.2 Properties of Determinants

39) Let

$$A = \begin{pmatrix} -1/2 & 3 & 1/3 \\ 1/4 & 2 & -1/6 \\ 1 & 2 & 3 \end{pmatrix}$$

Use Theorem 9 (b) and Definition 29 in the Workbook to evaluate $|A|$.

Solution:

$$\begin{aligned} |A| = \begin{vmatrix} -1/2 & 3 & 1/3 \\ 1/4 & 2 & -1/6 \\ 1 & 2 & 3 \end{vmatrix} &= (1/4) \begin{vmatrix} -2 & 3 & 1/3 \\ 1 & 2 & -1/6 \\ 4 & 2 & 3 \end{vmatrix} c_1 \cdot 4 \to c_1 \\ &= (1/4)(1/6) \begin{vmatrix} -2 & 3 & 2 \\ 1 & 2 & -1 \\ 4 & 2 & 18 \end{vmatrix} c_3 \cdot 6 \to c_3 \\ &= (1/24)(-154) \quad \text{(Using Exercise 38)} \\ &= -77/12. \end{aligned}$$

40) Let

$$A = \begin{pmatrix} -2 & 3 & 2 \\ 1 & 2 & -1 \\ 4 & 2 & 18 \end{pmatrix}$$

Use row operations to reduce $|A|$ to upper triangular form (using Theorem 9) and then use Theorem 8(c) to evaluate this determinant.

Solution:

$$\begin{aligned} |A| &= - \begin{vmatrix} 1 & 2 & -1 \\ -2 & 3 & 2 \\ 4 & 2 & 18 \end{vmatrix} r_1 \leftrightarrow r_2 \\ &= - \begin{vmatrix} 1 & 2 & -1 \\ 0 & 7 & 0 \\ 0 & -6 & 22 \end{vmatrix} \begin{matrix} r_2 + 2r_1 \to r_2 \\ r_3 - 4r_1 \to r_3 \end{matrix} \\ &= -7 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & -6 & 22 \end{vmatrix} r_2 \cdot (1/7) \to r_2 \\ &= -7 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 22 \end{vmatrix} r_3 + 6r_2 \to r_3 \\ &= (-7)(1)(1)(22) = -154, \end{aligned}$$

as we found in Exercise **38**).

41) Let

$$A = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

(a) Compute $|A|$ and $|A^{-1}|$.

(b) Make a conjecture about the determinant of the inverse of a matrix.

Solution:

(a) $|A| = 2(2) - 3(-1) = 4 + 3 = 7$. Then using the formula for the inverse of a 2×2 matrix (page 69 of the workbook) we have

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2/7 & 3/7 \\ -1/7 & 2/7 \end{pmatrix}.$$

Thus

$$|A^{-1}| = (2/7)^2 - (3/7)(-1/7) = (4+3)/49 = 1/7.$$

(b) We see that $|A^{-1}| = \frac{1}{|A|}$.

42) Let A and B be square matrices of order 3 such that $|A| = 4$ and $|B| = 5$.
(a) Find $|AB|$
(b) Find $|2A|$
(c) Are A and B singular or nonsingular? Explain.
(d) If A and B are nonsingular find $|A^{-1}|$ and $|B^{-1}|$
(e) Find $|(AB)^T|$

Solution:

(a) Using Theorem 10

$$|AB| = |A||B| = 4(5) = 20.$$

(b)

$$\begin{aligned} |2A| &= \begin{vmatrix} 2(\cdot) & 2(\cdot) & 2(\cdot) \\ 2(\cdot) & 2(\cdot) & 2(\cdot) \\ 2(\cdot) & 2(\cdot) & 2(\cdot) \end{vmatrix} \\ &= (2)(2)(2)\begin{vmatrix} (\cdot) & (\cdot) & (\cdot) \\ (\cdot) & (\cdot) & (\cdot) \\ (\cdot) & (\cdot) & (\cdot) \end{vmatrix} \quad \text{(Using Theorem 9(b) three times)} \\ &= 8|A| = 8(4) = 32. \end{aligned}$$

(c) From Theorem 11

$$|A|, |B| \neq 0 \iff A \text{ and } B \text{ are nonsingular.}$$

(d) From Corollary 1 (to Theorem 10) we have (after noting the conclusion of (c))

$$|A^{-1}| = \frac{1}{|A|} = \frac{1}{4}, \quad |B^{-1}| = \frac{1}{|B|} = \frac{1}{5}.$$

(e)

$$\begin{aligned} |(AB)^T| &= |B^T A^T| \quad \text{(Using Theorem 2 (part 11))} \\ &= |B^T||A^T| \quad \text{(Using Theorem 10)} \\ &= |B||A| \quad \text{(Using Theorem 7)} \\ &= 5(4) = 20. \end{aligned}$$

43)

$$\text{If} \qquad |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 2,$$

$$\text{find} \qquad |B| = \begin{vmatrix} (3a_1 - 6a_3) & a_2 & a_3 \\ (3b_1 - 6b_3) & b_2 & b_3 \\ (c_1 - 2c_3) & \frac{1}{3}c_2 & \frac{1}{3}c_3 \end{vmatrix}.$$

Hint: apply row and column operations (see Theorem 9). This is a harder problem.

Solution:

The method here is to apply the rules of Theorem 9 to convert $|A|$ into $|B|$, but keeping track of how the operations effect the determinants.

$$2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{1}{3} \begin{vmatrix} 3a_1 & a_2 & a_3 \\ 3b_1 & b_2 & b_3 \\ 3c_1 & c_2 & c_3 \end{vmatrix} 3c_1 \to c_1$$

$$= \frac{1}{3} \begin{vmatrix} (3a_1 - 6a_3) & a_2 & a_3 \\ (3b_1 - 6b_3) & b_2 & b_3 \\ (3c_1 - 6c_3) & c_2 & c_3 \end{vmatrix} c_1 - 6c_3 \to c_1$$

$$= 3(1/3) \begin{vmatrix} (3a_1 - 6a_3) & a_2 & a_3 \\ (3b_1 - 6b_3) & b_2 & b_3 \\ (c_1 - 2c_3) & (1/3)c_2 & (1/3)c_3 \end{vmatrix} \frac{1}{3} \cdot r_3 \to r_3.$$

I.e. we have $|B| = 2$.

3.3 Cofactor Expansions

44) Let

$$A = \begin{pmatrix} 1 & -3 & 4 \\ 0 & 2 & 5 \\ 6 & -1 & 7 \end{pmatrix}$$

Find $|M_{23}|$ (the minor of the entry a_{23}) and A_{23} (the cofactor of a_{23}).

Solution:

Crossing out row 2 and column 3 and taking the determinant of the resulting matrix yields:

$$|M_{23}| = \begin{vmatrix} 1 & -3 \\ 6 & -1 \end{vmatrix} = (1)(-1) - (6)(-3) = -1 + 18 = 17.$$

The cofactor of a_{23} (signed minor) is then

$$A_{23} = (-1)^{2+3}|M_{23}| = -17.$$

Caution: M_{23} is a matrix and A_{23} is a scalar!

45) Let A be the same as in Exercise **44**). Use a cofactor expansion along row 2 to evaluate $|A|$.

Solution:

$$\begin{aligned}|A| &= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \\ &= 0(-1)^{2+1}|M_{21}| + 2(-1)^{2+2}|M_{22}| + 5(-1)^{2+3}|M_{23}| \\ &= 0 + 2\begin{vmatrix}1 & 4 \\ 6 & 7\end{vmatrix} - 5\begin{vmatrix}1 & -3 \\ 6 & -1\end{vmatrix} \\ &= 2(7-24) - 5(-1+18) \\ &= 2(-17) - 5(17) \\ &= -119.\end{aligned}$$

46) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Use a cofactor expansion (along row 1 or column 2) to prove the usual formula for the determinant of A, namely

$$|A| = ad - bc.$$

Solution:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{pattern of signs: } \begin{matrix} + & - \\ - & + \end{matrix}.$$

Now do a cofactor expansion along row 1:

$$|A| = +a|d| - b|c| = ad - bc \quad \checkmark$$

Note:
(Here $|\cdot|$ does not represent absolute value, but a determinant of order 1 (see Definition 28).

Alternatively, expanding along column 2:

$$|A| = -b|c| + d|a| = ad - bc \quad \checkmark$$

47) Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

Use a cofactor expansion to verify

$$|A| = |A^T|.$$

Solution:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \text{pattern of signs: } \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Expanding along the 1st row:

$$|A| = +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

But

$$A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \qquad \text{pattern of signs: } \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Expanding along the 1st column:

$$\begin{aligned} |A^T| &= +a\begin{vmatrix} e & h \\ f & i \end{vmatrix} - b\begin{vmatrix} d & g \\ f & i \end{vmatrix} + c\begin{vmatrix} d & g \\ e & h \end{vmatrix} \\ &= +a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= |A|, \end{aligned}$$

where we used the fact that for an arbitrary determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = ad - cb = \begin{vmatrix} a & c \\ b & d \end{vmatrix}.$$

3.4 Inverse of a Matrix (via the Adjoint)

48) Find the adjoint of

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

.

Solution:
Reminder: the cofactor of a_{12} is

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -(0-1) = 1.$$

Finding the matrix of cofactors yields (exercise):

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 4 & 1 & 2 \\ 6 & 0 & 3 \\ 7 & 1 & 2 \end{pmatrix}.$$

The transpose of this matrix is the adjoint of A, that is:

$$\text{adj}(A) = \begin{pmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}.$$

49) Using $\mathbf{adj}(A)$, where A is the matrix in Exercise **48**), find A^{-1}.

Solution:

$$|A| = \begin{vmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 3 & 0 \end{vmatrix} r_3 + r_1 \to r_3$$
$$= -3 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} \quad \text{(Cofactor expansion along row 3)}$$
$$= -3(-1 - 0)$$
$$= 3.$$

Thus recalling the matrix $\mathbf{adj}(A)$ from Exercise **48**) we have that

$$A^{-1} = \frac{1}{|A|}\,\mathbf{adj}(A)$$
$$= \frac{1}{3}\begin{pmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 4/3 & 2 & 7/3 \\ 1/3 & 0 & 1/3 \\ 2/3 & 1 & 2/3 \end{pmatrix}.$$

50) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a nonsingular matrix. Use the formula

$$A^{-1} = \frac{1}{|A|}\mathbf{adj}(A)$$

to verify the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Solution:

We know that $|A| = ad - bc$, so all we need to show is that

$$\mathbf{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

From our notes we know that

$$\mathbf{adj}(A) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T.$$

Now with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{pattern of signs: } \begin{matrix} + & - \\ - & + \end{matrix}$$

we have

$$\begin{aligned} A_{11} &= +|d| = +d, \\ A_{12} &= -|c| = -c, \\ A_{21} &= -|b| = -b, \\ A_{22} &= +|a| = +a, \end{aligned}$$

thus

$$\operatorname{adj}(A) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^T = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \checkmark$$

(Here $|\cdot|$ does not represent absolute value, but a determinant of order 1 (see Definition 28).

Chapter 4

Real Vector Spaces

4.1 Vectors in the Plane and in 3-Space (generalized to n-Space)

51) Consider the vectors $u = (3, -4)$, $v = (9, 1)$, and $w = (-39, 0)$.

(a) Use directed line segments to represent u and v.

(b) Find $u + v$ and represent graphically.

(c) Find $2v - u$ and represent graphically.

(d) Write the vector w as a linear combination of u and v.

