Integrating illiquid assets into the portfolio decision process

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April 2010

Abstract

We consider the issues associated with modelling the decision to invest in an illiquid asset, such as real estate, over an extended period of time. Markets for illiquid assets tend to display certain characteristics: e.g. significant time-till-sale and correlation in the rates of return over time. More importantly, since the liquidity of a market cannot be an issue if an investor never needs to liquidate an asset, we focus on how the liquidity of a market interacts with an individual's uncertain need to liquidate.

We show that the optimal strategy is state-contingent, if possible. We also show that the penalty associated with an illiquid investment depends on the characteristics of other assets being held in the portfolio, on the characteristics of liquidity shocks and on the interaction between time and behavior. We show that borrowing to pay for a liquidity shock cannot overcome all of the costs of owning an illiquid asset. In contrast, borrowing at t= 0 benefits from the complementarity in the assets. In a simpler model, we show that the portfolio perspective makes illiquid assets more valuable to an investor with a longer time horizon.

J.E.L.: D20, F00

Key Words: liquidity, portfolio, real estate, holding period, autocorrelation, time

Theories of portfolio selection explore the relationship between the characteristics of assets and an investor's optimal mixture of those assets. These theories find it convenient to assume that the relevant characteristics of an asset can be summarized by a probability distribution of the rate of return (i.e. changes in the price). Some types of assets cannot be described so easily because of the time and expense needed to sell at any price.[1] These costs imply that the asset is traded in an illiquid market where, at any time, there is a significant trade off between the selling price and the time needed to achieve that selling price (e.g. Anglin, 2006). This classic definition of illiquidity certainly characterises the buying and selling of real estate assets and, at the level of a market equilibrium, a consequence of this characteristic is that the rate of return to holding the asset does not follow a random walk (Case and Shiller, 1989). This characteristic also needs to be given a context for an individual since an individual may not need to sell the asset. This paper notes how models of investment decisions can account for these issues and their consequences.

We create a series of models to help understand the behavior of a maximizing investor whose portfolio consists of a riskless asset and a risky asset that is also illiquid. We show that an investor who trades an asset where returns are correlated over time would like to use state-contingent strategy. If the market for this asset is illiquid, then the investor cannot trade their asset at their most preferred time. This constraint makes them worse off but the significance of this constraint, and how it causes risk and return to accumulate, is open to investigation.

For two reasons, we also introduce the concept of a "liquidity shock" to distinguish the aspect of liquidity that is special to a market and the aspect that is special to an individual. First, this concept helps to show that the liquidity of a market is not important if an investor does not want to liquidate. Second, this concept helps to differentiate illiquid assets based on the timing of the sale: given that such assets are not easy to sell, anything which reduces the ability of an individual to choose when to sell an asset would be significant. Practically, the effects of liquidity shocks are hard to predict because the cost cannot be summarized using a simple parameter. We show that it is essential to take a portfolio perspective because there is more than one way to pay for a liquidity shock and the preferred way varies with the time of the shock and the state of the market at that time. While developing this model, we show that the model needs to be described carefully in order for the predicted behavior to be considered reasonable. For example, it is possible for a model's specification to create a moral hazard situation where the investor seems to want to over-invest in the illiquid asset. This issue can be especially complex when an investor is allowed to borrow because a borrower may want money to overcome a short term liquidity shortfall but a lender may be concerned about the solvency of the investor.

The next section reviews some of the vast literature on liquidity and transaction costs to place our model in a context. After introducing the basic model, we solve for the optimal portfolio for assets traded in markets with various characteristics and compare the solutions. The following section introduces the idea that an investor may suffer a liquidity shock and discusses how this characteristic of an investor affects the optimal solution. The fifth section explores these issues using a simplified continuous time model. In particular, we show that illiquidity usually has less effect on investors with a longer time horizon. We also explore the role of borrowing in the investor's decision. An investor may borrow initially or after a liquidity shock occurs and the different reasons to borrow affect the benefits of borrowing. Some appendices discuss the more technical issues.

The concluding section discusses several contexts that should consider the liquidity dimension more substantially. For example, in the context of an institutional investor, we discuss how managerial decisions which are intended to affect the current value of operations might also increase the overall value of an illiquid asset. Or, in the context of public policy, we comment on how recent research offers insights into new regulations, such as FASB 157 and Basel II, which discuss how the "fair value" of an asset should be computed.

Some Related Literature

Fama (1970, Proposition 1) proved that, for a risk averse investor seeking to maximize expected utility from the stream of consumption over his lifetime, his choices in each period would be indistinguishable from that of a properly specified investor with a one period horizon. The power of this idea is evident in the fact that many undergraduate textbooks in finance consider only the case of a single period investment problem. This model is not unreasonable if, as proclaimed by the textbook model of perfect competition, anybody who wants to sell can immediately find a buyer at the market price at any time. Our discussion of illiquidity focuses on two features that are uncommon in perfectly competitive markets:

- Non-instantaneous time-till-sale and
- Rates of return do not necessarily follow a random walk (i.e. market frictions prevent prices from incorporating all information instantaneously).

These features imply that an investor needs to account for a third feature:

- Liquidity shocks to an individual.

The degree to which any one of these features is important may vary from time to time, place to place and from trader to trader but the literature cited below validates these characteristics. The discussion includes both empirically-oriented papers and theoretically-oriented papers.

It is well-known that real estate assets are hard to trade but the degree of difficulty may not be so well-known. Collett, Lizieri and Ward (2003) found that “round-trip transaction costs” were approximately 7 to 8 percent of the value of an asset. Jud, Wingler and Winkler (2006) reached a similar conclusion. Haurin and Gill (2002) estimated, using a unique kind of data set, that the cost of selling a residential house was about three percent of the house’s value plus four percent of a household’s earnings. Collett, Lizieri and Ward (2003) used numerical simulations to show that these costs are sufficiently high that a long holding period needs to pass before the internal rate of return is close to the official rate.

Several papers consider how the liquidation value of an asset affects corporate decision making (Benmelech, Garmaise and Moskowitz, 2005; Brown, Ciochetti and Riddiough, 2006; Giambona, Harding and Sirmans, 2008).[2] Using various proxy measures, these papers investigated whether an ability to attract a wider variety of buyer types affects a property’s liquidation value, in the event of a need to sell, and affects the initial use of debt or equity when buying. Benmelech, Germaise and Moskowitz (2005) and Giambona, Harding and Sirmans (2008) argued that, in an environment of asymmetric information and/or incomplete contracts, property which is easier to sell tends to use more leverage and a longer maturity on any loan. Using both analytical and empirical methods, Brown Ciochetti and Riddiough (2006) considered a different way to resolve a situation where a company is in financial distress: renegotiate a contract or restructure an organization. They found properties of distressed companies were sold at a discount relative to "fundamental value" of approximately 25 percent. They also found, in the event of trouble, that the decision to renegotiate varied with the market conditions, partly in anticipation of the outcome if the lender foreclosed on the property.

These costs are significant but investors have some discretion about when to incur the costs. Collett, Lizieri and Ward (2003) found that the average holding period of institutional property in the UK can be long, compared to the holding period for financial assets such as stocks and bonds, at about 11 years with some variation by the type of property. Brown and Guerts (2005) found that the average holding period for apartment buildings was shorter if the growth rate in prices was faster and that the average holding period was less than five years. They also concluded that the physical characteristics of the property, except for size, were less important than the characteristics of the investor, e.g. taxes. Baroni, Barthelemy and Mokrane (2007) discussed the optimal holding period of a real estate asset by analysing discounted cash flow. The solution for the optimal holding period represents a trade off between present value of the cash flow and the growth rate in the price of the asset. They also discussed how the optimal holding period varies with the parameters of the model, especially when the growth rate in the price is high or when the growth rate in the price is very low *and* the growth rate in cash flow is relatively high.

