UNIVERSITY of GUELPH

Winter 2018 MATH*1160 Test 2

Last name (print): ________________ **First** name (print): ________________

Student #: ____________________ Signature: ________________________

1. This is a 60 minute test. Do NOT start until instructed.
2. Please fill out your personal details above. Once the examination starts fill out your solutions in the spaces provided.
3. You may quote results used in your arguments from lecture notes without justification, unless asked to do otherwise. Notes or books are not permitted during the exam.
4. Give sufficient working as solutions with no justification will receive little or no credit.
5. You may use a calculator, but not a 'graphing calculator' that supports matrix algebra. No other electronic equipment is permitted. **Turn off or 'mute' your cell phone! Use of cell phones during the exam is prohibited.**
6. If you copy the work of your neighbour this is considered **Academic Misconduct** and will be reported to the appropriate university authority.
7. There are a total of **25** points to be awarded on this test.

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1. (8 points)

(a) Compute the following determinant by applying elementary row and/or column operations to reduce the determinant to triangular form:

$$|A| = \begin{vmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ -2 & 0 & 5 \end{vmatrix}.$$

Do NOT use a cofactor expansion. Show all the steps in your calculation.

Note: the operations used to get the determinant in triangular form may differ.

$$|A| \longrightarrow \begin{vmatrix} 1 & 1 & 7 \\ 1 & 3 & 2 \\ -2 & 0 & 5 \end{vmatrix} \begin{matrix} r_1 + r_3 \to r_1 \\ \\ \\ \end{matrix} \qquad \cdots\cdots \quad (1)$$

(method)

$$\longrightarrow \begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & -5 \\ 0 & 2 & 19 \end{vmatrix} \begin{matrix} \\ r_2 - r_1 \to r_2 \\ r_3 + 2r_1 \to r_3 \end{matrix} \qquad \cdots\cdots \quad (1)$$

(method)

$$\longrightarrow \begin{vmatrix} 1 & 1 & 7 \\ 0 & 2 & -5 \\ 0 & 0 & 24 \end{vmatrix} \begin{matrix} \\ \\ r_3 - r_2 \to r_3 \end{matrix} \qquad \cdots\cdots \quad (1)$$

(method)

$$= (1)(2)(24)$$
$$= 48. \qquad \cdots\cdots \quad (1)$$

(answer, follow through)

Question continued on next page

(b) For square matrices $\mathbf{A}$ and $\mathbf{B}$ such that $|\mathbf{A}| = -6$ and $|\mathbf{B}| = 3$ find $|(\mathbf{A}^{-1}\mathbf{B})^T|$. Justify each step of your calculation.

Using results from our lecture notes we have

$$\begin{aligned} |(A^{-1}B)^T| &= |A^{-1}B| && \cdots\cdots \ (1) \ \text{(justification)} \\ &= |A^{-1}||B| && \cdots\cdots \ (1) \ \text{(justification)} \\ &= \frac{1}{|A|} \cdot |B| && \cdots\cdots \ (1) \ \text{(justification)} \\ &= \frac{3}{-6} \\ &= -\frac{1}{2}. && \cdots\cdots \ (1) \ \text{(answer)} \end{aligned}$$

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2. (4 points)

(a) With $A = \begin{pmatrix} x^2 & 2x^2 & 3x^2 \\ 2x & 4x & 6x \\ 2 & 4 & 6 \end{pmatrix}$, $x \in \mathbb{R}$, apply a cofactor expansion along the **first row** of $\mathbf{A}$ to find $|\mathbf{A}|$.

$$\begin{aligned} |A| &= +x^2 \begin{vmatrix} 4x & 6x \\ 4 & 6 \end{vmatrix} - 2x^2 \begin{vmatrix} 2x & 6x \\ 2 & 6 \end{vmatrix} + 3x^2 \begin{vmatrix} 2x & 4x \\ 2 & 4 \end{vmatrix} && \cdots\cdots \ (1) \ \text{(method)} \\ &= x^2(24x - 24x) - 2x^2(12x - 12x) + 3x^2(8x - 8x) \\ &= x^2(0) - 2x^2(0) + 3x^2(0) && \cdots\cdots \ (1) \ \text{(method)} \\ &= 0. && \cdots\cdots \ (1) \ \text{(answer, follow through)} \end{aligned}$$

Question continued on next page

(b) Define the adjoint of a square matrix $\mathbf{A}$, namely $\mathbf{adj}\,\mathbf{A}$. (You may use words or math to answer this question.)