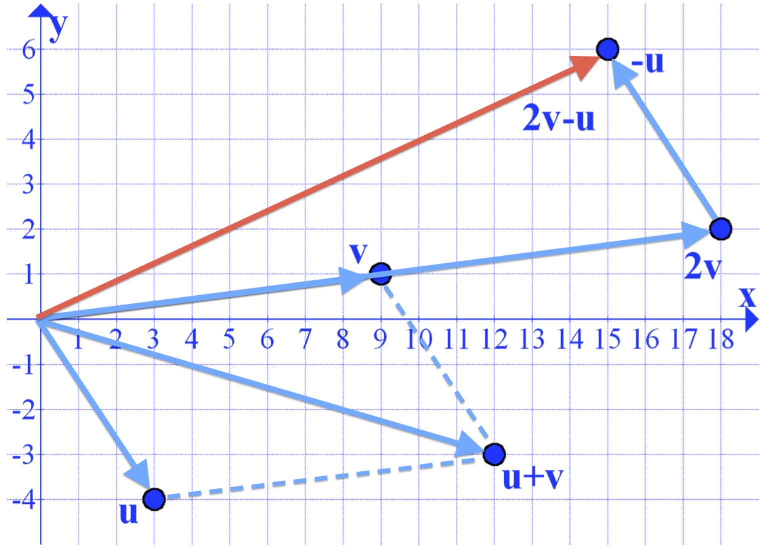
Solution:
(a) See the figure below.
(b) $u + v = (3, -4) + (9, 1) = (12, -3)$. See the figure below.
(c) $2v - u = 2(9, 1) - (3, -4) = (18, 2) - (3, -4) = (15, 6)$. See the figure below.
(d) We seek x and y s.t.

$$xu + yv = w$$
$$\text{i.e.} \quad x(3, -4) + y(9, 1) = (-39, 0),$$

or

$$\begin{cases} 3x + 9y = -39 \\ -4x + \ \ y = 0 \end{cases}$$

or



$$\begin{cases} \quad x + 3y = -13 \\ -4x + \quad y = 0 \end{cases}$$

which yields (exercise) $x = -1$ and $y = -4$.

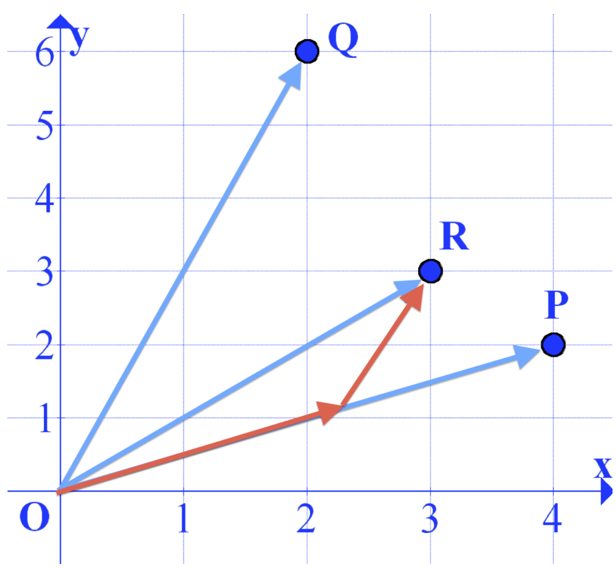
52) (a) Draw the vectors

$$\vec{OP} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \vec{OR} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \vec{OQ} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}.$$

(b) How do we get to R using the vectors $\vec{OP}$ and $\vec{OQ}$?

Solution:

(a) See the figure below:



(b) We seek x and y such that

$$\vec{OR} = x \cdot \vec{OP} + y \cdot \vec{OQ} \qquad (*)$$

i.e.

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} = x \begin{pmatrix} 4 \\ 2 \end{pmatrix} + y \begin{pmatrix} 2 \\ 6 \end{pmatrix},$$

or

$$\begin{cases} 4x + 2y = 3 \\ 2x + 6y = 3 \end{cases}$$

Solving these equations yields (exercise) $x = 3/5$, $y = 3/10$, so from $(*)$:

$$\vec{OR} = \frac{3}{5} \cdot \vec{OP} + \frac{3}{10} \cdot \vec{OQ},$$

illustrated in red above.

53) Let $\underline{x} = (-1, -2, -2)$, $\underline{u} = (0, 1, 4)$, $\underline{v} = (-1, 1, 2)$ and $\underline{w} = (3, 1, 2)$ be vectors in $\mathbb{R}^3$. Write (if possible) $\underline{x}$ as a linear combination of the vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$. (In other words, write $\underline{x}$ as a sum of constants times $\underline{u}$, $\underline{v}$ and $\underline{w}$.)

Solution:
We seek constants $a, b, c \in \mathbb{R}$ such that

$$\underline{x} = a\underline{u} + b\underline{v} + c\underline{w} \qquad (*)$$

(There may or may not be constants a, b and c such that $(*)$ holds.) I.e., we want

$$\begin{aligned} (-1, -2, -2) &= a(0, 1, 4) + b(-1, 1, 2) + c(3, 1, 2) \\ &= (0, a, 4a) + (-b, b, 2b) + (3c, c, 2c) \quad \text{(Using Definition 36)} \\ &= (-b + 3c, a + b + c, 4a + 2b + 2c) \quad \text{(Using Definition 36 again)} \end{aligned}$$

Equating corresponding components on both sides:

$$\begin{aligned} -b + 3c &= -1 \quad \text{(1st components)} \\ a + b + c &= -2 \quad \text{(2nd components)} \\ 4a + 2b + 2c &= -2 \quad \text{(3rd components)} \end{aligned}$$

We have 3 linear equations in 3 unknowns. Applying elementary row operations to get the associated augmented matrix into upper triangular form yields

$$\left(\begin{array}{ccc|c} 0 & -1 & 3 & -1 \\ 1 & 1 & 1 & -2 \\ 4 & 2 & 2 & -2 \end{array}\right) \longrightarrow \quad \cdots \quad \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -4 & 4 \end{array}\right).$$

(**Exercise**: Apply the operations: $r_1 \leftrightarrow r_2$, $r_3 - 4r_1 \rightarrow r_1$, $r_3 \cdot (1/2) \rightarrow r_3$, and $r_3 - r_2 \rightarrow r_3$.)

The associated linear system is

$$\begin{cases} a + b + \ \ c = -2 \\ \quad - b + 3c = -1 \\ \qquad\quad - 4c = 4 \end{cases}$$

Thus applying back-substitution yields

$$c = -1 \implies b = -2 \implies a = 1.$$

Thus from $(*)$

$$\underline{x} = \underline{u} - 2\underline{v} - \underline{w}.$$

4.2 Vector Spaces

54) Let

$$V = \text{ the set of integers with the standard operations of (vector) addition and (scalar) multiplication.}$$

Show that V is NOT a vector space.

Solution:
All we need is a single counter-example. We show that V is not closed with respect to scalar multiplication. I.e., if we multiply a member of V by a scalar then we don't necessarily get an integer as an answer (see the comments on page 150 of our notes). Now we are dealing with *real* vector spaces which means the scalars can be any number in $\mathbb{R}$. So just observe, e.g.:

$$\underbrace{\frac{1}{2}}_{\text{scalar}} \cdot \underbrace{(1)}_{\text{integer}} = \underbrace{\frac{1}{2}}_{\text{non-integer}}$$

According to Definition 37, V is clearly not closed with respect to scalar multiplication (as defined in this system) and so V is not a vector space.

Note: however, V *is* closed with respect to (vector) addition as

$$\text{integer} + \text{integer} = \text{integer},$$

always!

55)

$$V = \text{ the set of all second-degree polynomials.}$$

Show that V is NOT a vector space.

Solution:
Once again we find a single counter-example. We show that V is not closed with respect to (vector) addition. In other words, we find two polynomials such that when they are added together (using the usual method of adding polynomials together) we obtain a polynomial that is *not* of second degree.

Consider

$$p(x) = x^2, \quad \text{and} \quad q(x) = -x^2 + x + 1.$$

Then observe

$$p(x) + q(x) = x + 1,$$

which is a first-degree polynomial. Thus V is not closed with respect to (vector) addition and so according to Definition 37 V is not a vector space.

56) Let

$$V = \{(x, x-2) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}.$$

Prove that V is NOT a vector space.

Solution:

There are various ways we can answer this question. Let's start with the closure properties that a vector space must have. (Note: without any other information we must assume that the operations of (vector) addition and scalar multiplication are inherited from the usual ones for $\mathbb{R}^2$.)

Vector addition

Consider two arbitrary members of V, namely

$$v_1 = (x_1, x_1 - 2) \in V, \quad v_2 = (x_2, x_2 - 2) \in V, \quad x_1, x_2 \in \mathbb{R}.$$

Adding them together:

$$\begin{aligned} v_1 + v_2 &= (x_1, x_1 - 2) + (x_2, x_2 - 2) \\ &= (x_1 + x_2, x_1 + x_2 - 4) \\ &= \underbrace{(\widehat{x}, \widehat{x} - 4)}_{\notin V} \neq (\widehat{x}, \widehat{x} - 2), \end{aligned}$$

where $\widehat{x} := x_1 + x_2$. In other words when we add two arbitrary members of V together we get something which doesn't belong to V, i.e. V is not closed with respect to addition. And hence V is not a vector space according to Definition 37.

We could stop here. But just for clarity let's look at two other ways we could arrive at the same conclusion.

Scalar multiplication

Consider an arbitrary member of V and an arbitrary scalar:

$$v = (x, x - 2) \in V, \quad k \in \mathbb{R}.$$

Now do scalar multiplication:

$$\begin{aligned} kv &= k(x, x - 2) \\ &= (kx, kx - 2k) \\ &= \underbrace{(\widehat{x}, \widehat{x} - 2k)}_{\notin V} \neq (\widehat{x}, \widehat{x} - 2) \quad \text{unless } k = 1, \end{aligned}$$

($\widehat{x} := kx$). In other words scalar multiplication of a member of V doesn't produce something belonging to V (unless the scalar is equal to 1). And hence V is not a vector space.

Existence of a zero vector

According to Definition 37 (item A.2) if V is a vector space it must possess a zero vector. Suppose for the moment that V does have a zero vector, denoted $\mathbf{0}$. Then there must exist $x_0 \in \mathbb{R}$ such that

$$\mathbf{0} = (x_0, x_0 - 2) \in V,$$

so that for any member $(x, x-2) \in V$ we have

$$(x, x-2) + (x_0, x_0 - 2) = (x, x-2) \quad \text{(See item A.2)},$$

i.e.

$$(x + x_0, x + x_0 - 4) = (x, x-2).$$

Equating corresponding coefficients on both sides:

$$\begin{aligned} x + x_0 &= x \quad \text{(1st components)} \\ x + x_0 - 4 &= x - 2 \quad \text{(2nd components)} \end{aligned}$$

From the 1st equation we get

$$x_0 = 0.$$

But from the 2nd equation we get

$$x_0 = (x-2) - x + 4 = 2,$$

and so the equations are inconsistent, and thus there is no x_0 that will yield a zero vector. Hence V is not a vector space.

57) Verify that the standard examples in our notes (Examples 10 - 13) are vector spaces by checking that all the axioms of Definition 37 hold.

Solution:
This is a tedious exercise that is omitted from these solutions. However, you should still work through them on your own! If you get stuck get help at the Math & Stats Learning Centre.

4.3 Subspaces

58) Show that

$$W = \{(x, 0, z) \in \mathbb{R}^3 \mid x, z \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^3$ with the standard operations.

