UNIVERSITY of GUELPH

Winter 2018 MATH*1160 Midterm 1

Last name (print): ________________ **First** name (print): ________________

Student #: ____________________ Signature: ________________________

1. This is a 50 minute test. Do NOT start until instructed.
2. Please fill out your personal details above. Once the examination starts fill out your solutions in the spaces provided.
3. You may quote results used in your arguments from lecture notes without justification, unless asked to do otherwise. Notes or books are not permitted during the exam.
4. Give sufficient working as solutions with no justification will receive little or no credit.
5. You may use a calculator, but not a 'graphing calculator' that supports matrix algebra. No other electronic equipment is permitted. **Turn off or 'mute' your cell phone! Use of cell phones during the exam is prohibited.**
6. If you copy the work of your neighbour this is considered **Academic Misconduct** and will be reported to the appropriate university authority.
7. There are a total of **22** points to be awarded on this test.

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1. (8 points)

(a) If

$$\begin{pmatrix} a-c & c-d \\ c+2d & 3b+1 \end{pmatrix}^T = \begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix},$$

find $a, b, c,$ and d.

Taking the transpose yields

$$\begin{pmatrix} a-c & c+2d \\ c-d & 3b+1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 3 & 4 \end{pmatrix}. \qquad \cdots\cdots \ (1)$$

Equating corresponding elements on both sides:

$$\begin{cases} a-c &= 1 \qquad \ldots. \quad 1. \\ c+2d &= -3 \qquad \ldots. \quad 2. \\ c-d &= 3 \qquad \ldots. \quad 3. \\ 3b+1 &= 4 \qquad \ldots. \quad 4. \end{cases} \qquad \cdots\cdots \ (1)$$

($-1/2$ each error)
(follow through)

One possible solution route:

From $4.$: $3b = 3 \implies b = 1$.
$2. - 3.$: $3d = -6 \implies d = -2$.
From $3.$: $c + 2 = 3 \implies c = 1$.
From $1.$: $a - 1 = 1 \implies a = 2$.

(follow through)

Thus

$$(a, b, c, d) = (2, 1, 1, -2). \qquad \cdots\cdots \ (2)$$

($-1/2$ each error)

(Question continued on next page)

(b) Let

$$\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}^T.$$

If possible calculate $\mathbf{AB^T}$.

$$AB^T = \underbrace{\begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}}_{3\times 2} \underbrace{\begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}}_{2\times 3} \qquad \cdots\cdots \ (1)$$

(method)

$$= \begin{pmatrix} 1(-1)+(-3)(0) & 1(2)+(-3)(1) & 1(3)+(-3)(2) \\ 0(-1)+1(0) & 0(2)+1(1) & 0(3)+1(2) \\ 2(-1)+3(0) & 2(2)+3(1) & 2(3)+3(2) \end{pmatrix} \qquad \cdots\cdots \ (2)$$

(method, $-1/2$ each error, follow through)

$$= \underbrace{\begin{pmatrix} -1 & -1 & -3 \\ 0 & 1 & 2 \\ -2 & 7 & 12 \end{pmatrix}}_{3\times 3} \qquad \cdots\cdots \ (1)$$

(answer)

2. (4 points)

(a) Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be matrices and $r \in \mathbb{R}$. Find a simplified expression for

$$((\mathbf{AB})^T + r\mathbf{C}^T)^T,$$

clearly showing the algebraic properties of matrices you use.

$$((AB)^T + rC^T)^T = ((AB)^T)^T + (rC^T)^T \qquad \cdots\cdots \quad (1/2)$$

(method)

$$= AB + (rC^T)^T \qquad \cdots\cdots \quad (1/2)$$

(method)

$$= AB + r(C^T)^T \qquad \cdots\cdots \quad (1/2)$$

(method)

$$= AB + rC. \qquad \cdots\cdots \quad (1/2)$$

(method)