Because real estate assets are costly to trade,[3] market forces may not be sufficient to cause returns on real estate to follow a random walk. Using tests developed to identify the random walk associated with returns of stocks and bonds, Gu (2002) argued that excess volatility in returns implied positive correlation over time. Capozza, Hendershott and Mack (2004) studied the time series properties of residential prices in many cities and concluded that prices displayed serial correlation and mean reversion. Capozza and Israelsen (2007) reached the same conclusion for REITs. Their research identified some characteristics of markets which explain these dynamic properties. Fuguzza, Guidolin and Nicodano (2009) and MacKinnon and Zaman (2009) also noted that returns were correlated over time and investigated how the optimal portfolio varied with the time horizon of the investor.

The papers cited above identify characteristics of a market whereas our paper focuses on individual behavior. Fu (1995) focused on each person's conflict between the consumption motive to own real estate and the investment motive. He argued that, in an intertemporal model, liquidity constraints reduce the significance of permanent income as a determinant of behavior. Chinloy (1999) used a representative agent model to derive an Euler equation which emphasizes aspects of the mortgage contract and estimated it. In his model, if the loan-to-value ratio is sufficiently high, any problems arising from a homeowner's illiquidity can be overcome by borrowing at an interest rate that varies over time. Chinloy also noted that one of the advantages of being liquid is that the owner can act strategically, especially concerning pre-payment and refinancing decisions. These conclusions are reasonable but we wish to consider the special features of liquidity when a person cannot sell at some price at any time they choose, just because they want to sell at that price.

The papers closest in spirit to our paper are Lin and Vandell (2007), Bond, Hwang, Lin and Vandell (2007), Lin, Liu and Vandell (2009) and Cheng, Lin and Liu (2010).[4] The first paper showed how illiquidity creates a difference between "ex ante" and "ex post" returns. The ex ante condition is forward-looking and represents the value based on market conditions independent of whether a buyer has been found. The ex post condition is comparable to how data is collected and represents the value based on market conditions and on a seller having found a willing buyer. Lin and Vandell showed that the expectation of ex ante return is equal to the expectation of ex post return, assuming that the actual time of sale is equal on average to the expected time-till-sale. They also showed that the ex post risk is lower than the ex ante risk by a factor that varies with the expected time-till-sale, with the time between purchase and sale, and with the variance in returns. Some illustrative numbers demonstrate that this factor can be large even if the holding period is relative long (e.g. 10 years). Lin and Vandell also discussed the idea of "liquidation bias": that market prices, and the returns estimated from market prices, do not accurately represent the value to holding real estate because transaction prices represent sellers who are exceptionally motivated.[5]

Bond, Hwang, Lin and Vandell (2007) and Lin, Liu and Vandell (2009) implemented these ideas using a combination of data analysis and Monte Carlo methods. They argued that having at least 10 real estate (i.e. illiquid) assets is sufficient to avoid most of the costs of illiquidity, assuming that the price risk and the marketing period risks are independent across assets. Cheng, Lin and Liu (2010) used more formal methods to explore the importance of the difference between ex ante and ex post in the context of a portfolio. The familiar logic of diversification shows that adding more assets to a portfolio reduces the ex post risk, if the prices evolve independently. But, they added, the difference between the ex ante and ex post risk does not necessarily fall by adding assets if the expected time-till-sale of the added assets is the same as that of the portfolio itself. In an illiquid market, the expected time-on-market is significant and uncertainty about time-on-market becomes an important consideration, especially if an investor needs cash immediately. This risk is affected by the seller's motivation and is common to all properties being sold by the seller. When these risks are large, a strategy which reduces the cost of illiquidity becomes more important than a diversification strategy which exploits the correlation between assets to reduce the cost of financial risk.

Though motivated by this work, our paper differs in many ways. First, we consider a portfolio model with more than one type of asset where differences between the types can be used to increase the value of the portfolio. Second, we consider a wider range of trading strategies and more complex limitations on those strategies. Some of our results depend on an inability to adjust the portfolio easily, which causes an investor's portfolio to differ from what would be optimal at that time. The magnitude of the difference, and the cost of the difference, depends on the degree of correlation over time as well as the state of the market. More complex models would allow the holding period of the illiquid asset to be endogenous and, as a result, the reservation price for the illiquid asset would vary with a randomly-occurring liquidity shock, with the characteristics of the market, with the composition of the portfolio and with the passage of time since the initial investment.

A Common Decision Model

We consider an investor who starts with initial wealth W_0. We assume that T periods elapse between the first decision and the time when the portfolio is consumed. T is presumed to be longer than a reasonable holding period of an asset since, as we show later, the holding period may be random and because, in principle, decisions at the time of sale could affect later decisions. We further assume that there are no trading costs per se in order to focus on the implications of time and of the price-time trade-off.

Assumption 1: The investor has mean-variance preferences represented by

$$E(W_T) - k\ var(W_T) \qquad (1)$$

where $k \geq 0$ is a taste parameter, which can differ among investors, and W_T is the investor's wealth in final period, T.

Assumption 2: There is one risky asset. Let P_t be the price[6] of the risky asset in period t and $R_t = P_t/P_{t-1} - 1$ is the per-period rate of return. For any given t, R_t is Normally distributed. Serial correlation among R_t is represented by an AR(1) process parameterized by the correlation coefficient, ρ.

Assumption 2 implies that, between R_t and R_{t-1},

$$\begin{bmatrix} R_t \\ R_{t-1} \end{bmatrix} \sim N(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \sigma^2) \qquad (2)$$

for all t. When applying these results, it will be important to distinguish between the conditional and unconditional distributions. Based on the Normal distribution, the conditional density of R_t given R_{t-1} is

$$R_t \sim N(\mu + \rho(R_{t-1} - \mu), (1-\rho^2)\sigma^2). \qquad (3)$$

This dependence could be implemented in many other ways, such as by using a moving average process or an ARIMA process or by using a longer span of history. Fugazza, Guidolin and Nicodano (2009) and MacKinnon and Al Zaman (2009) show that implementation is a non-trivial exercise. In principle, using any non-independent process would imply that the risk per decision can no longer be summarized by something as simple or familiar as the unconditional variance, σ^2. A change in the value of another parameter, uncertainty about its value or the evolving value of R_t may both affect the expected return and the risk per decision. The magnitude of the effect would also depend on when and how often the investor is allowed to make investment decisions. We think that this characteristic helps to understand how the risk associated with investing in an illiquid asset varies over time and, more importantly, varies because an investor cannot liquidate the asset at a time of their choosing at a given market price.

Assumption 3: The investor can invest in one risk-free asset, with a per-period rate of return equal to $R_f > 0$. The investor can, within limits discussed below more precisely, borrow at a riskless per-period rate of R_b where $R_b > R_f$.

To introduce the notation, consider the simplest case with no borrowing. Wealth accumulates according to a simple process:

$$W_{t+1} = (W_t - a)(1 + R_f) + a(1 + R_t) \quad (4)$$

where a represents the amount invested in the illiquid asset in period t, either by choice or by involuntary accumulation starting from some prior decision. We consider several cases and, to distinguish the solutions to different cases, we will use slightly different symbols in the place of "a".