Solution:
Note: W can be interpreted as the xz-plane:

Closure with respect to addition ?:
Consider

$$w_1 = (x_1, 0, z_1) \in W, \quad w_2 = (x_2, 0, z_2) \in W,$$

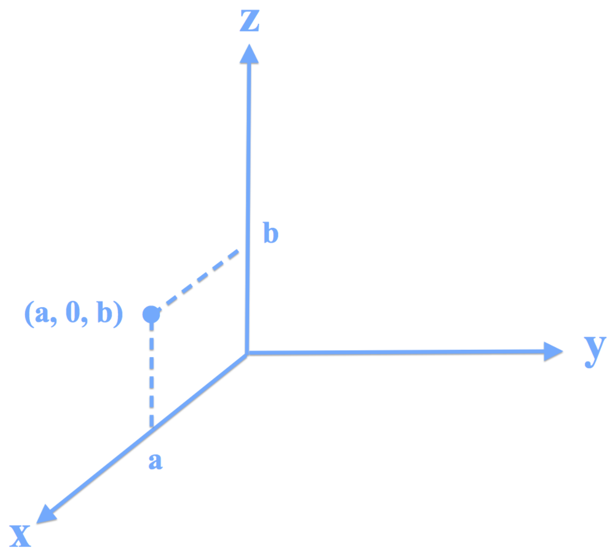
then

$$\begin{aligned} w_1 + w_2 &= (x_1, 0, z_1) + (x_2, 0, z_2) \\ &= (x_1 + x_2, 0, z_1 + z_2) \\ &= (x_3, 0, z_3) \in W \quad \checkmark, \end{aligned}$$

where $x_3 := x_1 + x_2$ and $z_3 := z_1 + z_2$. I.e., W is closed under (vector) addition.

Closure with respect to scalar multiplication ?:
Consider

$$w = (x, 0, z) \in W, \quad k \in \mathbb{R},$$



then

$$\begin{aligned} kw &= k(x, 0, z) \\ &= (kx, 0, kz) \\ &= (\widehat{x}, 0, \widehat{z}) \in W \quad \checkmark, \end{aligned}$$

where $\widehat{x} := kx$ and $\widehat{z} := kz$. I.e., W is closed under scalar multiplication. Thus according to Theorem 15 W is a subspace of $\mathbb{R}^3$ (and hence a vector space in its own right with the same operations as for $\mathbb{R}^3$).

59) Which of these two sets is a subspace of $\mathbb{R}^2$?
a) The set of points on the line $x + 2y = 0$.
b) The set of points on the line $x + 2y = 1$.

Solution:
(a) We are restricted to the vectors on the line $y = -\frac{1}{2}x$. I.e., the set

$$W = \left\{ \left(x, -\frac{1}{2}x \right) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.$$

Closure with respect to addition ?:
Consider

$$w_1 = (x_1, -\frac{1}{2}x_1) \in W, \quad w_2 = (x_2, -\frac{1}{2}x_2) \in W,$$

then

$$\begin{aligned} w_1 + w_2 &= (x_1, -\frac{1}{2}x_1) + (x_2, -\frac{1}{2}x_2) \\ &= (x_1 + x_2, -\frac{1}{2}(x_1 + x_2)) \\ &= (x_3, -\frac{1}{2}x_3) \in W \quad \checkmark, \end{aligned}$$

where $x_3 := x_1 + x_2$, I.e., W is closed under addition.

Closure with respect to scalar multiplication ?:
Consider

$$w = (x, -\frac{1}{2}x) \in W, \quad k \in \mathbb{R},$$

then

$$\begin{aligned} kw &= k(x, -\frac{1}{2}x) \\ &= (kx, k(-\frac{1}{2}x)) \\ &= (kx, -\frac{1}{2}(kx)) \\ &= (\widehat{x}, -\frac{1}{2}\widehat{x}) \in W \quad \checkmark, \end{aligned}$$

where $\widehat{x} := kx$, i.e., W is closed under scalar multiplication.

Thus W is a subspace of $\mathbb{R}^2$.

(b) We are restricted to vectors on the line $y = \frac{1}{2} - \frac{1}{2}x$. I.e., the set

$$W = \left\{ \left(x, \frac{1}{2} - \frac{1}{2}x \right) \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}.$$

Closure with respect to addition ?:
Consider

$$w_1 = (x_1, \frac{1}{2} - \frac{1}{2}x_1) \in W, \quad w_2 = (x_2, \frac{1}{2} - \frac{1}{2}x_2) \in W,$$

then

$$\begin{aligned} w_1 + w_2 &= (x_1, \frac{1}{2} - \frac{1}{2}x_1) + (x_2, \frac{1}{2} - \frac{1}{2}x_2) \\ &= (x_1 + x_2, 1 - \frac{1}{2}(x_1 + x_2)) \\ &= (x_3, 1 - \frac{1}{2}x_3) \notin W, \end{aligned}$$

where $x_3 := x_1 + x_2$, I.e., W is NOT closed under addition. We could stop here as all we need is one of the closure properties to not hold to know that W is not a vector space. For completeness we also check scalar multiplication.

Closure with respect to scalar multiplication ?:
Consider

$$w = (x, \frac{1}{2} - \frac{1}{2}x) \in W, \quad k \in \mathbb{R},$$

then

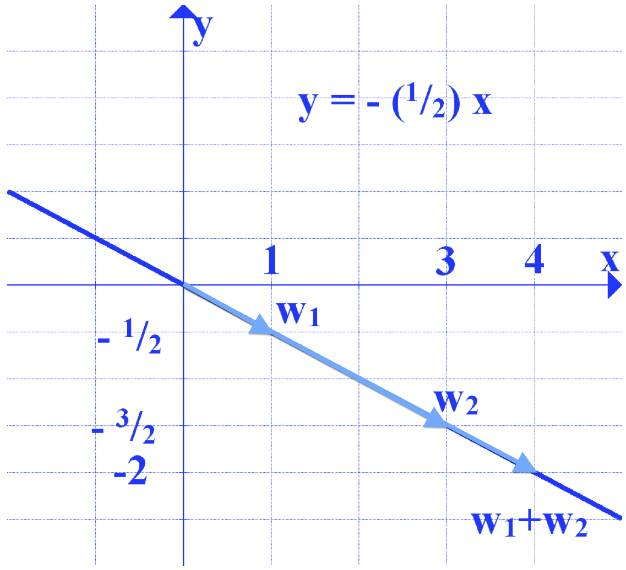
$$\begin{aligned} kw &= k(x, \frac{1}{2} - \frac{1}{2}x) \\ &= (kx, \frac{k}{2} - \frac{k}{2}x)) \\ &= (kx, \frac{k}{2} - \frac{1}{2}(kx)) \\ &= (\widehat{x}, \frac{k}{2} - \frac{1}{2}\widehat{x}) \notin W \quad \text{(unless } k = 1\text{)}, \end{aligned}$$

where $\widehat{\boldsymbol{x}} := \boldsymbol{kx}$, i.e., $\boldsymbol{W}$ is NOT closed under scalar multiplication, thus again we have proved that $\boldsymbol{W}$ is not a subspace of $\mathbb{R}^2$. (E.g., taking $\boldsymbol{k} = \boldsymbol{0}$ yields $\boldsymbol{kw} = (\boldsymbol{0}, \boldsymbol{0})$ which doesn't lie on the line - see below.)

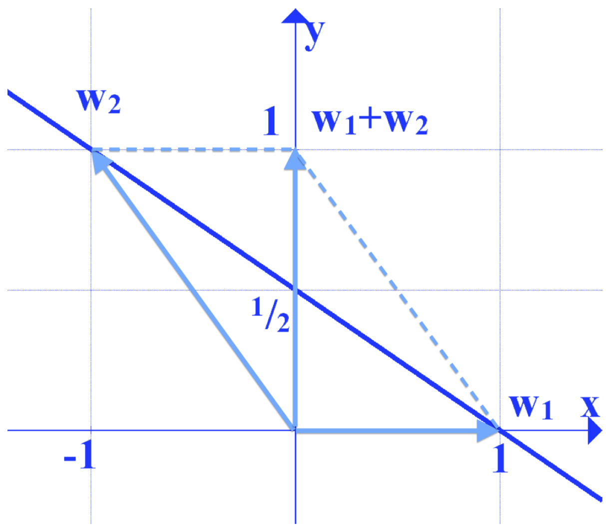
Existence of a zero vector ?

Alternatively, we know from lecture notes that if $\boldsymbol{W}$ is a subspace of $\mathbb{R}^2$ then $\boldsymbol{W}$ must possess the same zero vector as $\mathbb{R}^2$ (i.e., $(\boldsymbol{0}, \boldsymbol{0}) \equiv \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$), which it clearly doesn't as the set of points defining $\boldsymbol{W}$ lie on a straight line that *doesn't* pass through the origin. Hence once again we see that $\boldsymbol{W}$ cannot be a subspace of $\boldsymbol{W}$.

60) Draw diagrams to illustrate the results we found for (vector) addition in Exercise **59**). For part (a) choose $\boldsymbol{w_1} = (\boldsymbol{1}, -\frac{\boldsymbol{1}}{\boldsymbol{2}})$, $\boldsymbol{w_2} = (\boldsymbol{3}, -\frac{\boldsymbol{3}}{\boldsymbol{2}})$, and for part (b) choose $\boldsymbol{w_1} = (\boldsymbol{1}, \boldsymbol{0})$ and $\boldsymbol{w_2} = (-\boldsymbol{1}, \boldsymbol{1})$.



(a)



(b)

Solution:

(a) First observe that

$$w_1 + w_2 = (1, -\frac{1}{2}) + (3, -\frac{3}{2}) = (4, -2) \in W \quad \checkmark.$$

Showing this vector addition on a diagram (see below) we see that the resultant vector w_3 remains on the straight line, which illustrates that vector addition is closed. (The parallelogram law for vector addition still holds, but with a parallelogram of zero area.)

(b) Observe that

$$w_1 + w_2 = (1, 0) + (-1, 1) = (0, 1) \notin W.$$

Showing this vector addition on a diagram (see below) we see that the resultant vector w_3 is not on the straight line, which illustrates that vector addition is not closed. (The usual parallelogram law is clearly see.)

61) Prove that

$$W = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \quad \text{and} \quad x_2 \geq 0\}$$

with the standard operations is *not* a subspace of $\mathbb{R}^2$.

Solution:

Clearly W is closed under vector addition (two non-negative numbers added together always yields a non-negative number). However, for scalar multiplication observe (with a scalar $k = -1$):

$$-1 \cdot (1, 1) = (-1, -1) \notin W,$$

so W is not closed under scalar multiplication and hence is not a subspace of $\mathbb{R}^2$.

Note:

remember, to show that a set is *not* a closed only a single counter example is needed.

62) Prove that the set

$$2. \quad W = \left\{ \begin{pmatrix} x \\ 3x \end{pmatrix} \in \mathbb{R}^2 \mid x \in \mathbb{R} \right\}$$

with the standard operations is a subspace of $\mathbb{R}^2$.

Solution:

(i) Let

$$w_1 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} \in W, \quad w_2 = \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} \in W,$$

so

$$w_1 + w_2 = \begin{pmatrix} x_1 \\ 3x_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ 3x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 3(x_1 + x_2) \end{pmatrix} = \begin{pmatrix} x_3 \\ 3x_3 \end{pmatrix} \in W,$$

where $x_3 := x_1 + x_2$. So W is closed under vector addition.

(ii) Let

$$w = \begin{pmatrix} x \\ 3x \end{pmatrix} \in W, \quad k \in \mathbb{R},$$

then

$$k \begin{pmatrix} x \\ 3x \end{pmatrix} = \begin{pmatrix} kx \\ 3(kx) \end{pmatrix} = \begin{pmatrix} \widehat{x} \\ 3\widehat{x} \end{pmatrix} \in W,$$

where $\widehat{x} := kx$. So W is closed with scalar multiplication. Hence W is a subspace of $\mathbb{R}^2$.

