UNIVERSITY of GUELPH

Fall 2017 | MATH*1030 | Midterm 2

Last name (print): ________________ **First** name (print): ________________

Student #: ____________________ Signature: ________________________

1. This is a 50 minute test. Do NOT start until instructed.
2. Please fill out your personal details above. Once the examination starts fill out your solutions in the spaces provided.
3. You may quote results used in your arguments from lecture notes without justification, unless asked to do otherwise. Notes or books are not permitted during the exam.
4. Give sufficient working as solutions with no justification will receive little or no credit.
5. You may use a scientific calculator and/or a 'graphing calculator'. Note however that graphical answers require justification. No other electronic equipment is permitted. **Turn off or 'mute' your cell phone! The use of cell phones during the exam is prohibited.**
6. If you copy the work of your neighbour this is considered **Academic Misconduct** and will be reported to the appropriate university authority.
7. There are a total of **44** points to be awarded on this test.

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1. (16 marks)

(a) Producing x units of hot dogs costs

$$C(x) = 6x + 40 \quad \text{dollars.}$$

The revenue earned is given by $R(x) = 10x$ dollars.

(i) What is the break-even quantity?
(ii) What is the profit from selling 50 units?
(iii) How many units must be sold to give a profit of $\$100$?

(i) We seek x such that $R(x) = C(x)$, ······ (1) (method)

i.e., $10x = 6x + 40 \implies 4x = 40 \implies x = 10$ units ······ (1) (answer)

(ii) $P(x) = R(x) - C(x) = 10x - (6x + 40) = 4x - 40$ ······ (1) (method)

so $P(50) = 4(50) - 40 = 200 - 40 = \$160$ ······ (1) (answer)

(iii) We seek x such that $P(x) = 100$, ······ (1) (method)

i.e., $4x - 40 = 100 \implies 4x = 140 \implies x = 35$ units. ······ (1) (answer)

(Question continued on next page)

(b) Solve the equation

$$16^{-3x} = \left(\frac{1}{32}\right)^x$$

for x without the use of logarithms.

$$\begin{aligned} 16^{-3x} &= \left(\frac{1}{32}\right)^x \\ \Longrightarrow\ (2^4)^{-3x} &= \left(\frac{1}{2^5}\right)^x \\ \Longrightarrow\ 2^{-12x} &= (2^{-5})^x \\ \Longrightarrow\ 2^{-12x} &= 2^{-5x} \\ \Longrightarrow\ -12x &= -5x \\ \Longrightarrow\ 7x = 0 \ &\Longrightarrow\ x = 0 \end{aligned}$$

$\cdots\cdots$ (1) (answer)

$\cdots\cdots$ (4) (method)

(-1 each method error, follow through)

(c) Solve

$$\log(x) + \log(x-1) = \log(3x+12)$$

for x.

(Hint: don't forget to check your answers in the original equation.)

$$\begin{aligned} \log(x) + \log(x-1) &= \log(3x+12) \\ \Longrightarrow\ \log x(x-1) &= \log(3x+12) \\ \Longrightarrow\ x(x-1) &= 3x+12 \\ \Longrightarrow\ x^2 - x - 3x - 12 &= 0 \\ \Longrightarrow\ x^2 - 4x - 12 &= 0 \\ \Longrightarrow\ (x-6)(x+2) = 0 \ &\Longrightarrow\ x = 6 \quad \text{or} \quad x = -2. \end{aligned}$$

$\cdots\cdots$ (1) (answer)

$\cdots\cdots$ (2) (method)

(-1/2 each error, follow through)

Following the hint, we check if these answers satisfy the original equation (see next page):

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$\mathbf{x = 6}$: $\log(6) + \log(5) = \log(30)$ ✓ $\cdots\cdots$ (1/2)

$\mathbf{x = -2}$: $\log(-2) + \log(-3) = \log(6)$ ✗ $\cdots\cdots$ (1/2)

(we can't take logs of a negative number)

Thus the only solution is $x = 6$. $\cdots\cdots$ (1)
(answer)

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2. (10 marks)

(a) Consider the recursively defined sequence

$$a_1 = 1, \quad a_2 = 0, \quad \text{and} \quad a_{n+1} = 2\sqrt{a_n} + a_{n-1} \quad n = 2, 3, 4, \ldots.$$

List the 3rd, 4th and 5th terms of this sequence.
(**Note**: you are given the 1st two terms).