We use these results to solve for the optimal portfolio. Consider an investor who maximizes by making a series of decisions, represented by $A = \{a_0, a_1(.), \ldots, a_{T-1}(.)\}$ where $a_t(.)$ represents a decision taken in period t based on the information available to the investor in period t. Using the Bellman Principle, maximizing the value of final wealth implies that it is necessary to maximize the value at each interim period. We solve for A using a series of optimization problems: for each t, and given W_{t+1} and given R_t, choose a_t to maximize

$$V_t = E(W_{t+1}) - k\,Var(W_{t+1}).$$

Combining equations (3) and (4) to maximize V_t with respect to a_t reveals that the optimal portfolio in period t is:

$$a_t^*(R_{t-1}) = [(\mu + \rho(R_{t-1} - \mu)) - R_f]/[2\,k\,(1 - \rho^2)\,\sigma^2]. \quad (5)$$

Thus

$$\partial a_t^*/\partial R_{t-1} = \rho/[2\,k\,(1 - \rho^2)\,\sigma^2]. \quad (6)$$

When $\rho > 0$, the optimal investment in the risky asset increases in the period after an above-average rate of return. Each solution is independent of W_t, because of constant absolute risk aversion.

This formula introduces two changes to the common formula: $E(R_t; R_{t-1})$ replaces $E(R_t)$ and

$Var(R_t; R_{t-1})$ replaces $Var(R_t)$. The effect of the first change is relatively obvious. The effect of the second change is to reduce the risk *per decision*, if ρ differs from zero, as indicated by the denominator of equation (6).

If an asset must be held until T then final wealth is

$$W_T = (W_0 - a^0)(1+R_f)^T + a^0 (1+R_1)(1+R_2)\ldots(1+R_T). \quad (7)$$

where $R_1, R_2, \ldots, R_T$ are random variables and an investor chooses a^0 in period 0. Maximizing $E(W_T)$- k $Var(W_T)$ using a first order condition shows that the optimal solution for a^0 is described by

$$a^{0*} = [1 + CumMean_T - (1+R_f)^T]/[2\,k\,CumVar_T] \quad (8)$$

where $CumMean_T$ and $CumVar_T$ represent mean and variance of the return over T periods for the risky asset. Unfortunately, $CumMean_T$ and $CumVar_T$ cannot be expressed as simple functions of μ, σ^2 and ρ. Appendix 2 derives closed form solutions when T= 2 and an additional appendix available from the authors contains some numerical simulations to explore its significance. The following Lemma and Proposition give some general guidance to the kinds of answer that might be expected for T> 2.

Lemma 1

i) If $\rho \geq 0$ and $E(R_i) > R_f > 0$ then $E[(1+R_1)(1+R_2)\ldots(1+R_T)]/(1+R_f)^T$ increases as T increases.

ii) If T= 2, $var((1+R_1)(1+R_2)) = \sigma^2 (2(1+\rho)(1+\mu)^2 + (1+\rho^2)\sigma^2)$

All proofs are in Appendix 1.

Proposition 1

If $\rho \geq 0$ and $E(R_i) > R_f > 0$, then $a^{0*} > 0$.

Offering a more general Proposition with novel testable implications is difficult because, when an investor cannot trade freely, an asset's riskiness can no longer be summarized by σ^2.[7] We demonstrate this claim by showing that the effect of an increase in ρ on a^{0*}, even at ρ= 0 with T= 2, is ambiguous. If T= 2, then Appendix 2 shows that

$CumMean_2 = (1+\mu)^2 + \rho\, \sigma^2 - 1.$ (9)

An increase in ρ increases $CumMean_2$ and the magnitude of the increase increases with σ^2. Lemma 1 ii) implies that an increase in ρ increases $CumVar_2$ and that the magnitude of the increase increases with μ. When an increase in ρ increases both the expected cumulative return and the expected cumulative risk, the net effect on the optimal investment can be determined only with more precise knowledge. In general, $\partial a^{0*}/\partial \rho$ is more likely to be negative if μ is larger and if σ^2 is smaller.

This discussion has not explored the implications of borrowing or leveraging because they have been extensively studied in the context of assets which are risky and liquid. Equations (5) and (7) can be modified to recognize the effects of borrowing where B_t represents the amount borrowed in period t at a per-period interest rate of R_b. In practice, lenders often restrict the amount of borrowing based on the collateral offered by the investor. For example, if the investor could borrow no more than the fraction LTV of the purchase price, where 0< LTV< 1 is a fixed parameter, then the maximum initial investment in the risky asset would satisfy $0= W_0+ B_0- a^0$ and $B_0= a^0$ LTV: that is, $a^0= W_0/(1- LTV)$. For t> 0, the amount that can be borrowed would vary with the accumulated net equity in the asset to be used as collateral, and the timing of repayments on previous loans. In principle, B_t can vary each period but our analysis will limit borrowing to occur at t= 0 and allow the investor to borrow once when under duress. We assume that the initial loan is repaid at t= T and that a loan intended to pay for a liquidity shock would be offered only under the condition that it is repaid as soon as possible.[8]

The Effects of Liquidity Shocks

The discussion above shows that an investment choice varies with risk characteristics which depend on the ability to trade. But owning an illiquid asset is *not* inconvenient if the investor never wants to liquidate the asset. A constant flow of small scale trades in a market place can be explained by investors rebalancing their portfolios from time to time. Broad shocks, such as a change in economic conditions, can affect the market price without any trades occurring. Liquidity shocks are personal and have many sources: medical emergencies, unemployment or a new job in a different location. Investors in commercial real estate may also experience liquidity shocks: e.g. a major tenant may go bankrupt or stop paying their rent or a new law may force an owner to upgrade their building in a way that has little effect on its value as collateral. This section shows how to account for an uncertain need to access liquid assets in an interim period. We show that the effects of a liquidity shock vary with the timing of the shock.

The concept of a liquidity shock is not important in a liquid market because, at the market price, everybody always has the potential to invest or spend (i.e. liquidate) independently at any time. In an illiquid market, the investment and liquidation decisions cannot be separated and, when selling, an investor faces a trade off between the price and the time-till-sale. Even if not forced to sell an asset at a lower price, uncertainty in the timing of a liquidity shock creates a penalty that cannot be easily summarized by final wealth. Thus, a model with liquidity shocks and an illiquid market should consider the possibility of reduced consumption and the possibility of not being able to invest in high value opportunities.

An obvious modelling solution would be to assume that a liquidity shock raises the expected cost of holding all types of assets relative to value of consuming an ad hoc "liquidity good". This solution might use a variation on the Consumption-based CAPM model. We do not use this approach because the typical C-CAPM model assumes that the consumer faces a series of time-dated budget constraints with well-defined prices. Thus, this model cannot be used to study the implications of not finding a trading partner at any price. The presence of liquidity shocks also implies that the distribution of final wealth is unlikely to be well-described by anything like a Normal distribution. Therefore, to avoid the limitations of the mean-variance framework, we replace Assumption 1 by a natural generalization:

Assumption 1': The investor maximizes the expected utility of final wealth.

Since our model assumes that an investor's objective is based on final wealth, any additional insights that are associated with a consumption smoothing motive that could be derived from a similar C-CAPM model would add to our findings.