63) Is the set of all vectors of the form

$$\begin{pmatrix} a \\ b \\ a + 2b \end{pmatrix}$$

where $a, b \in \mathbb{R}$, a subspace of $\mathbb{R}^3$?

Solution:

Closure with respect to addition ?:

Just add to vectors of the given form together:

$$\begin{pmatrix} a_1 \\ b_1 \\ a_1 + 2b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \\ a_2 + 2b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ a_1 + a_2 + 2(b_1 + b_2) \end{pmatrix} = \begin{pmatrix} a_3 \\ b_3 \\ a_3 + 2b_3 \end{pmatrix},$$

where $a_3 := a_1 + a_2$ and $b_3 := b_1 + b_2$. So yes, the set is closed with respect to addition.

Closure with respect to scalar multiplication ?:
Consider $k \in \mathbb{R}$ times a representative member of the given set:

$$k \begin{pmatrix} a \\ b \\ a+2b \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ k(a+2b) \end{pmatrix} = \begin{pmatrix} ka \\ kb \\ ka+2(kb) \end{pmatrix} = \begin{pmatrix} \widehat{a} \\ \widehat{b} \\ \widehat{a}+2\widehat{b} \end{pmatrix},$$

where $\widehat{a} := ka$ and $\widehat{b} := kb$. So yes, the set is closed with respect to scalar multiplication. Thus according to Theorem 15 the given set of vectors is a subspace of $\mathbb{R}^3$.

64) Let W be the set of all nonsingular 2-by-2 matrices. Prove that W is NOT a subspace of M_{22} (the vector space of all 2-by-2 matrices with the usual operations)?
(**Hint**: all you need is a single counter-example for the closure property of vector addition, i.e., choose two nonsingular matrices whose sum is singular. Recall that we can tell easily if a matrix is singular or not from Theorem 11.)

Solution:
First recall from Theorem 11 that

$$A \quad \text{nonsingular} \quad \iff \quad |A| \neq 0.$$

We prove that W is not closed under (vector) addition. So using the hint we choose two nonsingular matrices whose sum is singular. Observe that with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have $|A| = |B| = 1$ (i.e., A and B are nonsingular), but

$$|A+B| = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$$

I.e., $A+B$ is singular. Thus W not closed under addition and so W is not a subspace of M_{22}.
Exercise: replace B with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to get the same conclusion.

65) Prove the corresponding result to Exercise **64**) for *singular* matrices.

Solution:
First recall from Theorem 11 that

$$A \quad \text{singular} \quad \iff \quad |A| = 0,$$

(the contrapositive of the statement given in Exercise **64**). We prove that W is not closed under (vector) addition. So using the hint we choose two singular matrices whose sum is nonsingular. Observe that with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have $|A| = |B| = 0$ (i.e., A and B are singular), but

$$|A+B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

I.e., $A+B$ is nonsingular. Thus W not closed under addition and so W is not a subspace of M_{22}.

66) Give a geometrical interpretation of why the set of vectors on the unit circle (centred at the origin) is not a subspace of $\mathbb{R}^2$.

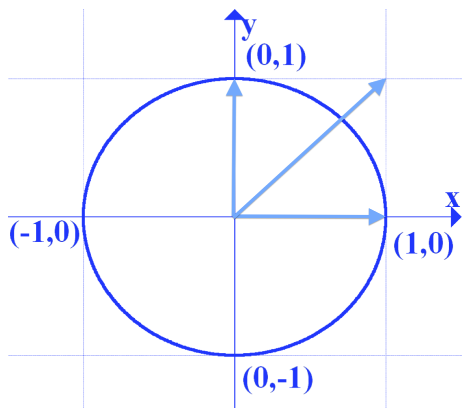
Solution:
Consider the two vectors on the unit circle:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Illustrated below. Adding them together

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

yields a vector that is clearly not on the circle. Hence the given subset of vector is not closed under vector addition and so cannot be a subspace of $\mathbb{R}^2$. We could also note that the set of vectors on the unit circle do not possess the zero vector for $\mathbb{R}^2$ (namely, the origin), and so (by Corollary 4 to Theorem 15) it cannot be a subspace.



4.4 Span

67) Consider the following set of vector in $\mathbb{R}^3$:

$$S = \{\underbrace{(1,3,1)}_{v_1}, \underbrace{(0,1,2)}_{v_2}, \underbrace{(1,0,-5)}_{v_3}\}.$$

Verify that the vector v_1 can be written as a linear combination of v_2 and v_3, in the form $v_1 = 3v_2 + v_3$.

Solution:
Just observe

$$\begin{aligned} 3v_2 + v_3 &= 3(0,1,2) + (1,0,-5) \\ &= (1,3,1) \\ &\equiv v_1 \quad \checkmark \end{aligned}$$

68) Consider the following set of vectors in M_{22}:

$$S = \left\{ \underbrace{\begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}}_{v_3}, \underbrace{\begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix}}_{v_4} \right\}.$$

Verify that the vector v_1 can be written as a linear combination of v_2, v_3, and v_4, in the form $v_1 = v_2 + 2v_3 - v_4$.

Solution:

$$\begin{aligned} v_2 + 2v_3 - v_4 &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} + 2\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} -2 & 0 \\ 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 8 \\ 2 & 1 \end{pmatrix} \\ &\equiv v_1 \quad \checkmark \end{aligned}$$

69) If possible, write the vector $w = (1, 1, 1)$ as a linear combination of the vectors in the set S:

$$S = \{\underbrace{(1,2,3)}_{v_1}, \quad \underbrace{(0,1,2)}_{v_2}, \quad \underbrace{(-1,0,1)}_{v_3}\}.$$

If there is more than one solution find one particular solution.

Solution:

We seek scalars c_1, c_2, and c_3 such that

$$w = c_1 v_1 + c_2 v_2 + c_3 v_3 \tag{$*$}$$

I.e., we need

$$\begin{aligned} (1,1,1) &= c_1(1,2,3) + c_2(0,1,2) + c_3(-1,0,1) \\ &= (c_1, 2c_1, 3c_1) + (0, c_2, 2c_2) + (-c_3, 0, c_3) \\ &= (c_1 - c_3, 2c_1 + c_2, 3c_1 + 2c_2 + c_3). \end{aligned}$$

Equating corresponding components on both sides yields the linear equations:

$$\begin{cases} c_1 \qquad\quad - c_3 = 1 \\ 2c_1 + \ \ c_2 \qquad\ = 1 \\ 3c_1 + 2c_2 + c_3 = 1 \end{cases}$$

Using elementary row operations, the associated augmented matrix of this system row reduces to (Exercise):

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the system has an infinite number of solutions. The associated linear system is

$$\begin{cases} c_1 \quad - \quad c_3 = 1 \\ \quad c_2 + 2c_3 = -1 \end{cases}$$

Starting from the lasr equation we have $c_2 = -2c_3 - 1$, where c_3 is a free variable. Set $c_3 = \alpha$, so $c_2 = -2\alpha - 1$, and from the first equation we get $c_1 = \alpha + 1$. I.e., we have the infinite solution set

$$\{(\underbrace{\alpha + 1}_{c_1}, \underbrace{-2\alpha - 1}_{c_2}, \underbrace{\alpha}_{c_3}) \mid \alpha \in \mathbb{R}\}.$$

E.g., choosing $\alpha = 1$ yields the particular solution

$$(c_1, c_2, c_3) = (2, -3, 1).$$

So from $(*)$ we have

$$w = 2v_1 - 3v_2 + v_3.$$

70) Try and write the vector

$$w := (1, -2, 2)$$

as a linear combination of the vectors in the set

$$S = \{\underbrace{(1,2,3)}_{v_1}, \quad \underbrace{(0,1,2)}_{v_2}, \quad \underbrace{(-1,0,1)}_{v_3}\}.$$

Why can this not be done? (see Exercise **69**)).

Solution:
Following the procedure from Exercise **69**) yields

$$\begin{cases} c_1 \qquad\quad - c_3 = 1 \\ 2c_1 + \ \ c_2 \qquad = -2 \\ 3c_1 + 2c_2 + c_3 = 2 \end{cases}$$

Using elementary row operations, the associated augmented matrix of this system row reduces to (Exercise):

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Clearly from the last row we see that the associated system of equations is inconsistent, thus there is no solution. Consequently W cannot be written as a linear combination of v_1, v_2 and v_3.

71) Consider the following vectors in $\mathbb{R}^3$:

$$\underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix}-1\\0\\5\end{pmatrix}}_{v_2}, \quad \underbrace{\begin{pmatrix}6\\7\\2\end{pmatrix}}_{v_3}.$$

Using Theorem 1 find an expression for the span (a noun !) of these vectors involving a matrix-vector product.

Solution:

$$\begin{aligned} \operatorname{span}\{v_1, v_2, v_3\} &= \{\alpha v_1 + \beta v_2 + \gamma v_3 \mid \alpha, \beta, \gamma \in \mathbb{R}\} \\ &= \left\{\alpha \begin{pmatrix}1\\2\\3\end{pmatrix} + \beta \begin{pmatrix}-1\\0\\5\end{pmatrix} + \gamma \begin{pmatrix}6\\7\\2\end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R}\right\} \\ &= \left\{\begin{pmatrix}1 & -1 & 6\\2 & 0 & 7\\3 & 5 & 2\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R}\right\}. \end{aligned}$$

72) By considering the span (a noun) of the following vectors in $\mathbb{R}^3$

$$\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}}_{v_1}, \quad \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{v_2}, \quad \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{v_3},$$

show that these vectors span (a verb!) $\mathbb{R}^3$. (If you haven't guessed already, this question is more about terminology and notation than actual math - the math result is actually quite trivial.)

Solution:
We are being asked to show that

$$\text{span}\{v_1, v_2, v_3\} = \mathbb{R}^3.$$

Now

$$\begin{aligned}\text{span}\{v_1, v_2, v_3\} &= \left\{\alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}0\\0\\1\end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R}\right\} \\ &= \left\{\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{R}\right\}.\end{aligned}$$

But α, β, γ are completely arbitrary, so every vector in $\mathbb{R}^3$ can be represented in the span of the three vectors.

73) Find sets of vectors that span
(a) $\mathbb{R}^2$
(b) P_2
(c) M_{22}

Solution:
(a) The set $S = \{(1,0), (0,1)\}$ spans $\mathbb{R}^2$ because any vector $(a, b) \in \mathbb{R}^2$ can be written as

$$(a, b) = a(1, 0) + b(0, 1).$$

(b) The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial $p(x) = a + bx + cx^2 \in P_2$ can be written as

$$p(x) = a(1) + b(x) + c(x^2) = a + bx + cx^2.$$

(c) The set

$$S = \left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$$

spans M_{22} because any $\begin{pmatrix}a & b\\c & d\end{pmatrix} \in M_{22}$ can be written as

$$\begin{pmatrix}a & b\\c & d\end{pmatrix} = a\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + b\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} + c\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + d\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}.$$

74) Do the vectors

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \quad \begin{pmatrix}0\\1\\2\end{pmatrix}, \quad \begin{pmatrix}-2\\0\\1\end{pmatrix}$$

span $\mathbb{R}^3$?

Solution:

Let $\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ (for arbitrary $a, b, c \in \mathbb{R}$). We seek scalars x, y and z such that

$$\begin{aligned} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= x \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \end{aligned} \qquad (*)$$

(Using Theorem 1). Now

$$\begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = -1 \neq 0 \quad \text{(exercise)}$$

so the system $(*)$ has a unique solution for *every* right hand side vector $(a, b, c)^T$. And as the numbers a, b and c are arbitrary, we can write *every* vector in $\mathbb{R}^3$ as a linear combination of the given vectors.

Important note:

Students often ask me

> "what about the case of an infinite number of solutions?"