$\mathbf{n = 2}$: $a_3 = 2\sqrt{a_2} + a_1 = 2\sqrt{0} + 1 = 1$, $\cdots\cdots$ (1)
(answer)

$\mathbf{n = 3}$: $a_4 = 2\sqrt{a_3} + a_2 = 2\sqrt{1} + 0 = 2$, $\cdots\cdots$ (1)
(answer)

$\mathbf{n = 4}$: $a_5 = 2\sqrt{a_4} + a_3 = 2\sqrt{2} + 1$, or 3.83 (2 d.p.), $\cdots\cdots$ (1)
(answer)

(Question continued on next page)

(b) Consider the following sequence

$$\left\{\frac{3}{10}, -\frac{3}{35}, \frac{6}{245}, -\frac{12}{1715}, \ldots\right\}$$

(i) identify what type of sequence this is
(ii) determine a recursive formula for this sequence
(iii) determine a formula for the nth term, and
(iv) determine the 8th term (expressed as a fraction)

(i) Geometric, with $a_1 = \frac{3}{10}$, $r = -\frac{2}{7}$ ...... (1)

(ii) $a_{n+1} = r a_n$, $n = 1, 2, \ldots$ ...... (1/2) (formula)

$= \left(-\frac{2}{7}\right) a_n$, $n = 1, 2, \ldots$ ...... (1/2) (answer)

(iii) $a_n = a_1 r^{n-1}$ ...... (1/2) (formula)

$= \left(\frac{3}{10}\right)\left(-\frac{2}{7}\right)^{n-1}$ ...... (1/2) (answer)

(iv) $a_8 = \left(\frac{3}{10}\right)\left(-\frac{2}{7}\right)^7 = -\frac{7}{150125}$. ...... (1) (answer)

(follow through)

(Question continued on next page)

(c) Evaluate

$$S = \lim_{n\to\infty}\left(\frac{2n^4 + n^3 + 2}{n^3 + 2n^2 + n - 1}\right).$$

Be careful to justify your answer by showing the rules of limits that you use!

As shown in the homework, we start by dividing the numerator and denominator of the fraction we are taking a limit of by the highest power of n in the denominator (i.e., n^3). We then use the rules of limits:

$$S = \lim_{n\to\infty}\left(\frac{2n + 1 + \frac{2}{n^3}}{1 + \frac{2}{n} + \frac{1}{n^2} - \frac{1}{n^3}}\right) \qquad \cdots\cdots \quad (1)$$

(method)

$$= \frac{\lim_{n\to\infty}\left(2n + 1 + \frac{2}{n^3}\right)}{\lim_{n\to\infty}\left(1 + \frac{2}{n} + \frac{1}{n^2} - \frac{1}{n^3}\right)} \qquad \cdots\cdots \quad (1)$$

(method)

$$= \frac{\infty + 1 + 0}{1 + 0 + 0 - 0}$$

$$= +\infty. \qquad \cdots\cdots \quad (1)$$

(answer)

(Series diverges)

3. (6 marks)

(a) Evaluate $S = \sum_{i=2}^{3} \frac{2i^2}{\sin\left[(2i-1)\frac{\pi}{2}\right]}$.

$$S = \frac{2(2)^2}{\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1}} + \frac{2(3)^2}{\underbrace{\sin\left(\frac{5\pi}{2}\right)}_{=+1}} \qquad \cdots\cdots \quad (1)$$

(method)

$$= -2(4) + 2(9)$$
$$= 10 \qquad \cdots\cdots \quad (1)$$

(answer)