The essential characteristic of a liquidity shock is that the shock is specific to the investor and not to a particular asset. Using a one period model, Gray and Parkin (1973) considered the portfolio problem facing an investor who can pay for a shock using a range of assets with different rates of return and different costs of liquidation. Their model was developed to study the demand for money based on the "precautionary motive". More relevant to our research is their prediction that the cost of liquidating an asset has no effect on the investment in that type of asset, if no liquidity shock is large enough to require its liquidation. Unfortunately, their model has limited applicability to the issue considered here because, in a one-period model, a liquidity shock is always paid for at the same time as consumption. Our research aims to understand two dimensions of the portfolio strategy: the investment strategy showing how much of the risky illiquid asset to buy in the beginning, α, and the spending strategy used later if a liquidity shock occurs.

Figure 1: Sequence of Events

0 --------------------------	t ---------------	τ -------------------	T
Start	Liquidity Shock	Sale of Illiquid Asset	Wealth is Consumed

Figure 1 shows the sequence of events. A liquidity shock may occur during any interim period. Formally, liquidity shocks are modelled using a Poisson process where the probability that it occurs in period t is $\lambda\,(1-\lambda)^t$.[9] We assume that no shock can occur in period T in order to close the model and to account for the final consumption decisions simply. The probability of an investor experiencing no shock before consuming goods in period T is $(1-\lambda)^{T-1}$ and, for large T, the expected time until the investor experiences a liquidity shock is approximately equal to $1/\lambda$. We also assume that all shocks are the same size.

Unlike other aspects of consumption, liquidity shocks must be paid as soon as possible. If an investor does not have enough liquid assets to pay for a shock immediately, they must liquidate the illiquid asset. An investor has enough liquid assets to pay for a liquidity shock in any period t if the initial wealth plus any borrowing less the investment in the illiquid asset is sufficiently large. In other words, if $t \geq t^*$ such that

$$(W_0 + B_0 - \alpha)(1+R_f)^{t^*} = z$$

i.e.

$$t > t^*(\alpha) = \ln(z/(W_0 + B_0 - \alpha))/(1+R_f). \quad (10)$$

If $z \leq W_0 + B_0 - \alpha$ then define $t^* = 0$. If $t^* > T$ then the investor would never have enough liquid assets to pay for a shock without being forced to sell the illiquid asset, before the model ends in period T; in such cases, the computations below assume that t* is replaced by T. More formally, we suppose that an investor uses a motivated ("M") selling strategy where, if the asset has not yet been sold, it sells with probability β_M in a period at a price of δP_t where $\delta < 1$.[10]

We assume that the parameters β_M and δ are fixed characteristics of the property. Some properties may be naturally more attractive and could have a higher price or a lower time on market or both. In the context of models which analyse those characteristics and the selling strategy of an owner, the degree of market liquidity is often summarized by the trade off between time and money (e.g. Anglin, 2006). For a relatively liquid market, δ could be close to 1. In a liquid market, β_M could be large and the probability of sale using an unmotivated ("U") selling strategy (i.e β_U) could also be large. More importantly, the *trade off* would be characterized either by a low value of δ or by a large difference between β_U and β_M, or both. For large T- t, the expected time-till-sale for a motivated seller would be approximately equal to $1/\beta_M$. Allowing an investor who has not experienced a liquidity shock to use an unmotivated selling strategy, where the asset sells with a lower probability $\beta_U < \beta_M$ at a higher price of P_t, would make this model compatible with other work which investigates the selling process. An appendix available from the authors shows that introducing this option has few surprising insights for the initial investment decision.

Though δ represents the discount for selling quickly at a particular time, it is an imperfect measure of the severity of the penalty associated with a liquidity shock. Even without a price penalty to involuntary sales, i.e. $\delta = 1$, liquidity shocks penalize an investor because the best available price may be much lower than the investor would otherwise accept.

Borrowing may seem to be a lower cost substitute to selling quickly at a discount but its practical implications are unclear. Borrowing in the event of a liquidity shock should be considered in the context of borrowing for other purposes and in the context of whether the investor is solvent as an on-going concern. First, since an investor can borrow at t= 0, they may prefer to borrow in advance of needing the funds and a lender may be unwilling to lend *more* at the same interest rate at the later time. Second, borrowing with a high loan-to-value ratio introduces the possibility that the investor may become insolvent. Even if the investor is likely to be solvent by period T, the net wealth of the investor may be negative at the time of the liquidity shock. Even if net wealth is not negative, being forced to raise liquid funds using a motivated sales strategy may change a portfolio with slightly-positive net wealth into a position of negative net wealth. Montes-Negret (2009, p. 1) used the term "heavenly twins" because the issues of solvency and of liquidity "interact in complex ways" and because "it is difficult - particularly at times of crisis - to distinguish between them". A more complete investigation of this issue must consider the problem of information asymmetry and the types of contracts which are feasible. In this sense, our model does not really address the familiar paradox that banks prefer to lend to people who do not need to borrow.[11] To account for borrowing with the simplest possible presentation, our model uses the more familiar presentation of assuming that the investor knows that they can borrow at a fixed interest rate and that the loan is offered on the condition that the loan is repaid as soon as the illiquid asset is sold.

To identify the actions and choices of a forward-looking investor, it is necessary to compute the relevant objective function for each state and time. If a liquidity shock occurs in period t, and if the investor owns a sufficient amount of the liquid asset to pay for the shock, then they sell it.[12] Since the values of the remaining assets evolve according to market conditions, the investor's final wealth would be

$$W_T = ((W_0 + B_0 - \alpha)(1+R_f)^t - z)\,(1+R_f)^{(T-t)} + \alpha\,(P_T/P_0) - B_0(1+R_b)^T \qquad (11)$$

where $P_T/P_0 = (1+R_1)(1+R_2)\ldots(1+R_T)$ is random. If the amount of the liquid asset is insufficient then the investor uses all of the liquid assets immediately and tries to sells the illiquid asset using the motivated strategy. When the illiquid asset is sold in period τ, for some $\tau > t$, the investor uses the proceeds of the sale to pay the balance of the shock (plus the interest which accumulated at the per-period rate of $R_b > R_f$) and holds the balance in the form of the riskless and perfectly liquid asset until period T. Thus, if the sale occurs in period τ then final wealth would be

$$W_T = [((W_0 + B_0 - \alpha)(1+R_f)^t) - z)(1+R_b)^{\tau-t} + \alpha\,(\delta P_\tau)/P_0](1+R_f)^{(T-\tau-t)} - B_0(1+R_b)^T. \qquad (12)$$

(If the investor experiences a liquidity shock but does not sell before period T, then we close the model by assuming that sale takes place in period $\tau = T$ at the price δP_T: the relevant level of wealth is expressed in equation (12).) The final utility associated with a liquidity shock that occurs in period t and a sale in period τ would be

$O_{t\tau}(\alpha)= U([(W_0+ B_0- \alpha)(1+ R_f)^t- z]\ (1+ R_f)^{(T- t)}+ \alpha\ P_T/ P_0- B_0(1+ R_b)^T)$

if $(W_0+ B_0- \alpha)(1+ R_f)^t- z> 0$

$= U([((W_0+ B_0- \alpha)(1+ R_f)^t)- z)(1+ R_b)^{\tau- t}+ \alpha\ (\delta P_\tau)/P_0](1+ R_f)^{(T- \tau- \tau)}- B_0(1+ R_b)^T)$

otherwise (13)

where the utility function, U(.), is an increasing concave function.