Remember, if the determinant of the coefficient matrix was *equal* to zero there are two possibilities - either no solution, or an infinite number of solutions. So it's natural to wonder for these sorts of problems if it is sufficient (for the vectors to span the space) just to check if the determinant of the coefficient matrix is non-zero. The answer is "yes". Why? Suppose we are going to get an infinite number of solutions associated with the augmented matrix of the form

$$\left(\begin{array}{ccc|c} * & * & * & a \\ * & * & * & b \\ * & * & * & c \end{array}\right).$$

For there to be an infinite number of solutions we must have free variables because one or more rows during the row reduction procedure are zero. However, this will *always* lead to a restriction on the parameters a, b and c. E.g., suppose (for the sake of argument) that we apply the row operations $r_2 - 3r_1 \to r_2$, $r_3 + 2r_1 \to r_3$, and $r_3 - r_2 \to$ (typical sort of operations used to get the matrix in upper triangular form) leading to the last row of the augmented matrix having all zeros. Then the equivalent augmented matrix will be in the form

$$\left(\begin{array}{ccc|c} * & * & * & a \\ * & * & * & b - 3a \\ 0 & 0 & 0 & c - b + 5a \end{array}\right).$$

And for a consistent set of equations we must impose the restriction $c - b + 5a = 0$. So all vectors $(a, b, c)^T$ that don't lie on this plane (i.e. $c - b + 5a \neq 0$) will yield an inconsistent system, and so can't be written as a linear combination of the given vectors.

Well, you asked!

4.5 Linear Independence

75) Determine whether the following set of vectors in $\mathbb{R}^3$ is linearly independent or linearly dependent:

$$S = \left\{ \underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}0\\1\\2\end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix}-2\\0\\1\end{pmatrix}}_{v_2} \right\}.$$

Solution:

We seek scalars α_1, α_2 and $\alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0, \qquad (*)$$

i.e.,

$$\alpha_1 \begin{pmatrix}1\\2\\3\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\2\end{pmatrix} + \alpha_3 \begin{pmatrix}-2\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\text{i.e.} \quad \begin{pmatrix}1 & 0 & -2\\2 & 1 & 0\\3 & 2 & 1\end{pmatrix} \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Applying Gauss-Jordon elimination to the associated augmented matrix yields (exercise):

$$\left(\begin{array}{ccc|c}1 & 0 & -2 & 0\\2 & 1 & 0 & 0\\3 & 2 & 1 & 0\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right).$$

Thus we have the unique ('trivial') solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$ to $(*)$ and so the set S is linearly independent.

76) Determine whether the following set of vectors in $\mathbb{R}^2$ is linearly independent or linearly dependent:

$$S = \left\{ \underbrace{\begin{pmatrix}2\\4\end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix}-3\\-6\end{pmatrix}}_{v_2} \right.$$

Solution:

You may have spotted that these vectors are clearly scalar multiples of each other, e.g. $v_1 = (-2/3)v_2$ (hence the vectors are l.d.), but the point here is to learn a consistent methodology to find all possible solutions, even for more complicated problems involving multiple vectors.

We seek scalars α_1 and $\alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \alpha_2 v_2 = 0, \qquad (*)$$

i.e.,

$$\alpha_1 \begin{pmatrix}2\\4\end{pmatrix} + \alpha_2 \begin{pmatrix}-3\\-6\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\text{i.e.} \quad \begin{pmatrix}2 & -3\\4 & -6\end{pmatrix} \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Applying Gauss-Jordon elimination to the associated augmented matrix yields (exercise):

$$\left(\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & -6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

I.e., we have the single equation $\alpha_1 - (3/2)\alpha_2$, where α_2 is free. Thus we have an infinite number of solutions to $(*)$. E.g. if $\alpha_2 = 2/3$ then $\alpha_1 = 1$, so the particular solution to $(*)$ is $v_1 + (2/3)v_2 = 0$, or $v_1 = (-2/3)v_2$, as we (may have) guessed from the start.

77) Determine whether the set of vectors given below is linearly independent or linearly dependent:

$$S = \{\underbrace{1 + x - 2x^2}_{v_1}, \underbrace{2 + 5x - x^2}_{v_2}, \underbrace{x + x^2}_{v_3}\} \subset P_2.$$

Solution:
The method for polynomials is the same as for other vectors. As usual, we seek $c_1, c_2, c_3 \in \mathbb{R}$ such that*

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \qquad (\star)$$

(Here of course we understand that $0 = 0 + 0x + 0x^2$). So we have

$$c_1(1 + x - 2x^2) + c_2(2 + 5x - x^2) + c_3(x + x^2) = 0,$$

or, after a bit of tidying up

$$(c_1 + 2c_2) + (c_1 + 5c_2 + c_3)x + (-2c_1 - c_2 + c_3)x^2 = 0.$$

Now equating coefficients of like powers of x on both sides yields:

$$\begin{array}{lrcrcrcl} x^0: & c_1 & + & 2c_2 & & & = & 0 \\ x^1: & c_1 & + & 5c_2 & + & c_3 & = & 0 \\ x^2: & -2c_1 & - & c_2 & + & c_3 & = & 0 \end{array}$$

Writing these equations as a single matrix equation:

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Applying Gaussian elimination to to the augmented matrix yields (exercise):

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The last row of zeros tells us we have free variables (more unknowns than equations) so there will be an infinite number of solutions. So the equation $(\star)$ must have a non-trivial solution, i.e., the original vectors are linearly dependent.

78) Determine whether the set of vectors given below is linearly independent or linearly dependent:

$$S = \left\{ \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}}_{v_3} \right\} \subset M_{22}$$

*I'm getting lazy, so I'm using c's instead of α's.

Solution:
The same method applies. We seek scalars $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \qquad (\star)$$

I.e.,

$$c_1 \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 2c_1 & c_1 \\ 0 & c_1 \end{pmatrix} + \begin{pmatrix} 3c_2 & 0 \\ 2c_2 & c_2 \end{pmatrix} + \begin{pmatrix} c_3 & 0 \\ 2c_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

or

$$\begin{pmatrix} 2c_1 + 3c_2 + c_3 & c_1 \\ 2c_2 + 2c_3 & c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Equating corresponding entries of the matrices on both sides:

$$\begin{cases} 2c_1 + 3c_2 + c_3 = 0 \\ c_1 = 0 \\ 2c_2 + 2c_3 = 0 \\ c_1 + c_2 = 0 \end{cases}$$

Note that there is no need to apply row operations to the associated augmented matrix as we can solve this system easily as it stands. The 2nd equation yields $c_1 = 0 \implies c_2 = 0$ (using 4th equation) $\implies c_3 = 0$ (using the 3rd equation). Thus we see from $(\star)$ that the set S is linearly independent.

4.6 Basis and Dimension

Exercise 62 in the Workbook is a key example showing how to verify that a given set of vectors is a basis for a vector space. Once we know the dimension of the vector space concerned, according to Theorem 19, if the number of vectors is equal to the dimension then all we need to check if we have a basis is that either the set of vectors is linear independence or, the vectors span the space (not both). If the number of vectors is not equal to the dimension of the space then the vectors can't be a basis!

We initially give two simple exercises:

79) Prove that

$$\left\{ \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^2$.

Solution:
According to Theorem 19, as $\dim \mathbb{R}^2 = 2$, and the number of vectors we consider is also 2, we need only show that the vectors are linearly independent (or that they span the space). Consider linear independence:

Seek α_1 and α_2 such that

$$\alpha_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.

$$\begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now we can show using Gauss-Jordon elimination that

$$\begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(exercise), or

$$\begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4 \neq 0,$$

and so we have the unique ('trivial solution') $\alpha_1 = \alpha_2 = 0$, i.e., the vectors are linearly independent, and hence by Theorem 19 (i) the vectors are a basis for $\mathbb{R}^2$. So we can stop here, but for completeness we do the argument that also shows that the vectors span $\mathbb{R}^2$ (whether you go for showing linear independence, or span is up to you).

We seek α_1, α_2 such that for any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ we have

$$\alpha_1 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix},$$

i.e.

$$\begin{pmatrix} 3 & -1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

I.e., instead of the homogeneous system $A\underline{x} = \underline{0}$ that we got while considering linear independence, we have the associated nonhomogeneous system $A\underline{x} = \underline{b}$, which has a unique solution (for every right hand side $\underline{b}$) as we already showed as A is nonsingular (because $|A| \neq 0$).

Note:
If the given two vectors were not a basis for $\mathbb{R}^2$ then we would have found that $|A| = 0$, indicating that A is a singular matrix, which tells us that the given two vectors are not linear independent and don't span the space. We now look at two examples of bases for vector spaces other than $\mathbb{R}^n$.

80) Show that the set

$$S = \{2, x-1, x^2+1\}$$

is a basis for P_2.

Solution:
This may look like a different problem, but it is done in exactly the same way. The dimension of P_2 is three, and as we are considering three vectors as a possible basis, according to Theorem 19 to show that we have a basis, all we need do is show the given vectors are linearly independent (or, span P_2).

To check linear independence, we seek $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$2c_1 + c_2(x-1) + c_3(x^2+1) = 0 \equiv 0 + 0x + 0x^2. \qquad (*)$$

This simplifies to:

$$(2c_1 - c_2 + c_3) + c_2 x + c_3 x^2 = 0.$$

Equating corresponding coefficients of like powers on both sides yields:

$$\begin{cases} 2c_1 - c_2 + c_3 = 0 & (x^0) \\ \quad\quad c_2 \quad\quad = 0 & (x^1) \\ \quad\quad\quad\quad c_3 = 0 & (x^2) \end{cases}$$

This readily yields $c_3 = c_2 = 0 \implies c_1 = 0$, hence from $(*)$ we see that the vectors are linearly independent. Alternatively, we could have verified that the vectors span P_2, as shown below:

We seek $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$2c_1 + c_2(x-1) + c_3(x^2+1) = a_0 + a_1 x + a_2 x^2, \qquad (*)$$

for fixed, but arbitrary $a_0, a_1, a_2 \in \mathbb{R}$. This simplifies to:

$$(2c_1 - c_2 + c_3 - a_0) + (c_2 - a_1)x + (c_3 - a_2)x^2 = 0.$$

Equating corresponding coefficients of like powers on both sides yields:

$$\begin{cases} 2c_1 - c_2 + c_3 = a_0 & (x^0) \\ \quad\quad c_2 \quad\quad = a_1 & (x^1) \\ \quad\quad\quad\quad c_3 = a_3 & (x^2) \end{cases}$$

Back-substitution yields a unique solution for any a_0, a_1, a_2. In other words, regardless of the values of a_0, a_1, a_2 we can always find coefficients c_1, c_2, c_3 so that $(*)$ is true, that is the given vectors span P_2. And again, according to Theorem 19 (part (ii)), the vectors are a basis for P_2.

81) Show that the set

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} \right\}$$

is a basis for M_{22}.

Solution:

We do this problem in exactly the same way. Recall from notes that M_{22} has dimension equal to 4, and as the set S has exactly 4 vectors, according to Theorem 19 to show that we have a basis, all we need do is show the given vectors are linearly independent (or, span M_{22}).