(b) Determine all values of a scalar $k \in \mathbb{R}$ such that $(k\mathbf{A})^T(k\mathbf{B}) = 125$, where

$$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

$$(kA)^T(kB) = (kA^T)(kB) \qquad \cdots\cdots \quad (1/2)$$

(method)

$$= k^2(A^TB) \qquad \cdots\cdots \quad (1/2)$$

(method)

$$= k^2 \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= k^2(-1 + 0 + 6)$$

$$= 5k^2 = 125 \quad \text{(using the given information)} \qquad \cdots\cdots \quad (1/2)$$

(method)

$$\implies k^2 = 25 \implies k = \pm 5. \qquad \cdots\cdots \quad (1/2)$$

(answer)

Please Turn Over

3. (3 points)

(a) Let

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Solve the matrix equation $\mathbf{A}^2\mathbf{x} = \mathbf{b}$ for the vector $\mathbf{x}$.

$$\boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{b} \implies \boldsymbol{A}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$ ······ (1/2) (method)

$$\implies \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$ (Multiply both sides on the left by $\boldsymbol{A}^{-1}$) ······ (1/2) (method)

$$\implies \boldsymbol{x} = (\boldsymbol{A}^{-1})^2\boldsymbol{b}$$ (Multiply both sides on the left by $\boldsymbol{A}^{-1}$) ······ (1/2) (method)

$$= \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$ ······ (1/2) (answer)

using the given matrices.

(b) A cook wants a mixture of flour and milk powder that contains $\mathbf{20}$ grams of protein and $\mathbf{40}$ grams of carbohydrate. The flour contains $\mathbf{14\%}$ protein and $\mathbf{73\%}$ carbohydrate, and the milk powder contains $\mathbf{3\%}$ protein and $\mathbf{5\%}$ carbohydrate. Write down (do not solve) two equation in two unknowns for the amount of flour ($\boldsymbol{x}$) in grams and the amount of milk powder ($\boldsymbol{y}$) in grams needed.

We formulate two equation for the total protein and the total carbohydrate content of the mixture:

Protein: $$\frac{14}{100}x + \frac{3}{100}y = 20$$ ······ (1/2)

Carbohydrate: $$\frac{73}{100}x + \frac{5}{100}y = 40$$ ······ (1/2)

I.e.,

$$\begin{cases} 0.14x + 0.03y = 20 \\ 0.73x + 0.05y = 40 \end{cases}$$

4. (7 points)
Use the Gauss-Jordon elimination method to find the infinite solution set for the following system of linear equations:

$$\begin{cases} x_1 + \ \ x_2 + 2x_3 = 4 \\ 2x_1 + 3x_2 - \ \ x_3 = 1 \end{cases}$$

Clearly indicate all the steps in your argument including the row operations you use (answers with NO justification receive NO credit).

Starting from the associated augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & 3 & -1 & 1 \end{array}\right) \qquad \cdots\cdots \ (1)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & -5 & -7 \end{array}\right) r_2 - 2r_1 \to r_2 \qquad \cdots\cdots \ (1)$$

(method)

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 7 & 11 \\ 0 & 1 & -5 & -7 \end{array}\right) r_1 - r_2 \to r_1 \qquad \cdots\cdots \ (1)$$

(method)

The system is in RREF. The associated linear system of equations is

$$\begin{cases} x_1 \qquad + 7x_3 = 11 \\ \qquad x_2 - 5x_3 = -7 \end{cases}$$

$\cdots\cdots$ (1)

From the 2nd equation we have $x_2 = 5x_3 - 7$, x_3 free. Let $x_3 = \alpha \in \mathbb{R}$ so $x_2 = 5\alpha - 7$. Then from the 1st equations $x_1 = -7x_3 + 11 = -7\alpha + 11$.

$\cdots\cdots$ (2)

($-1/2$ each error, follow through)

Thus the infinite solution set is

$$\{(-7\alpha + 11, 5\alpha - 7, \alpha) \mid \alpha \in \mathbb{R}\}. \qquad \cdots\cdots \ (1)$$

(answer)

END OF TEST

TA Initials (print) ________ Total: / **22**