UNIVERSITY of GUELPH

Winter 2018	MATH*1030	Midterm 2

Last name (print): ________________ **First** name (print): ________________

Student #: ____________________ Signature: ________________________

1. This is a 50 minute test. Do NOT start until instructed.

2. Please fill out your personal details above. Once the examination starts fill out your solutions in the spaces provided.

3. You may quote results used in your arguments from lecture notes without justification, unless asked to do otherwise. Notes or books are not permitted during the exam.

4. Give sufficient working as solutions with no justification will receive little or no credit.

5. You may use a scientific calculator and/or a 'graphing calculator'. Note however that graphical answers require justification. No other electronic equipment is permitted. **Turn off or 'mute' your cell phone! The use of cell phones during the exam is prohibited.**

6. If you copy the work of your neighbour this is considered **Academic Misconduct** and will be reported to the appropriate university authority.

7. There are a total of **35** points to be awarded on this test.

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1. (13 marks)

(a) The daily cost (in dollars $\$$) of a company to print x paperback fantasy novels is given by

$$C(x) = 3.5x + 1200 \quad \text{dollars.}$$

(i) What is the marginal cost of production?
(ii) Give two different interpretations of the marginal cost
(iii) What is the fixed cost of production?

(i) $m = \$3.5$ per item ······ (1)

(ii) The slope of the cost line, or ······ (1/2)

the cost to produce one additional item at any level of production. ······ (1/2)

(iii) $b = \$1200$ ······ (1)

(Question continued on next page)

(b) Solve the equation

$$4^{5-9x} = \frac{1}{8^{x-2}}$$

for x without the use of logarithms.

$$\begin{aligned} 4^{5-9x} &= \frac{1}{8^{x-2}} \\ \implies (2^2)^{5-9x} &= \frac{1}{(2^3)^{x-2}} \\ \implies 2^{10-18x} &= \frac{1}{2^{3x-6}} \\ &= 2^{-(3x-6)} \\ &= 2^{6-3x} \\ \implies 10 - 18x &= 6 - 3x \\ \implies 18x - 3x &= 10 - 6 \\ \implies 15x &= 4 \\ \implies x &= \frac{4}{15}. \end{aligned}$$

$\cdots\cdots$ (5)

(method)

(-1 each error, follow through)

(Question continued on next page)

(c) Solve

$$\log_2(x^2 - 6x) = 3 + \log_2(1 - x)$$

for x.

(Hint: don't forget to check your answers in the original equation.)

$$\begin{aligned} \log_2(x^2 - 6x) &= 3 + \log_2(1 - x) \\ \implies \log_2(x^2 - 6x) - \log_2(1 - x) &= 3 \\ \implies \frac{x^2 - 6x}{1 - x} &= 2^3 = 8 \\ \implies x^2 - 6x &= 8(1 - x) = 8 - 8x \\ \implies x^2 + 2x - 8 &= 0 \\ \implies (x + 4)(x - 2) &= 0 \\ \implies x = -4 \quad \text{or} \quad x &= 2. \end{aligned}$$

...... (3)
(method)
($-1/2$ each error, follow through)

Following the hint, we check if these answers satisfy the original equation

x = -4: $\log_2(16 + 24) = 3 + \log_2(5)$ ✓ ...... (1/2)
x = 2: $\log_2(4 - 12) = 3 + \log_2(-1)$ ✗ ...... (1/2)
(we can't take logs of a negative number)

Thus the only solution is $x = -4$. ...... (1)
(answer)

2. (7 marks)

(a) Consider the recursively defined sequence

$$a_1 = 3, \quad a_2 = 2, \quad a_3 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_{n-2}}{a_n + a_{n-1}} \quad n = 3, 4, 5, \ldots.$$

List the 4rd, 5th and 6th terms of this sequence.

n = 3: $a_4 = \dfrac{a_1}{a_3 + a_2} = \dfrac{3}{1+2} = 1$, ... (1)

(answer)

n = 4: $a_5 = \dfrac{a_2}{a_4 + a_3} = \dfrac{2}{1+1} = 1$, ... (1)

(answer)

n = 5: $a_6 = \dfrac{a_3}{a_5 + a_4} = \dfrac{1}{1+1} = \dfrac{1}{2}$. ...... (1)

(answer)

(Question continued on next page)

(b) Evaluate

$$S = \lim_{n\to\infty}\left(\frac{3}{n} - \frac{8}{5^n}\right).$$

Be careful to justify your answer by showing the rules of limits and other results that you use!