For linearly independence we seek $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1 \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} + c_4 \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., after applying the usual operations for matrices (vector addition and scalar multiplication) we have

$$\begin{pmatrix} c_1 - c_2 + c_3 + 3c_4 & 3c_2 + c_3 + 5c_4 \\ -c_1 + 2c_2 - c_3 + c_4 & 2c_1 - c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and so equating corresponding entries in both matrices yields

$$\begin{cases} c_1 - c_2 + c_3 + 3c_4 = 0 \\ -c_1 + 2c_2 - c_3 + c_4 = 0 \\ 3c_2 + c_3 + 5c_4 = 0 \\ 2c_1 - c_4 = 0 \end{cases}$$

Then applying Gauss-Jordon to the associated coefficient matrix yields

$$\begin{pmatrix} 1 & -1 & 1 & 3 \\ -1 & 2 & -1 & 1 \\ 0 & 3 & 1 & 5 \\ 2 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4,$$

i.e., the linear system has the unique ('trivial') solution $c_1 = c_2 = c_3 = c_4 = 0$, i.e. the vectors are linearly independent and hence according to Theorem 19 the given vectors are indeed a basis for M_{22}.

Alternatively, if we decided to show instead that the vectors span M_{22}, we go through almost the same steps (exercise) leading to

$$\begin{cases} c_1 - c_2 + c_3 + 3c_4 = a \\ -c_1 + 2c_2 - c_3 + c_4 = c \\ 3c_2 + c_3 + 5c_4 = b \\ 2c_1 - c_4 = d \end{cases}$$

(with the assumption that a linear combination of the given vectors can be used to construct an arbitrary vector $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{22}$). And as we saw with the linear independence argument the coefficient matrix is non-singular, and so we obtain a unique solution for every right-hand-side vector $(a, c, b, d)^T$. I.e., the vectors do span M_{22}, and hence according to Theorem 19 again we have that the given vectors are a basis for M_{22}.

82) Let

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 7 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 6 \end{pmatrix} \right\}$$

(a) Without doing any calculations why do we know that S is not a basis for $\mathbb{R}^3$?
(b) Show from first principles that S is not a basis for $\mathbb{R}^3$.

Hint: for (b) consider linear independence, but do not apply any row operations to the resulting matrix equation.

Solution:
(a) S is not a basis for $\mathbb{R}^3$ because we are given 4 vectors when $\dim \mathbb{R}^3 = 3$.

(b) Applying the usual argument for checking linear independence: we seek $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -3 \\ 7 \end{pmatrix} + c_4 \begin{pmatrix} -4 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (\star)$$

or (after using Theorem 1)

$$\begin{pmatrix} 1 & 2 & 0 & -4 \\ 2 & 0 & -3 & 1 \\ 3 & -4 & 7 & 6 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We see that the number of equations is $<$ the number of unknowns $\implies$ we have free variables $\implies$ a non-trivial solution $\implies$ from ($\star$) that the given vectors are linearly dependent, and so the vectors are NOT a basis for $\mathbb{R}^3$.

Comment:
However, the vectors do span $\mathbb{R}^3$. In fact the linear system resulting from the usual argument has an infinite number of solutions, so there are an infinite number of ways of constructing an arbitrary vector $(a,\ b,\ c)^T \in \mathbb{R}^3$ using the given vectors (Exercise).

4.7 Homogeneous Systems

83) Consider the homogeneous linear system $A\underline{x} = \underline{0}$ where

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 3 & 6 & -5 & 4 \\ 1 & 2 & 0 & 3 \end{pmatrix}.$$

Find a basis for the null space of A (i.e., $N(A)$). What is the dimension of $N(A)$?

Solution:
As shown in Algorithm 1, applying elementary row operations yields

$$A \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(RREF)}.$$

With $n = 4$ unknowns and $r = 2$ pivot columns yields $p = n - r = 2$ free variables. (It's good to do this little check first so you know how many free variables we will get in the next step.) The associated system of equations is

$$\begin{cases} x_1 + 2x_2 \quad\ + 3x_4 = 0 \\ \qquad\qquad x_3 + \ \ x_4 = 0 \end{cases}$$

The variables x_4 and x_2 are free. Set $x_4 = \alpha$, $x_2 = \beta$ so $x_3 = -\alpha$ and $x_1 = -2\beta - -3\alpha$. So any solution has the form

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2\beta - -3\alpha \\ \beta \\ -\alpha \\ \alpha \end{pmatrix}.$$

Now to find the basis vectors that span the null-space of A observe that

$$\underline{x} = \alpha \underbrace{\begin{pmatrix} -3 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{\underline{x}_1} + \beta \underbrace{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\underline{x}_2}.$$

So $\underline{x}_1$ and $\underline{x}_2$ clearly span the solution space, and as stated in the Algorithm they are also linearly independent. Thus a basis for $N(A)$ is $\{\underline{x}_1, \underline{x}_2\}$ and so the dimension of $N(A)$ is 2.

4.8 Rank of a Matrix

84) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

(a) Find expressions for the row space and the column space of $\boldsymbol{A}$.
(b) What are the dimensions of the row space and the column space of $\boldsymbol{A}$?

Solution:
(a) The row space of $\boldsymbol{A}$ is given by the set

$$\textbf{span}\{(\mathbf{1},\mathbf{0},\mathbf{0}),\ (\mathbf{0},\mathbf{1},\mathbf{0})\} = \{\underbrace{\alpha(\mathbf{1},\mathbf{0},\mathbf{0}) + \beta(\mathbf{0},\mathbf{1},\mathbf{0})}_{=(\alpha,\beta,\mathbf{0})} \mid \alpha,\beta \in \mathbb{R}\}.$$

The column space of $\boldsymbol{A}$ is given by the set

$$\begin{aligned} \textbf{span}\left\{ \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix},\ \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix},\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \right\} &= \left\{\alpha \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \beta \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} + \gamma \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mid \alpha,\beta,\gamma \in \mathbb{R}\right\} \\ &= \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mid \alpha,\beta \in \mathbb{R}\right\} \\ &= \left\{\alpha \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} + \beta \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} \mid \alpha,\beta \in \mathbb{R}\right\}. \end{aligned}$$

(b) The rows of $\boldsymbol{A}$ are linearly independent (exercise) and as they span the row space (by definition of row space) the rows of $\boldsymbol{A}$ are also a basis for the row space. The column space is spanned by $(\mathbf{1},\mathbf{0})^{\boldsymbol{T}}$ and $(\mathbf{0},\mathbf{1})^{\boldsymbol{T}}$ (the vector $(\mathbf{0},\mathbf{0})^{\boldsymbol{T}}$ is redundant as it plays no part in the span of the columns). And as $(\mathbf{1},\mathbf{0})^{\boldsymbol{T}}$ and $(\mathbf{0},\mathbf{1})^{\boldsymbol{T}}$ are also linearly independent (exercise) they constitute a basis for the column space of $\boldsymbol{A}$. So for both the row space and the column space the dimensions are equal, illustrating Theorem 25.

85) Let

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{0} & \mathbf{6} & \mathbf{7} & \mathbf{14} \end{pmatrix}$$

(a) Find the dimension of the row space of $\boldsymbol{A}$ (i.e., the row rank of $\boldsymbol{A}$).

(b) Find the dimension of the column space of $\boldsymbol{A}$ (i.e., the column rank of $\boldsymbol{A}$).

Solution:
(a) We use Theorem 24 and just count the number of non-zero rows in the RREF matrix. So applying elementary row operations to $\boldsymbol{A}$ we get

$$\boldsymbol{A} \longrightarrow \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} \end{pmatrix} \quad \text{(RREF)},$$

thus the dimension of the row space is $\mathbf{2}$.

(b) To find the column rank we apply elementary row operations to $\boldsymbol{A}^{\boldsymbol{T}}$ and count the number of non-zero rows in the RREF matrix:

$$\boldsymbol{A}^{\boldsymbol{T}} \longrightarrow \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{(RREF)},$$

thus the dimension of the row space is $\mathbf{2}$.

Note:
This problem illustrates Theorem 25.

86) Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for arbitrary $a, b, c, d \in \mathbb{R}$. The matrix

$$B = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix},$$

is obtained by applying the row operation $r_1 + r_2 \to r_1$ to A. Show that

$$\text{row space of } A = \text{row space of } B.$$

Solution:
Row space of A:
All entries have the form

$$\alpha_1(a, b) + \alpha_2(c, d) = (\alpha_1 a + \alpha_2 c, \alpha_1 b + \alpha_2 d).$$

Row space of B:
All entries have the form

$$\begin{aligned} \alpha_1(a+c, b+d) + \alpha_2(c, d) &= (\alpha_1(a+c) + \alpha_2 c, \alpha_1(b+d) + \alpha_2 d) \\ &= \big(\alpha_1 a + \underbrace{(\alpha_1 + \alpha_2)}_{\alpha_3} c, \alpha_1 b + \underbrace{(\alpha_1 + \alpha_2)}_{\alpha_3} d\big). \end{aligned}$$

But the α's are completely arbitrary, so the form of entries in the row space of A is the same form in the row space of B. Thus A and B have the same row space.

Note:
This problem can be generalized to prove Theorem 23.

87) Let

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{pmatrix}$$

Find a basis for the row space of A.

Solution:
Applying elementary row operations yields (exercise)

$$A \longrightarrow \begin{pmatrix} 1 & 0 & 13 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(RREF)}$$

Thus from Theorem 24 the nonzero rows of the matrix in RREF, i.e., $(1, 0, 13)$ and $(0, 1, 5)$ form a basis for the row space of A.

88) Find a basis for the subspace V of $\mathbb{R}^3$ spanned by $S = \{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$, where

$$\underline{u}_1 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \underline{u}_2 = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \quad \underline{u}_3 = \begin{pmatrix} 1 \\ -4 \\ -7 \end{pmatrix} \quad (V = \operatorname{span} S),$$

(Hint: see Exercise 68 in the notes.)

Solution:
Now we have that V is equal to the row space of the matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -5 & 1 \\ 1 & -4 & -7 \end{pmatrix}.$$

Thus from Exercise **87**) the first and second rows of the matrix in RREF form a basis for the row space of A. Thus $\{\underline{v}_1, \underline{v}_2\}$ where

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 13 \end{pmatrix}, \quad \underline{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$$

is a basis for $V = \operatorname{span} S$.

89) Find the rank of the following matrices:

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 5 & -1 \\ 0 & 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Solution:
As we saw from Theorem 24 and Theorem 25 in order to find the rank of these matrices we reduce them to RREF and just count the number of non-zero rows:

$$A \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(RREF)},$$

(exercise) and so $\operatorname{rank} A = 3$. Similarly

$$B \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(RREF)},$$

(exercise) and so $\operatorname{rank} B = 2$.

90) Consider the linear system $A\underline{x} = \underline{b}$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 3 & 6 & 1 & -1 \\ 1 & 2 & -2 & -5 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Consider the solvability of this system using the concept of rank.

Solution:
Applying elementary row operations to the augmented matrix yields

$$[A|\underline{b}] \longrightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \quad \text{(RREF)}.$$

Thus we see using Theorem 24 that

$$2 = \text{rank } A \neq \text{rank}[A|\underline{b}] = 3,$$

thus Theorem 28 tells us that the system is inconsistent.

Chapter 5

Inner Product Spaces

5.1 Length and Direction in $\mathbb{R}^n$ $(n \geq 2)$

91) Consider the following vectors in $\mathbb{R}^4$:

$$\underline{u} = (1, -1, 0, 4), \quad \underline{v} = (3, 2, 1, 0).$$

(a) Find the norms of $\underline{u}$ and $\underline{v}$
(b) Find the distance between $\underline{u}$ and $\underline{v}$
(c) Find the standard inner product of $\underline{u}$ and $\underline{v}$
(d) Find $\cos(\theta)$ where θ is the angle between $\underline{u}$ and $\underline{v}$
(e) Find the normalized vector $\widehat{\underline{u}}$ corresponding to $\underline{u}$
(f) What value do we need to change the last component of $\underline{v}$ to in order for $\underline{u}$ and $\underline{v}$ to be orthogonal?