The adjoint of a matrix is the transpose of the matrix of cofactors $\cdots\cdots$ (1).

Please turn over

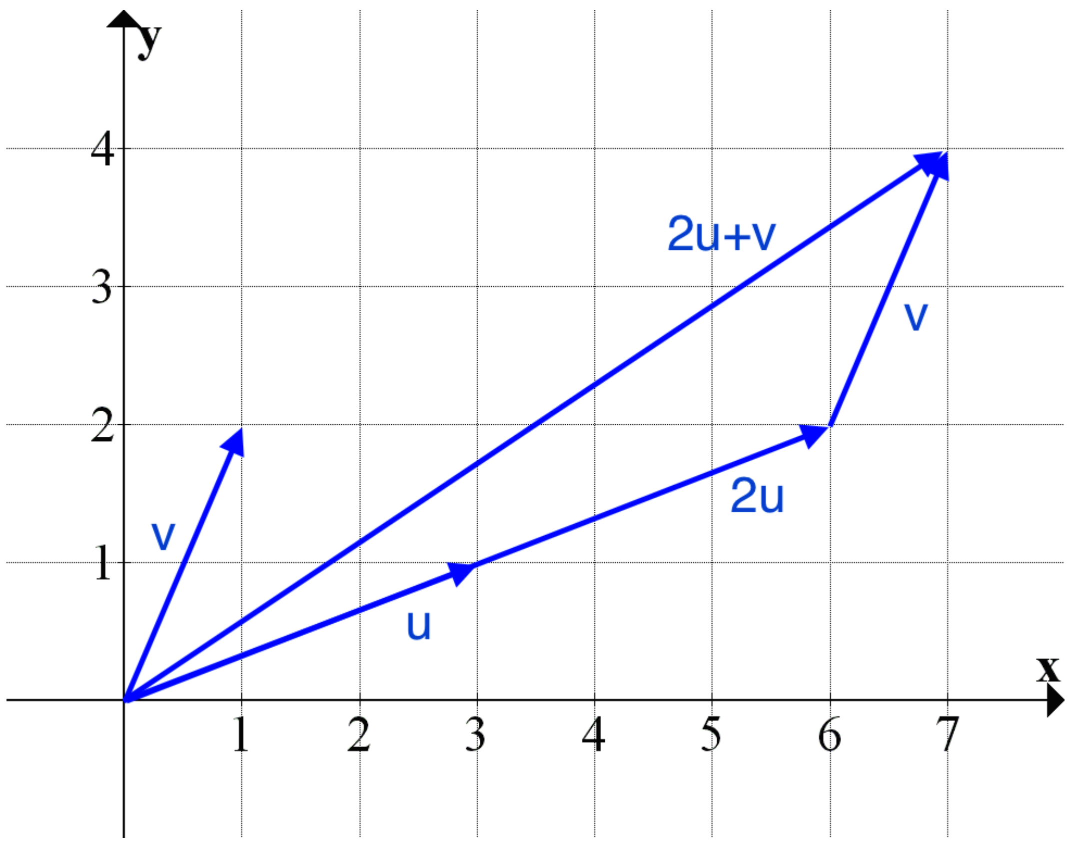
3. (7 points)

(a) Let

$$\mathbf{u} = \begin{pmatrix} \mathbf{3} \\ \mathbf{1} \end{pmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix}.$$

(i) Use directed line segments to represent the vectors $\mathbf{u}$ and $\mathbf{v}$ on the diagram below.

(ii) Calculate $\mathbf{2u} + \mathbf{v}$ and indicate the vector sum on your diagram from (i).