After a liquidity shock in period t, a sale occurs in period τ with probability $\beta_M\ (1- \beta_M)^{\tau- t}$. Thus, the expected utility to an investor immediately after experiencing a liquidity shock in period t is

$L_t(\alpha)= \sum_{t\leq \tau\leq T} \beta_M\ (1- \beta_M)^{\tau- t}\ O_{t\tau+ 1}(\alpha) + (1- \beta_M)^{T- t- 1}\ O_{tT}(\alpha)$ (14)

If the investor experiences no liquidity shock before T then the utility payoff is:

$L_T(\alpha)= U((W_0+ B_0- \alpha)\ (1+ R_f)^T+ \alpha\ (1+ R_1)(1+ R_2)\ldots(1+ R_T) - B_0(1+ R_b)^T).$ (15)

The assumed probability distribution of liquidity shocks reveals the probability of a shock occurring in period t. Therefore, using equations (14) and (15), the expected utility at period 0 must be:

$EU_0(\alpha)= \sum_t \lambda\ (1- \lambda)^{t- 1}\ E(L_{t+ 1}(\alpha)) + (1- \lambda)^{T- 1}\ E(L_T(\alpha))$ (16)

where the expectation concerning rates of return, E(.), is with respect to information available in period 0.

Though these expressions are sufficiently complicated that we are not able to express the optimal solution for the initial investment choice in a closed form, they are simple enough that some properties can be described easily. Intuition suggests that an increase in α changes the degree of risk to an investor and, if they are risk averse or if α is close to an optimum, implies that expected utility is a locally concave function of α. Proposition 2 shows that this intuition is valid if t*(α) is not an integer.

Proposition 2

If t*(α) is not an integer, EU_0 is a locally concave function of α.

This intuition is incomplete when there are liquidity shocks because a change in α changes risk, return *and* the probability of being forced to liquidate. Although a decrease in α changes the expected return and the financial risk of the portfolio smoothly, a small decrease in α could increase the investor's payoff by reducing the number of periods during which an investor could be forced to liquidate the illiquid asset. Proposition 3 shows that, when investing optimally, the liquidity constraint is binding in one of the periods. The condition on R_f enables this intuition to apply to an arbitrarily small change in α which, in turn, allows for arbitrarily small deviations from the optimal balancing of the returns and the risks not associated with liquidation. Let α^* be the solution which maximizes EU_0.

Proposition 3

If R_f is small enough, $t^*(\alpha^*)$ is an integer.

The following proposition shows that predicting the level of investment requires knowing both the liquidity of the market and an investor's need to liquidate. It implies that the level of investment predicted by a model which uses the characteristics of a perfectly liquid market represents an upper bound.

Proposition 4

α^* is a non-increasing function of λ and z. If $\lambda > 0$, α^* is a non-decreasing function of β_M and δ.

Corollary

Because an increase in R_f decreases t^*, an increase in the rates of return, holding the spreads $\mu - R_f$ and $R_b - R_f$ constant, increases α^*.

This Corollary applies the idea that, regardless of the relative returns of the two types of assets, an increase in R_f decreases the probability that an investor would be forced to liquidate the illiquid asset.

A Model in Continuous Time

The model above is algebraically complex because uncertainty about the timing of the resolution of the liquidity shock creates a sequence of state-contingent functions. This complexity is not surprising since there is no market price to summarize the state of the market at any point in time and since risk can accumulate in odd ways over time. Though important, risk is not the only aspect worth studying. If we remove the characteristics of the model associated with financial risk then we can develop a continuous time version of the model and use it to investigate how an investor's decisions over time combine to affect the value of illiquid assets, independent of any risk-sharing or diversification effects of the market mechanism.

To focus attention, we suppose
Assumption 1'': The investor maximizes expected final wealth (i.e. the investor is risk neutral).

We also suppose that the market price of the illiquid asset grows at a rate μ with certainty. In this continuous time model, λ and β_M continue to summarize the uncertainty about the timing of a liquidity shock and the time-on-market and we can represent the uncertainty using Exponential distributions (as opposed to its cousin, the Poisson distribution, used above). The advantages of a continuous time model are that the investor's objective function depends on t* visibly and that t* can vary continuously. We maintain the assumption that an investor seeks to maximize expected final wealth, based on expectations concerning the liquidity shocks and the time-on-market if forced to sell.

Under the stated conditions, the investor's objective is

$$EU_0= (W_0+ B_0- \alpha)\ RtLA\ \alpha\ RtIA- z\ FC- B_0\ (1+ R_b)^T. \tag{17}$$

where

$RtLA= \{[\beta_M\ (1+ R_f)^T/\ln(K_1)\{\lambda\ (K_1)^T\ ((K_1^{-1}\ \exp(-\lambda))^{t^*}-1)/\ln(K_1^{-1}\ \exp(-\lambda))- (1- \exp(-\lambda t^*))\}]+ [(1+ R_b)^T\ \exp(-\beta_M T)\ \lambda\ (K_3^{t^*}- 1)/\ln(K_3)]+ (1+ R_f)^T\ \exp(-\lambda t^*)\}$

$RtIA= \{[\delta\ ((1+ R_f)^T\ \lambda\ \beta_M/\ln(K_2))\ \{K_2^T\ (K_7^{t^*}- 1)/\ln(K_7)- (K_8^{t^*}- 1)/\ln(K_8)\}]+ [\delta(1+ \mu)^T\ \exp(-\beta_M T)\ (\lambda/(\lambda- \beta_M))(1- \exp(-(\lambda- \beta_M)t^*))]+ (1+ \mu)^T\ \exp(-\lambda t^*)\}$

$FC= \{[(\beta_M\ \lambda\ (1+ R_f)^T/\ln(K_1))\ ((K_1)^T\ ((K_4^{t^*}- 1)/\ln(K_4)- (K_5^{t^*}- 1)/\ln(K_5))]+ [(1+ R_b)^T\ \exp(-\beta_M T)\ \lambda\ (K_6^{t^*}- 1)/\ln(K_6)]+ [(1+ R_f)^T\ \lambda\ [(\exp(-\lambda T)/(1+ R_f)^T- \exp(-\lambda t^*)/(1+ R_f)^{t^*})/\ln(\exp(-\lambda)/(1+ R_f))]]\}$

and

$K_1= (1+ R_b)\ (1+ R_f)^{-1}\ \exp(-\beta_M)$

$K_2= (1+ \mu)\ (1+ R_f)^{-1}\ \exp(-\beta_M)$

$K_3= (1+ R_f)\ (1+ R_b)^{-1}\ \exp(+\beta_M)\ \exp(-\lambda) = \exp(-\lambda)(K_1)^{-1}$

$K_4= (1+ R_f)^{-1}\ \exp(-\lambda)\ K_1^{-1} = \exp(-\lambda)\ (1+ R_b)^{-1}\ \exp(+\beta_M)= (1+ R_f)\ K_3$

$K_5 = (1+R_f)^{-1}\exp(-\lambda)$

$K_6 = (1+R_b)^{-1}\exp(-\lambda)\exp(+\beta_M) = K_4$

$K_7 = (1+\mu)(1+R_f)^{-1}\exp(-\lambda)(K_2)^{-1} = \exp(-\lambda-\beta_M)$

$K_8 = (1+R_f)^{-1}(1+\mu)\exp(-\lambda)$.