Solution:
(a) The norm of $\underline{u}$ is

$$||\underline{u}|| = \sqrt{1^2 + (-1)^2 + 0^2 + 4^2} = \sqrt{18} \approx 4.24.$$

The norm of $\underline{v}$ is

$$||\underline{v}|| = \sqrt{3^2 + 2^2 + 1^2 + 0^2} = \sqrt{14} \approx 3.74.$$

(b) The distance between $\underline{u}$ and $\underline{v}$ is

$$||\underline{u} - \underline{v}|| = \sqrt{(1-3)^2 + (-1-2)^2 + (0-1)^2 + (4-0)^2} = \sqrt{30} \approx 5.48.$$

(c) The (standard) inner product is given by

$$\underline{u} \cdot \underline{v} = (1)(3) + (-1)(2) + (0)(1) + (4)(0) = 1.$$

(d)

$$\cos(\theta) = \frac{\underline{u} \cdot \underline{v}}{||\underline{u}||\,||\underline{v}||} = \frac{1}{\sqrt{18}\sqrt{14}} = \frac{1}{3\sqrt{2}\sqrt{14}}.$$

(e) All we do is divide each component of $\underline{u}$ by the norm of $\underline{u}$ (yielding a vector of unit length). Thus from (a):

$$\widehat{\underline{u}} = (1/\sqrt{18}, -1/\sqrt{18}, 0, 4/\sqrt{18}).$$

(f) Let $\underline{v} = (3, 2, 1, x)$, where x is to be determined. For $\underline{u}$ and $\underline{v}$ to be orthogonal we need:

$$\underline{u} \cdot \underline{v} = (1)(3) + (-1)(2) + (0)(1) + (4)(x) = 0,$$

i.e.

$$3 - 2 + 4x = 0 \implies 4x = -1 \implies x = -1/4.$$

5.2 Inner Product Spaces

92) Let u and v belong to an inner product space V. Given that $\|u\| = 2$, $\|v\| = 3$ and $(u, v) = 4$ calculate $\|u + 2v\|^2$.

Solution:
We use the axioms of an inner product listed in Definition 55 and the definition of an 'induced' norm in the comments of page 239:

$$\begin{aligned}
\|u + 2v\|^2 &= (u + 2v, u + 2v) \quad \text{(definition of the norm)} \\
&= (u, u + 2v) + (2v, u + 2v) \quad \text{(axiom (iii))} \\
&= (u + 2v, u) + (u + 2v, 2v) \quad \text{(axiom (ii))} \\
&= (u, u) + (2v, u) + (u, 2v) + (2v, 2v) \quad \text{(axiom (iii))} \\
&= (u, u) + (2v, u) + (2v, u) + 2(v, 2v) \quad \text{(axiom (ii) and (iv))} \\
&= (u, u) + 2(v, u) + 2(v, u) + 2(2v, v) \quad \text{(axiom (ii) and (iv))} \\
&= (u, u) + 4(u, v) + 4(v, v) \quad \text{(axiom (ii) and (iv))} \\
&= \|u\|^2 + 4(u, v) + 4\|v\|^2 \quad \text{(definition of the norm)} \\
&= 2^2 + 4(4) + 4(3)^2 \\
&= 56.
\end{aligned}$$

Note:
The above argument is twice as long as it needs to be because of the explicit use of the 'symmetry' axiom (ii). From now on we assume it applies, e.g. $(u, cv) = c(u, v)$ (instead of $(u, cv) = (cv, u) = c(v, u) = c(u, v)$).

93) Consider the standard inner product $(u, v) := u \cdot v$, where $u, v \in \mathbb{R}^3$.
(a) Show that $(u + v) \cdot (u - v) = \|u\|^2 - \|v\|^2$.
(b) Verify the result in (a) for

$$u = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Solution:
(a)

$$\begin{aligned}
(u + v) \cdot (u - v) &= u \cdot u - u \cdot v + v \cdot u - v \cdot v \quad \text{(Using Definition 55)} \\
&= \|u\|^2 - u \cdot v + u \cdot v - \|v\|^2 \quad \text{(Definition of the norm)} \\
&= \|u\|^2 - \|v\|^2 \quad \checkmark
\end{aligned}$$

(b)

$$\begin{aligned}(\boldsymbol{u}+\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v}) &= \left(\begin{pmatrix}1\\-1\\0\end{pmatrix}+\begin{pmatrix}2\\1\\-1\end{pmatrix}\right)\cdot\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}-\begin{pmatrix}2\\1\\-1\end{pmatrix}\right)\\ &= \begin{pmatrix}3\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\1\end{pmatrix}\\ &= -3+0-1\\ &= -4.\end{aligned}$$

And

$$\begin{aligned}\|\boldsymbol{u}\|^2-\|\boldsymbol{v}\|^2 &= \left(1^2+(-1)^2+0^2\right)-\left(2^2+1^2+(-1)^2\right)\\ &= 2-6\\ &= -4 \quad \checkmark\end{aligned}$$

94) Let $\boldsymbol{V}=\boldsymbol{C}[0,1]^*$ with the inner product

$$(\boldsymbol{f},\boldsymbol{g}) := \int_0^1 \boldsymbol{f}(x)\boldsymbol{g}(x)\,dx,$$

and $\boldsymbol{f}(t)=1$ and $\boldsymbol{g}(x)=x$.
(a) Evaluate $(\boldsymbol{f},\boldsymbol{g})$
(b) Without performing any calculations explain why $(\boldsymbol{g},\boldsymbol{g})>0$
(c) Define an induced norm for the inner product space $\boldsymbol{V}$ and hence calculate $\|\boldsymbol{f}\|$ and $\|\boldsymbol{g}\|$
(d) Verify the Cauchy-Schwartz inequality $|(\boldsymbol{f},\boldsymbol{g})|\le\|\boldsymbol{f}\|\|\boldsymbol{g}\|$

Solution:
(a)

$$(\boldsymbol{f},\boldsymbol{g}) = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

(b)

$$(\boldsymbol{g},\boldsymbol{g}) = \int_0^1 x^2\,dx > 0,$$

as this is just the area under the graph of $y=x^2$ from $x=0$ to $x=1$, which is positive.

(c) See the comment on page 239 of the Workbook:

$$\|\boldsymbol{f}\| = \sqrt{(\boldsymbol{f},\boldsymbol{f})} = \sqrt{\int_0^1 1\,dx} = \sqrt{[x]_0^1} = \sqrt{1} = 1.$$

$$\|\boldsymbol{g}\| = \sqrt{(\boldsymbol{g},\boldsymbol{g})} = \sqrt{\int_0^1 x^2\,dx} = \sqrt{\left[\frac{x^3}{3}\right]_0^1} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}.$$

(d) Using the results from (a) and (c) we have

$$0.5 = \frac{1}{2} = |(\boldsymbol{f},\boldsymbol{g})| \le \|\boldsymbol{f}\|\|\boldsymbol{g}\| = 1\cdot\frac{1}{\sqrt{3}} \approx 0.5774 \quad \checkmark$$

*Set of continuous function on the interval $[0,1]$.

95) Complete Exercise 76 (page 243) in the Workbook by deriving formulae for a_0 and a_m corresponding to Definition 57 for Fourier Series.
(Hint: to derive the formulae for a_0 and a_m, multiply $(*)$ (page 240 of the Workbook) by 1 and $\cos(mx)$ respectively, integrate from $-\pi$ to π, and then use the Calculus results on page 241 of the Workbook.)

Solution:

Derivation of the formula for a_0:
The formula $(*)$ is:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Following the hint we integrate both sides from $-\pi$ to π yielding:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\, dx &= a_0 \int_{-\pi}^{\pi} 1\, dx + \sum_{n=1}^{\infty} \left\{ a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\, dx}_{=0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\, dx}_{=0} \right\} \\ &= a_0 [x]_{-\pi}^{\pi} = a_0 \cdot 2\pi, \\ \text{thus} \quad a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx. \end{aligned}$$

Derivation of the formula for a_m:
Following the hint (and using the Calculus results again) yields

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx &= a_0 \underbrace{\int_{-\pi}^{\pi} \cos(mx)\, dx}_{=0} + \sum_{n=1}^{\infty} \left\{ a_n \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx \right. \\ &\qquad \left. + b_n \int_{-\pi}^{\pi} \underbrace{\sin(nx)\cos(mx)\, dx}_{=0} \right\} \\ &= a_m \underbrace{\int_{-\pi}^{\pi} \cos(mx)\cos(mx)\, dx}_{=\pi} \\ \text{thus} \quad a_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx)\, dx. \end{aligned}$$

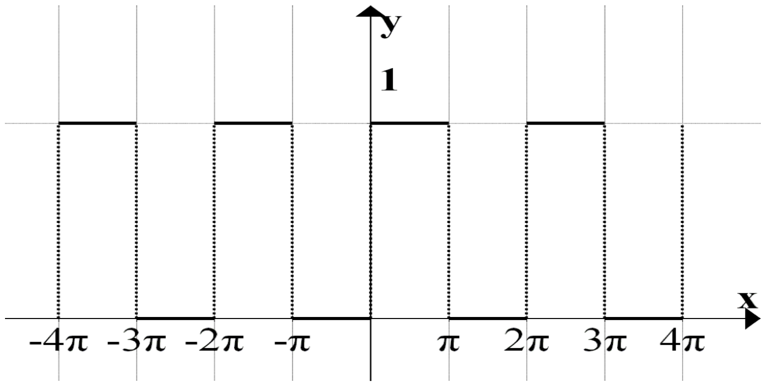
Note on the 2nd to last line above that we used the fact that when $n \neq m$ in the sum then $\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\, dx = 0$. We also can observe that when $m = 0$ the formula for a_m is consistent with the formula for a_0.

96) (a) Using Definition 57 in your Workbook find the Fourier coefficients and Fourier series of the 'square wave' function

$$f(x) = \begin{cases} 0 & \text{if } -\pi \le x < 0 \\ 1 & \text{if } 0 \le x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x),$$

i.e. f is periodic with period 2π and has the shape shown below (only shown on $[-4\pi, 4\pi]$):

(b) Graph several terms of your Fourier series against the square wave to see how well successive numbers of terms approximates the given function.