(i) $\boldsymbol{u}$ and $\boldsymbol{v}$ correctly drawn. ······ (1)

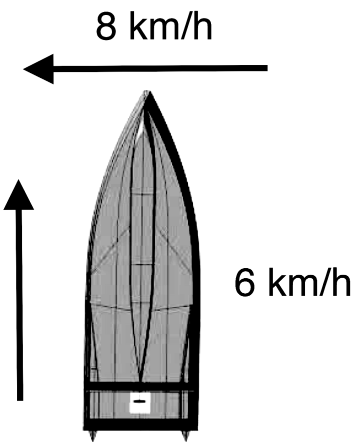
(ii) $\mathbf{2}\boldsymbol{u} + \boldsymbol{v} = \mathbf{2}\begin{pmatrix} \mathbf{3} \\ \mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{6} \\ \mathbf{2} \end{pmatrix} + \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \mathbf{7} \\ \mathbf{4} \end{pmatrix}$ ······ (1)

$\mathbf{2}\boldsymbol{u} + \boldsymbol{v}$ correctly drawn. ······ (2)

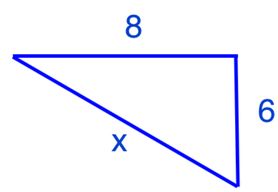
($-\mathbf{1/2}$ each error)

Question continued on next page

(b) A boat travels north with a constant velocity of 6 km/h. If the current pushes the boat west with a constant velocity of 8 km/h for 3 minutes (see the illustration below), how far does the boat travel in **metres** (m) during the 3 minutes? (Note: 1 km $= 1000$ m.)



Using pythagorus we first find the velocity of the boat in the direction of motion:



Thus $x = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$ km/h ······ (1) (method)

Then (need time in hours!)

$$\text{distance} = \text{speed} \times \text{time}$$ ······ (1/2) (formula)

$$= 10 \times \left(\frac{3}{60}\right)$$

$$= \frac{3}{6} \quad \text{or} \quad \frac{1}{2} \text{ km}$$ ······ (1/2) (method)

$$= 500 \text{ m.}$$ ······ (1) (answer)

Please Turn Over

4. (6 points)

(a) Consider the set of vectors in $\mathbb{R}^n$, with the usual rules for addition and scalar multiplication. For any $\mathbf{u} \in \mathbb{R}^n$, prove that the **3**rd axiom of a vector space holds, namely, that

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0},$$

where $\mathbf{0}$ is the zero vector for $\mathbb{R}^n$. Clearly show *all* the steps in your argument.

(Hint: use the 'coordinate' form to represent vectors, e.g., $\mathbf{u} = (u_1, u_2, \ldots, u_n)$.

We have

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{u} + (-1 \cdot \mathbf{u}) \qquad \cdots\cdots \quad (1/2)$$

(explicit demonstration of what we mean by a negative vector)

$$= (u_1, u_2, \ldots, u_n) + (-u_1, -u_2, \ldots, -u_n) \qquad \cdots\cdots \quad (1/2)$$

(using the scalar product in $\mathbb{R}^n$)

$$= (u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n) \qquad \cdots\cdots \quad (1/2)$$

(using the vector sum in $\mathbb{R}^n$)

$$= (0, 0, \ldots, 0)$$

$$= \mathbf{0} \quad \text{(the zero vector for } \mathbb{R}^n\text{)} \qquad \cdots\cdots \quad (1/2)$$

(understanding what the zero vector in $\mathbb{R}^n$ is)

Question continued on next page

(b) Let W be the set of all $n \times n$ symmetric matrices. Prove whether or not W is a subspace of M_{nn} (the vector space of all $n \times n$ matrices.)

Closure with respect to vector addition:
Let $A, B \in W$ i.e. $A^T = A$ and $B^T = B$. Then ······ (1/2)

$$(A+B)^T = A^T + B^T \quad \text{(using algebraic properties of matrices)} \quad \cdots\cdots \quad (1/2)$$

$$= A + B \quad \checkmark \quad \text{(as } A \text{ and } B \text{ belong to } W\text{)} \quad \cdots\cdots \quad (1/2)$$

Closure with respect to scalar multiplication:
Let $A \in W$ i.e. $A^T = A$, and $k \in \mathbb{R}$. Then ······ (1/2)

$$(kA)^T = kA^T \quad \text{(using algebraic properties of matrices)} \quad \cdots\cdots \quad (1/2)$$

$$= kA \quad \checkmark \quad \text{(as } A \text{ belongs to } W\text{)} \quad \cdots\cdots \quad (1/2)$$

Thus by a theorem in our notes W is a subspace of M_{nn}. ······ (1)
(correct conclusion)

END OF TEST

TA Initials (print) ________ Total: / **25**