Using the rules of limits we have

$$S = \lim_{n\to\infty}\frac{3}{n} - \lim_{n\to\infty}\frac{8}{5^n} \qquad \cdots\cdots \quad (1)$$

(method)

$$= 3\lim_{n\to\infty}\frac{1}{n} - 8\lim_{n\to\infty}\left(\frac{1}{5}\right)^n \qquad \cdots\cdots \quad (1)$$

(method)

$$= 3(0) - 8(0) = 0, \qquad \cdots\cdots \quad (1)$$

(answer)

using the result in our notes that states

$$\lim_{n\to\infty} b^n = 0 \quad \text{if} \quad |b| < 1 \qquad \cdots\cdots \quad (1)$$

(justification)

(Note: it is OK if students simply state that $\lim_{n\to\infty}\left(\frac{1}{5}\right)^n = 0$ because $\frac{1}{5}$ is less than 1.)

3. (5 marks)

(a) Suppose $\sum_{i=1}^{n} i = 55$ and $\sum_{i=1}^{n} i^2 = 385$. Then evaluate

$$\sum_{i=1}^{n}(3i - 2i^2).$$

Be careful to justify your answer by showing the algebra of series that you use!

Using the algebra of series we have

$$\sum_{i=1}^{n}(3i - 2i^2) = \sum_{i=1}^{n} 3i - \sum_{i=1}^{n} 2i^2 \qquad \cdots\cdots \quad (1)$$

(method)

$$= 3\left(\sum_{i=1}^{n} i\right) - 2\left(\sum_{i=1}^{n} i^2\right) \qquad \cdots\cdots \quad (1)$$

(method)

$$= 3(55) - 2(385) \quad \text{(using the given information)}$$

$$= -605 \qquad \cdots\cdots \quad (1)$$

(answer)

Comment:
If the test was longer I would have asked what the value of n was. To figure this out just recall the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 55$. This leads to the solution of the quadratic $n^2 + n - 110 = 0$. Using the quadratic formula you will arrive at two solutions: $n = 10$, or $n = -11$. We know that n is a positive integer, so we conclude $n = 10$. Maybe I will ask this question next time I teach this course!

(Question continued on next page)

(b) Determine whether the following infinite Geometric series is convergent or divergent. If it is convergent, find its sum:

$$59049 - 19683 + 6561 - 2187 + \dots$$

(Make sure you identify the value of r in this series.)

We have $a = 59049$ and $r = \dfrac{-19683}{59049} = -\dfrac{1}{3}$. And as $|r| < 1$ the series is convergent. $\cdots\cdots$ (1)

The sum is given by

$$S = \frac{a}{1-r} \qquad \cdots\cdots \ (1/2)$$

(formula)

$$= \frac{59049}{1-(-1/3)}$$
$$= \frac{59049}{4/3}$$
$$= 59049 \cdot \frac{3}{4}$$
$$= \frac{177147}{4} \quad \text{or} \quad 44286.75 \qquad \cdots\cdots \ (1/2)$$

(answer)

4. (10 marks)

(a) Find the maturity value (i.e., the future value) of a deposit of $\$5000$ invested at a simple interest rate of 1.75% for 1 year and 3 months.

$t = 1 + \frac{3}{12} = 1 + \frac{1}{4} = \frac{5}{4}$ years, $r = 0.0175$ $\cdots\cdots$ (1)

$$A = P(1 + rt)$$ $\cdots\cdots$ (1) (formula)

$$= 5000\left(1 + 0.0175\left(\frac{5}{4}\right)\right)$$

$$= 5109.375\ldots$$

$$= \$5109.38$$ $\cdots\cdots$ (1) (answer)

(Question continued on next page)

(b) Sara borrows $\$1000$ at 2.75% annual interest compounded monthly for 42 weeks. How much money must Sara eventually pay back?

$r = 0.0275$, $t = \frac{42}{52} = \frac{21}{26}$ years, $m = 12$ ...... (1)

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$ ...... (1)

(formula)

$$= 1000\left(1 + \frac{0.0275}{12}\right)^{12(21/26)}$$
$$= 1022.434\ldots$$
$$= \$1022.43$$ ...... (1)

(answer)

(Question continued on next page)

(c) What is the future value of $\$0.10$ deposited in a bank account paying 3% annual interest, compounded continuously for 100 years?

$r = 0.03$, $t = 100$ years

$$A = Pe^{rt} \qquad \cdots\cdots \quad (1)$$

(formula)

$$= 0.10e^{0.03(100)}$$
$$= 0.10e^{3}$$
$$= 2.0085\ldots$$
$$= \$2.01 \qquad \cdots\cdots \quad (1)$$

(answer)

(Question continued on next page)

(d) What is the effective interest rate corresponding to $\mathbf{1.25\%}$ compounded continuously? (Give your answer as a percent to $\mathbf{2}$ decimal places.)

$r = 0.0125$, $(t = 1)$

$$r_E = e^r - 1 \qquad \cdots\cdots \quad (1)$$

(formula)

$$= e^{0.0125} - 1$$
$$= 0.012578\ldots$$
$$= 1.26\% \qquad \cdots\cdots \quad (1)$$

(answer)

END OF TEST

TA Initials (print) ________ Total: / **35**