Solution:

(a) **Finding the Fourier coefficients**:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\,dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dx \\ &= 0 + \frac{1}{2\pi}(\pi) \\ &= \frac{1}{2}. \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\cos(nx)\,dx \\ &= 0 + \frac{1}{\pi}\left[\frac{\sin(nx)}{n}\right]_0^{\pi} \\ &= \frac{1}{n\pi}(\sin(n\pi) - \sin(0)) \\ &= 0. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(nx)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cdot\sin(nx)\,dx \\ &= 0 + \frac{1}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} \\ &= -\frac{1}{n\pi}(\cos(n\pi) - \cos(0)) \qquad (*) \end{aligned}$$

Now

$$\begin{aligned} \cos(0) = 1, &\quad \cos(\pi) = -1, \\ \cos(2\pi) = 1, &\quad cos(3\pi) = -1, \quad \text{etc.} \end{aligned}$$

Thus from $(*)$ we have (recall, $n \geq 1$)

$$b_n = \begin{cases} -\frac{1}{n\pi}(+1-1) & n \text{ even} \\ -\frac{1}{n\pi}(-1-1) & n \text{ odd} \end{cases} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd} \end{cases}.$$

Finding the Fourier Series:
We have

$$\begin{aligned} f(x) = a_0 &+ a_1 \cos(x) + b_1 \sin(x) \\ &+ a_2 \cos(2x) + b_2 \sin(2x) \\ &+ a_3 \cos(3x) + b_3 \sin(3x) + \dots \end{aligned}$$

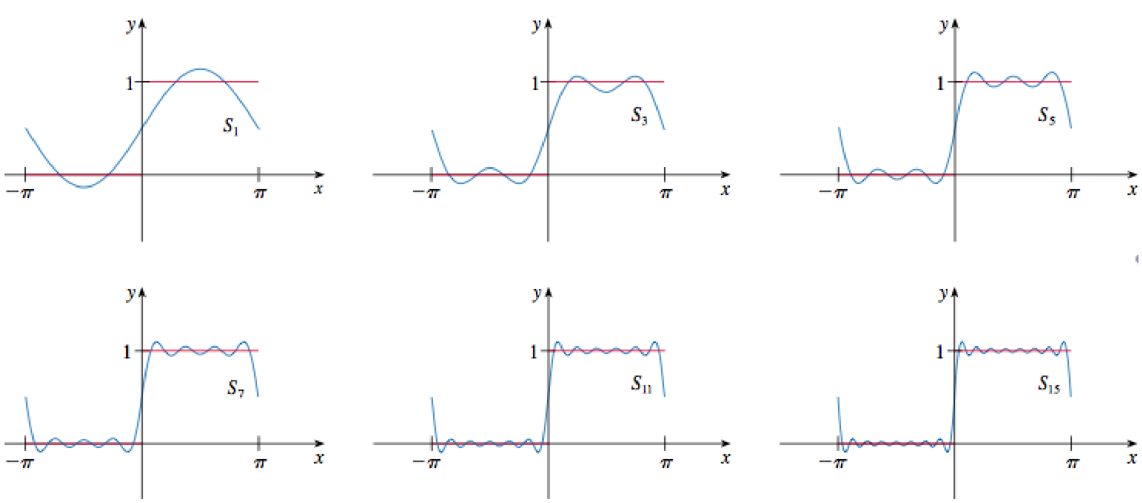
Now we saw that all terms that involve cosine, and sine terms with even multiples of x in the argument are zero. So we have

$$\begin{aligned} f(x) &= \frac{1}{2} + b_1 \sin(x) + b_3 \sin(3x) + b_5 \sin(5x) + \dots \\ &= \frac{1}{2} + \frac{2}{\pi} \sin(x) + \frac{2}{3\pi} \sin(3x) + \frac{2}{5\pi} \sin(5x) + \dots \end{aligned}$$

(b) It is natural to wonder how well the series approximates the square wave when only a few terms are used. In the graphs below† we denote

$$S_n = \frac{1}{2} + \frac{2}{\pi} \sin(x) + \frac{2}{3\pi} \sin(3x) + \dots + \frac{2}{n\pi} \sin(nx).$$

Even though the square wave is discontinuous and the partial sums S_n are continuous, the more terms in the Fourier series we use the more closely we approximate the square wave.



97) Let u and v be vectors in an inner product space V. Prove that

$$\|u + v\| \leq \|u\| + \|v\|$$

with equality if and only if u and v are orthogonal.

†Reproduced from 'Stewart: Calculus, Sixth Edition.

(**Hint**: start by expanding $\|\boldsymbol{u}+\boldsymbol{v}\|^2$ with the aid of the rules in Definition 55, use the definition of an induced norm (see page 239 in the Workbook), and then use the Cauchy-Schwartz inequality.)

Solution:

$$\begin{aligned}\|\boldsymbol{u}+\boldsymbol{v}\|^2 &= (\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v})\\ &= (\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u})+(\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v})\\ &= (\boldsymbol{u},\boldsymbol{u})+(\boldsymbol{v},\boldsymbol{u})+(\boldsymbol{u},\boldsymbol{v})+(\boldsymbol{v},\boldsymbol{v})\\ &= \|\boldsymbol{u}\|^2+2(\boldsymbol{u},\boldsymbol{v})+\|\boldsymbol{v}\|^2 \qquad (*)\end{aligned}$$

Using the Cauchy-Schwartz inequality (see Theorem 33) we have

$$\|\boldsymbol{u}+\boldsymbol{v}\|^2 \le \|\boldsymbol{u}\|^2+2\|\boldsymbol{u}\|\|\boldsymbol{v}\|+\|\boldsymbol{v}\|^2 = (\|\boldsymbol{u}\|+\|\boldsymbol{v}\|)^2,$$

and so taking the square root of both sides yields

$$\|\boldsymbol{u}+\boldsymbol{v}\| \le \|\boldsymbol{u}\|+\|\boldsymbol{v}\| \quad \checkmark$$

We also observe from $(*)$ that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal (i.e. $(\boldsymbol{u},\boldsymbol{v})=\boldsymbol{0}$) then we get the equality case: $\|\boldsymbol{u}+\boldsymbol{v}\|=\|\boldsymbol{u}\|+\|\boldsymbol{v}\|$.

Chapter 6

Eigenvalues and Eigenvectors

6.1 Eigenvalues and Eigenvectors

98) Prove Theorem 35 in the Workbook which states: if $\boldsymbol{A}$ is a square $\boldsymbol{n}\times\boldsymbol{n}$ matrix and $\boldsymbol{\lambda}$ is an eigenvalue of $\boldsymbol{A}$, then the eigenspace of $\boldsymbol{\lambda}$ is a subspace of $\mathbb{R}^{\boldsymbol{n}}$.

Solution:
The eigenspace of $\boldsymbol{\lambda}$ is given by

$$\boldsymbol{V}=\{\boldsymbol{x}\in\mathbb{R}^{\boldsymbol{n}} \mid \boldsymbol{A}\boldsymbol{x}=\boldsymbol{\lambda}\boldsymbol{x}\}.$$

So $\boldsymbol{x}_1,\boldsymbol{x}_2\in\boldsymbol{V}$ then $\boldsymbol{A}\boldsymbol{x}_1=\boldsymbol{\lambda}\boldsymbol{x}_1$ and $\boldsymbol{A}\boldsymbol{x}_2=\boldsymbol{\lambda}\boldsymbol{x}_2$. Now consider

$$\boldsymbol{A}(\boldsymbol{x}_1+\boldsymbol{x}_2)=\boldsymbol{A}\boldsymbol{x}_1+\boldsymbol{A}\boldsymbol{x}_2=\boldsymbol{\lambda}\boldsymbol{x}_1+\boldsymbol{\lambda}\boldsymbol{x}_2=\boldsymbol{\lambda}(\boldsymbol{x}_1+\boldsymbol{x}_2),$$

hence $\boldsymbol{x}_1+\boldsymbol{x}_2\in\boldsymbol{V}$, i.e., $\boldsymbol{V}$ is closed under vector addition. Also consider $\boldsymbol{x}\in\boldsymbol{V}$ and $\boldsymbol{k}\in\mathbb{R}$, then

$$\boldsymbol{A}(\boldsymbol{k}\boldsymbol{x})=\boldsymbol{k}\boldsymbol{A}\boldsymbol{x}=\boldsymbol{k}\boldsymbol{\lambda}\boldsymbol{x}=\boldsymbol{\lambda}(\boldsymbol{k}\boldsymbol{x}),$$

hence $\boldsymbol{k}\boldsymbol{x}\in\boldsymbol{V}$, i.e., $\boldsymbol{V}$ is closed under scalar multiplication. Thus according to Theorem 15 $\boldsymbol{V}$ is a subspace of $\mathbb{R}^{\boldsymbol{n}}$.

A note about notation:
Notice how I didn't underline vectors in this problem, i.e. I used x instead of $\underline{x}$. Now most advanced textbooks don't bother with the underlying as the context should make it clear when we are dealing with a vector and when we are dealing with a scalar.* However, for the standard eigenvector-eigenvector problems (see next exercise) we really do need to distinguish between a vector and its components, so I do underline vectors.

99) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}.$$

Solution:
We seek the solution of $A\underline{x} = \lambda\underline{x}$, i.e. find the eigenvalues λ and the associated eigenvectors $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such that

$$\begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

I.e.

$$\begin{cases} 2x_1 - 12x_2 = \lambda x_1 \\ x_1 - 5x_2 = \lambda x_2 \end{cases}$$

or,

$$\begin{cases} (2-\lambda)x_1 - \qquad\quad 12x_2 = 0 \\ \qquad\quad x_1 + (-5-\lambda)x_2 = 0 \end{cases}$$

which in matrix form is

$$\begin{pmatrix} (2-\lambda) & -12 \\ 1 & (-5-\lambda) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (*)$$

We seek the non-trivial solutions ($\underline{x} \neq \underline{0}$) so we know from lecture notes that we need

$$|A - \lambda I| = \begin{vmatrix} (2-\lambda) & -12 \\ 1 & (-5-\lambda) \end{vmatrix} = 0,$$

i.e.

$$\begin{aligned} &(2-\lambda)(-5-\lambda) + 12 = 0 \\ &\implies -10 - 2\lambda + 5\lambda + \lambda^2 + 12 = 0 \\ &\implies \lambda^2 + 3\lambda + 2 = 0 \\ &\implies (\lambda+1)(\lambda+2) = 0. \end{aligned}$$

Hence we have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. To find the associated eigenvectors we solve $(*)$ for $\underline{x}$ for each value of λ.

*It also makes it quicker for me to type up the solutions :).

Case $\lambda_1 = -1$:
From $(*)$ we have

$$\begin{pmatrix} 3 & -12 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Elementary row operations yield (exercise)

$$\begin{pmatrix} 3 & -12 \\ 1 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix},$$

thus $x_1 - 4x_2 = 0$, or $x_1 = 4x_2$ where x_2 is free. Set $x_2 = \alpha$ so the eigenvectors associated with the eigenvalue λ_1 have the form

$$\underline{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

E.g., for $\alpha = 1$ we get the particular eigenvector $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

Case $\lambda_2 = -2$:
From $(*)$ we have

$$\begin{pmatrix} 4 & -12 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Elementary row operations yield (exercise)

$$\begin{pmatrix} 4 & -12 \\ 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix},$$

thus $x_1 - 3x_2 = 0$, or $x_1 = 3x_2$ where x_2 is free. Set $x_2 = \beta$ so the eigenvectors associated with the eigenvalue λ_2 have the form

$$\underline{x}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3\beta \\ \beta \end{pmatrix} = \beta \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

E.g., for $\beta = 2$ we get the particular eigenvector $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

6.2 Diagonalization and Similar Matrices

100) Diagonalize the matrix A from Exercise **99**). Justify this process.

Solution:
We saw in Exercise **99**) that the eigenvectors of A were distinct (i.e., different), thus we know from Corollary 6 of Theorem 38 that A is indeed diagonalizable. We had $A = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}$ with $\lambda_1 = -1$ and $\lambda_2 = -2$. Corresponding to λ_1 and λ_2 we found the particular eigenvalues

$$\underline{x}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \text{and} \quad \underline{x}_2 = \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

(Note: to diagonalize A it doesn't matter which particular eigenvectors you use.) We let

$$P = [\underline{x}_1 | \underline{x}_2] = \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Let's check our answers. First observe that $|\boldsymbol{P}| = 8 - 6 = 2 \neq 0$ so $\boldsymbol{P}$ is nonsingular. Then using Theorem 37

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -6 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} \quad \checkmark.$$

101) Use diagonalization of the matrix $\boldsymbol{A}$ in Exercise **99**) to calculate $\boldsymbol{A}^{10}$.

Solution:

Recall from Exercise **100**) we found that

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -6 \\ -1 & 4 \end{pmatrix}.$$

Thus using Theorem 39 we have

$$\begin{aligned} \boldsymbol{A}^{10} = \boldsymbol{P}\boldsymbol{D}^{10}\boldsymbol{P}^{-1} &= \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^{10} & 0 \\ 0 & (-2)^{10} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -6 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 6 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1024 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 2 & -6 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} -3068 & 12276 \\ -1023 & 4093 \end{pmatrix}. \end{aligned}$$

Note:

This is easily checked in Matlab.