The steps used to derive this objective are available in an appendix available from the authors; the steps are not complex, but are tedious. This complex expression can be made slightly more transparent if a common approximation (i.e. that ln(1+ R)= R is R is small) is used:

$$\mathrm{RtLA} = \Bigg\{(1+R_b)^T\exp(-\beta_M T)\frac{R_b-R_f}{R_b-R_f-\beta_M}\frac{\lambda\left(K_3^{t^*}-1\right)}{R_f-R_b+\beta_M-\lambda} + (1 + R_f)^T\left(\exp(-\lambda t^*)\frac{R_b-R_f}{R_b-R_f-\beta_M} - \frac{\beta_M}{R_b-R_f-\beta_M}\right)\Bigg\}$$

$$\mathrm{RtIA} = (1+\mu)^T\Bigg\{\delta\frac{\beta_M}{\mu-R_f-\beta_M}\lambda\exp(-\beta_M T)\frac{\left(K_7^{t^*}-1\right)}{-(\beta_M+\lambda)} - \delta\lambda\frac{\beta_M}{\mu-R_f-\beta_M}\frac{(1+R_f)^T}{(1+\mu)^T}\frac{\left(K_8^{t^*}-1\right)}{\mu-R_f-\lambda} + \delta\exp(-\beta_M T)\frac{\lambda}{(\lambda-\beta_M)}(1 - \exp(-(\lambda-\beta_M)t^*)) + \exp(-\lambda t^*)\Bigg\}$$

$$\mathrm{FC} = \Bigg\{\Big[\frac{\beta_M}{R_b-R_f-\beta_M}(1+R_b)^T\lambda\,(K_1^{\,T}\frac{K_4^{t^*}-1}{-R_b+\beta_M-\lambda} - \frac{K_5^{t^*}-1}{-R_f-\lambda})\Big] + \Big[(1+R_b)^T\exp(-\beta_M T)\lambda\frac{\left(K_6^{t^*}-1\right)}{-R_b+\beta_M-\lambda}\Big] + \Big[(1+R_f)^T\frac{\lambda}{\lambda+R_f}\Big[\frac{\exp(-\lambda t^*)}{(1+R_f)^{t^*}} - \frac{\exp(-\lambda T)}{(1+R_f)^T}\Big]\Big]\Bigg\}$$

(18)

The complexity of these expressions is directly connected to the issue of accounting for time when behaviour varies over time. If liquidity shocks are not an issue, i.e. z= λ= 0, then equation (17) simplifies to something familiar,

$$(W_0+B_0-\alpha)(1+R_f)^T + \alpha(1+\mu)^T - B_0(1+R_b)^T, \quad (19)$$

with the implication that the investor would prefer to invest as much as possible in the asset with the higher rate of return. If $R_b = R_f$, these expressions can be simplified only a little.

z FC shows the expected cost associated with the possibility of experiencing a liquidity shock. This cost is fixed in the sense that we are more concerned with the decision margins of an investor. If a shock were paid immediately using the liquid asset then it would be easy to calculate the effect on final wealth as

$$\int_0^T z\,(1+R_f)^{T-t}\,\lambda\,\exp(-\lambda t)dt$$

which is equal to $z\,(1+R_f)^T\,(\lambda/\ln(\exp(-\lambda)(1+R_f)^{-1}))(1-(\exp(-\lambda T)/(1+R_f)^T)$. Comparing this expression to z FC shows that the full cost of a liquidity shock reflects timing issues and investor behaviour after t= 0; i.e. the effects of β_M and R_b and the relative importance of T vs. t*. Through t*, changes in z also affect the other coefficients.

The complexity of the return to the liquid asset (RtLA) shows how the shadow value of liquid assets varies with both R_b and R_f. If an investor is unlikely to use liquid assets, e.g. t* is close to 0 then RtLA is close to the nominal return of $(1+R_f)^T$. If β_M is close to 0 then the asset is hard to sell, the investor is more likely to be forced to borrow, and, adjusting for the likelihood that a liquidity shock occurs before t*, the shadow value of liquid assets is closer to $(1+R_b)^T$.

RtIA shows the shadow value to investing in the illiquid asset. It differs from $(1+\mu)^T$ because of the extra costs due to trading barriers and the possibility of being forced to liquidate at an unfortunate time. $\exp(-\lambda t^*)$ shows the probability that the liquidity shock occurs after t* and that the investor will not be forced to liquidate the illiquid asset. If it is so hard to trade the illiquid asset that all sales occur at T, i.e. $\beta_M = 0$, then RtIA simplifies to

$$(1+\mu)^T\,(\exp(-\lambda t^*)+\delta\,(1-\exp(-\lambda t^*))) \qquad (20)$$

as expected. If it is easy to find a trading partner, i.e. β_M approaches infinity, then RtIA approaches a limiting value of

$$(1+\mu)^T\,\exp(-\lambda t^*)+\delta\,\lambda\,[(1+R_f)^T-(1+\mu)^{t^*}\,(1+R_f)^{T-t^*}\,(\exp(\lambda t^*)]/\ln(K_8). \qquad (21)$$

The variable B_0 appears in equation (17) twice, as well as in the expression for t* and the interest rate on borrowed funds appears in many of the K_i. Deriving equation (17) shows that the trade offs involved in the decision to borrow varies at different times. Borrowing at t= 0 enables an investor to use the money to invest more in the asset with the higher rate of return and to invest more in the liquid asset. Since investing more in the liquid asset reduces the likelihood of being forced to liquidate at a discount, these actions complement one another and increase the final payoff. Borrowing at a later time to pay for a liquidity shock does not produce the same value-adding complementarity because the loan is offered on the condition of a quick sale. Further, being forced to sell implies that t< t* and that a loan intended to be used to overcome a liquidity shock is made soon after another loan that was considered optimal at t= 0.

Our final Proposition concerns the effects of a longer time horizon. While equation (17) indicates the expected value of investing for a fixed T, market forces tend to select investors based on differences in their personal characteristics. Two differences seem most relevant when the "term structure of risk" (MacKinnon and Al Zaman, 2009) is not an issue, differences in the likelihood of experiencing a liquidity shock sometime or the time horizon over which to earn a rate of return. Proposition 5 uses this model to show that investors with a longer time horizon react by increasing their initial investment.

Proposition 5

If T≥ t* then an increase in T increases α*.

The alternate case, of an investor who invests so much in the illiquid asset that T< t*, is more noteworthy because of what it reveals about the application of models of investor behavior. The proof of Proposition 5 considers the cases of T> t* and T= t* because α* is not too big. If the parameters of the model are such that α* is sufficiently large that t*> T then the investor would be forced to liquidate the illiquid asset if they experienced a liquidity shock, regardless of when the shock occurred. Since we assumed that the investor must liquidate *all* of the illiquid asset whenever they liquidate some of it, a risk neutral investor receives no extra benefit from investing in anything but the illiquid asset. Thus the case of T< t* is an extreme solution and it is matched with the unrealistic decision of being fully invested in the illiquid asset.

A more flexible model of investor liquidation strategies might avoid this moral hazard result by altering the extreme investment behavior but the way in which the liquidation strategy uses time would remain an important characteristic. In an investment model, two aspects of time seem relevant: the length of time which an investment is owned and the time horizon before wealth is consumed. If the model assumed that the investor's decision horizon were equal to the average holding period then the model would underestimate the benefits of holding the asset: the probability that the investor experiences a liquidity shock is $1- \exp(-\lambda T)$. In this model and except for an upper bound T, the *expected* time before a liquidity shock occurs would be approximately $1/\lambda$. Thus, in a model which assumes $T= 1/\lambda$, the probability that an investor experiences a liquidity shock can be relatively large: e.g. $1- \exp(-\lambda (1/\lambda))= 63.2$ percent.