(b) Evaluate

$$S = 100 + 101 + 102 + \cdots + 150.$$

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Method 1:
This is a Arithmetic series. Using the formula $\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$ we have

$$S = \sum_{i=1}^{150} i - \sum_{i=1}^{99} i \qquad \cdots\cdots \quad (1)$$

(method)

$$= \frac{150(151)}{2} - \frac{99(100)}{2}$$
$$= 75(151) - 99(50) = 6375 \qquad \cdots\cdots \quad (1)$$

(answer)

alternatively, use (see next page)

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Alternatively, use **Method 2**:
We use the formula for a general Arithmetic series where $a = 100$, $d = 1$, $N = 51$:

$$S = \frac{N}{2}\left[2a + (N-1)d\right] \qquad \cdots\cdots \quad (1)$$

(formula)

$$= \frac{51}{2}\left[2(100) + 50(1)\right]$$

$$= \frac{51}{2}(250) = 6375 \quad \text{again} \qquad \cdots\cdots \quad (1)$$

(answer)

Note: marks awarded either for Method 1 or for Method 2, but not both!

(c) Determine whether the infinite geometric series is convergent or divergent. If it is convergent, find its sum:

$$\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots$$

(Make sure you identify the value of r in this series.)

We have $a = 1/3$, $r = 1/2$. And as $|r| < 1$
the series is convergent. ······ (1)

The sum is given by

$$S = \frac{a}{1-r} \qquad \cdots\cdots \quad (1/2)$$

(formula)

$$= \frac{1/3}{1-1/2}$$

$$= \frac{1/3}{1/2}$$

$$= \frac{1}{3}\cdot 2$$

$$= \frac{2}{3} \qquad \cdots\cdots \quad (1/2)$$

(answer)

4. (12 marks)

(a) Determine how much money must be invested for $\mathbf{240}$ days at a simple interest rate of $\mathbf{1.75\%}$ to earn $\mathbf{\$100.00}$?

$t = \frac{240}{365} = \frac{48}{73}$ years, $r = 0.0175$ ······ (1)

$$I = Prt \quad \text{so}$$ ······ (1) (formula)

$$100 = P(0.0175) \cdot \left(\frac{48}{73}\right)$$

$$\implies P = \frac{100}{(0.0175) \cdot \left(\frac{48}{73}\right)}$$ ······ (1) (method)

$$= 8690.476\ldots$$

$$= \$8690.48$$ ······ (1) (answer)

(Question continued on next page)

(b) John borrows $\$2000$ at 2% annual interest compounded bi-weekly (i.e., every 2 weeks) for 32 months. How much money must John eventually pay back?

$r = 0.02$, $t = \frac{32}{12} = \frac{8}{3}$ years, $m = \frac{52}{2} = 26$ ······ (1)

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$ ······ (1)

(formula)

$$= 2000\left(1 + \frac{0.02}{26}\right)^{26(8/3)}$$
$$= 2109.519\ldots$$
$$= \$2109.52$$ ······ (1)

(answer)

(Question continued on next page)

(c) Find the present value of $\$5000$ due in 3 years at 3.75% compounded continuously.

$t = 3$, $r = 0.0375$

$$A = Pe^{rt} \qquad \cdots\cdots \quad (1)$$

(formula)

$$\implies 5000 = Pe^{0.0375(3)}$$

$$\implies P = \frac{5000}{e^{0.0375(3)}} \qquad \cdots\cdots \quad (1)$$

(method)

$$= 4467.986\ldots$$

$$= \$4467.99 \qquad \cdots\cdots \quad (1)$$

(answer)

(Question continued on next page)

(d) Find the effective interest rate corresponding to $\mathbf{3\%}$ compounded monthly. (Give your answer as a percent to $\mathbf{2}$ decimal places.)

$r = 0.03$, $m = 12$, $(t = 1)$

$$r_E = \left(1 + \frac{r}{m}\right)^m - 1 \qquad \cdots\cdots \ (1)$$

(formula)

$$= \left(1 + \frac{0.03}{12}\right)^{12} - 1$$
$$= 0.03041\ldots$$
$$= 3.04\% \qquad \cdots\cdots \ (1)$$

(answer)

END OF TEST

TA Initials (print) ________ Total: / **44**