Concluding Comments

This paper explores how the liquidity of an asset can affect an individual's investment decisions. Our analysis focused on the implications of two aspects of an illiquid market, i.e. autocorrelation and an inability to trade easily, and one aspect of an investor, an uncertain need to liquidate. It is well-known that the barriers to the flow of information in an illiquid market can create correlations in the rate of return and we show that this fact implies that the optimal portfolio is state-contingent. We also showed that the fact that an illiquid asset cannot be traded at any time changes its risk profile. The magnitude of this change depends on the degree and type of correlation over time, the holding period of the asset and the accumulation of imperfections in the risk-return trade off. Numerical simulations available from the authors show that the magnitudes can be surprisingly large. Finally, since the illiquidity of a market is unimportant if an investor never needs to liquidate an asset, we showed that the characteristics of other assets can reduce the cost of holding an illiquid asset. A surprising aspect of this analysis may be that it does not seem possible to summarize the decision to invest using any familiar metric of liquidity, such as a trade off between price and time-on-market at the time of sale. Independent of the holding period, the time horizon of the investor matters. For all of these reasons, it is important to understand the costs of holding illiquid assets in the context of a portfolio.

Borrowing offers an interesting application of these ideas because borrowing enables an investor to increase their holdings of both liquid and illiquid assets. When borrowing initially, this combination of actions adds value by increasing the thing which reduces the potential costly liquidation of the asset which earns the higher rate of return. Borrowing after a liquidity shock does not have this benefit because borrowing at this time is conditional on selling the property using a motivated selling strategy and because it occurs relatively soon after the initial borrowing. Even if the investor can convince a lender that they are solvent, an investor seeking to pay a liquidity shock may be required to pay a higher interest rate and expect a lower loan-to-value ratio than for an initial investment. Thus, discussions of liquidity and solvency should be connected more closely than is implied by the familiar assumption that the interest rate on loans is "given" at R_b. Montes-Negret (2009) indirectly suggested that some common accounting indicators, such as the "quick ratio" or the "current ratio", might offer practical insights.

Some of the parameters can be estimated directly: e.g. many researchers estimate parameters similar to β_U and β_M (see Anglin, Rutherford and Springer (2003) and its bibliography). Commonly-available data sets may include few observations of a rarely-seen liquidity shock. It may be possible to jointly estimate z and λ for an investor by using something like a "1% Value at Risk (VaR)". A 1% VaR is a number such that, 99 percent of the time, the loss will be less than this number: i.e. $F(z)= \lambda= 0.01$ where F(.) is the distribution of net profit for this investor.[13] Many banks and other financial institutions compute a VaR based on daily variations where real estate investors may prefer to base the calculation on quarterly or annual data. Alternatively, Buckles (2008) showed how to estimate a Liquidity Index which combines a measure of the difference in valuations between buyers and sellers and the matching rate. A bigger problem is that the precision of an estimate may vary with market conditions, since a "cold" market has fewer transactions than a "hot" market. With little relevant data, valuing an asset based on a model with imprecisely-estimated parameters can lead to opportunistic arbitrariness (Stulz, 2007). Our model should help to add the liquidity dimension into valuation models coherently rather than adding it in an ad hoc fashion or letting it be a source of model risk.

Our model focused on an investor's final wealth while ignoring any operating decisions and any cash flowing from an asset during the interim periods. We did so to focus on the issues related to liquidity and to avoid issues related to consumption smoothing which have been studied previously. This focus means that our model can be extended to consider issues related to the cash flow produced by an asset (in the case of owning commercial real estate) or a flow of utility that is more stable than the current conditions of the market (in the case of owning a residence). The issues associated with a continuing flow of cash may be especially important for an *on-going* business which must pay workers and to pay shareholders on a continuing basis. Operating cash flows represent an alternative way to pay for a liquidity shock and understanding the operating dimension of an asset may reveal other ways in which an investor can use time to respond to liquidity concerns. For example, a commercial operator can become aware of a tenant in difficulty and can act before a liquidity issue becomes serious enough to affect the investor's portfolio strategy. Or, if the liquidity shock cannot be pre-empted, an owner can reduce operating expenses, such as maintenance, to increase the available cash temporarily. Thus, it may be possible to extend our model to show that operational changes which affect the cash flowing directly from an asset also affect the capital value of the asset within an investor's portfolio.

Our model also focused on the investment strategy of a single trader in a market where the evolution of prices is described by μ, σ and ρ. A more complete model would solve for the investment strategies of many investors as part of a market equilibrium, though Fugazza, Guidolin and Nicodano (2009) and MacKinnon and Al Zaman (2009) show that this computation is a non-trivial exercise. Intuitively, it is easy to conceive of situations where investors anticipate that prices are falling, and will continue to fall, at the same time as investors want more liquid assets: for example, if liquidity shocks occur because investors must satisfy stock margin requirements. Considering one investor in isolation, the combined effect of experiencing a liquidity shock when market conditions are cold may be found by combining the effects discussed in equation (5) and Proposition 4. Although the relevant combination would depend on the nature of the relationship between λ for an investor and the state of the market, we conjecture that any kind of positive relationship would tend to lower the expected benefit to owning an illiquid asset relative to our model of stable conditions.

In a larger sense, proposing a connection between liquidity shocks and the state of the market shows one of the ways in which illiquidity of a market can *amplify* the effect of any other change

(Brunnermeier, 2008; Novy-Marx, 2009). Recent events showed that such analysis should also address the question of whether the liquidity of a market is endogenous and over what time period. It is not immediately clear from equation (17) whether the investor's payoff is concave or convex in β_M, i.e. whether they like or dislike variability in liquidity. The answer seems to vary with t* and T. An equilibrium analysis would identify a preferred selling strategy (i.e. select β_M and δ from a set of possibilities which could depend on the types and motives of the buyers and sellers who are active at a point in time) if the investor were forced to sell. An equilibrium analysis would also show how the market value of an asset depends on both the solvency of the asset if owned until it matures and its ease of trading at any time before. Many authors have studied parts of these questions[14] and it is beyond the scope of our analysis.

Finally, we note several recent changes in important financial institutions which suggest that understanding the characteristics of illiquidity and of investor behavior will become increasingly important. First, new accounting standards will require assets to be valued more frequently. Though the advantages of using a market price are familiar if the market is liquid, Plantin, Sapra and Shin (2008) showed that using the market value of an asset can inject "artificial risk that degrades the information value of prices" (p. 1) and effectively changes the incentives facing a portfolio manager. In their model, "the damage done by marking to market is greatest when claims are (i) long-lived [and] (ii) illiquid" (p. 1). Second, the Basel II Accord, which sets the standard for determining how much capital a bank or financial institution must have, is becoming more significant. Interestingly, while the first pillar of this Accord focuses on questions of valuation and capital adequacy, the third pillar focuses on providing the kind of information that would make the market process more liquid for all traders. Third, a recent regulation in the United States (Financial Accounting Standards Board (FASB) Statement 157) sets forth a hierarchy of preferred valuation methods. People can use valuation methods that are more or less based on a formula or a model but FASB 157 forces firms to declare the proportions of assets being valued by different methods. These declarations may encourage researchers to carefully probe the foundations of the valuation models used by investors both from the perspective of a market equilibrium and from the perspective of an individual investor.

Appendix 1

Lemma 1

i) If $\rho \geq 0$ and $E(R_i) > R_f > 0$ then $E[(1+R_1)(1+R_2)\ldots(1+R_T)]/(1+R_f)^T$ increases as T increases.

ii) If T= 2, $\mathrm{var}((1+R_1)(1+R_2)) = \sigma^2 (2(1+\rho)(1+\mu)^2 + (1+\rho^2)\sigma^2)$

Proofs

i) (by induction) Let $Z(T) = E[(1+R_1)(1+R_2)\ldots(1+R_T)]/(1+R_f)^T$. If T= 1 then $E(R_i) > R_f$ implies that $E(1+R_1) > (1+R_f)$ and $Z(1) > 1$. If T= 2 then equation (A5) in Appendix 2 implies that $Z(2) \geq (Z(1))^2$. $(Z(1))^2 > Z(1)$ since $E(R_i) > R_f > 0$.

Consider the general case.

$Z(T+1)\,(1+R_f)^{T+1} = E[(1+R_1)(1+R_2)\ldots(1+R_T)\;(1+R_{T+1})]$

$= E[(1+R_1)(1+R_2)\ldots(1+R_T)(1+R_f+R_{T+1}-R_f)]$

$= E[(1+R_1)(1+R_2)\ldots(1+R_T)(1+R_f)] + E[(1+R_1)(1+R_2)\ldots(1+R_T)(R_{T+1}-R_f)]$

$> Z(T)\,(1+R_f)^T(1+R_f) + E[(1+R_1)(1+R_2)\ldots(1+R_T)]\,E[(R_{T+1}-R_f)] +$

$\mathrm{cov}\,[(1+R_1)(1+R_2)\ldots(1+R_T), (R_{T+1}-R_f)]$ (A1)

where the extra terms follow from a covariance expansion and the inequality is valid due to the induction hypothesis. $E[(1+R_1)(1+R_2)\ldots(1+R_T)]\,E[(R_{T+1}-R_f)] > 0$ since $E(R_i) > R_f$. $\mathrm{cov}[(1+R_1)(1+R_2)\ldots(1+R_T), (R_{T+1}-R_f)] \geq 0$ because $\rho \geq 0$. Thus,

$Z(T+1) = E[(1+R_1)(1+R_2)\ldots(1+R_{T+1})]/(1+R_f)^{T+1} > Z(T).$ (A2)

ii) See Equation (A17) in Appendix 2.

Q.E.D.

Proposition 1

If $\rho \geq 0$ and $E(R_i) > R_f > 0$, then $a^{0*} > 0$.

Proof

If a^0 is near 0 then an investor is approximately risk neutral (Rabin, 2000). Therefore the variance of wealth has second order importance to the investor and the effect on the expected rate of return dominates the choice of a^0. Lemma 1 i) shows that the cumulative mean return to holding the type 2 asset is greater than the risk-free rate of return. Thus, the investor will hold some of the type 2 asset. Q.E.D.

Proposition 2

If $t^*(\alpha)$ is not an integer, EU_0 is a locally concave function of α.

Proof

EU_0 is a linear combination of time-dated functions each of which depends on α. In each period, investor risk-aversion implies that utility is a strictly concave function of α. If $t^*(\alpha)$ is not an integer, then concavity of U(.) is complemented by the fact that, for lower values of α, wealth after liquidation grows at the risky rate which is higher on average and, for high values of α, wealth after liquidation grows at the riskless rate. Thus, $E(O_{t\tau}(\alpha))$, $E(L_{t+1}(\alpha))$ and EU_0 are concave. Q.E.D.

Lemma A1

If $t^*(\alpha)$ is an integer, then an increase in α never increases EU_0 discretely.

Proof

If $t^*(\alpha)$ is an integer then a small increase in α changes the condition in equation (13) which applies. But this change in the application of the liquidity constraint never benefits the investor. Thus a small increase in α never increases $O_{t\tau}(\alpha)$ or EU_0 discretely, for any t. (The size of the jump varies with δ, μ and R_f.) Q.E.D.

Proposition 3

If R_f is small enough, $t^*(\alpha^*)$ is an integer.

Proof

Equations (11)- (16) show that an investor's choice of α^* depends on balancing the expected return to holding the risky asset against its riskiness and the risk of being forced to liquidate an illiquid asset at a discount. If $\partial EU_0(\alpha)/\partial\alpha = 0$ but $t^*(\alpha)$ is not an integer then there is no reason to increase α but, because the possibility of liquidity shocks imply that $EU_0(.)$ is discontinuous in a discrete time model, there is a reason to decrease α if R_f is small enough.

Specifically, the equation $(W_0+ B_0- \alpha)(1+ R_f)^{t^*}= z$ shows the connection between t* and α. This equation shows that, as R_f approaches 0, an increasingly large change in α is needed to produce a given change in t*. (A change in R_f also changes the level of t* for a given α but that effect is not important for the later steps in the argument, especially since all of the relevant functions are continuously differentiable in R_f.) Therefore, as R_f approaches 0, an increasingly small change in α suffices to reduce by one the number of periods for which the investor is exposed to the risk of being forced to liquidate. The traditional risk-return trade off invoked by the first order condition applies locally, where a marginal change in α has only a second order effect on risk-adjusted returns. Being able to reduce the likelihood of being forced to liquidate by a single period increases EU_0 discretely. Therefore, for R_f close enough to 0, the solution to a first order condition is dominated by a solution which is slightly less and for which $t^*(\alpha^*)$ is an integer. Q.E.D

Proposition 4

α^* is a non-increasing function of λ and z. If $\lambda> 0$, α^* is a non-decreasing function of β_M and δ.

Proof

The proof used to find the effects of a change in λ and β_M are similar to each other while the proofs used to find the effects of a change in z and δ are similar to each other. (A change in either β_M or δ has no direct effect on t*.) Therefore, we consider the proofs showing the effect of an increase in λ and of an increase in z.

Given an optimum, α^*, Proposition 3 shows that $t^*(\alpha^*)= \ln(z/(W_0+ B_0- \alpha^*))/(1+ R_f)$ is an integer. If $EU(\alpha^*)> EU(\alpha_1)$ for all α_1, then an increase in z decreases α^* without changing $t^*(\alpha^*)$: α^* must decrease. If, given z, $EU(\alpha^*)= EU(\alpha_1)$ for some α_1 then let $\ln(z/(W_0+ B_0- \alpha_1))/(1+ R_f)= t_1$ for some other integer t_1. The choice of α depends on balancing the higher expected rate of return on the illiquid asset against the risk of liquidating at a discounted price; even if $EU(\alpha^*)= EU(\alpha_1)$, an increase in z changes this trade off and causes a discrete change in the optimal solution. Specifically, an increase in z increases the probability of a forced liquidation. This fact implies that an increase in z decreases both $EU(\alpha^*)$ and $EU(\alpha_1)$ but it has a larger effect on the solution with the larger t*: i.e. if $t^*(\alpha^*) > t^*(\alpha_1)$ then an increase in z causes $EU(\alpha^*) < EU(\alpha_1)$. Thus, the optimal solution becomes $\alpha_1 < \alpha^*$.

To find the effect of a change in λ, we note that λ has no effect on $O_{t\tau}$ for any t and τ or on t*. The derivative of EU_0 with respect to α is a weighted average of derivatives of expected value of L_t, which is a weighted average of derivative of the expected value of $O_{t\tau}$:

$$\partial O_{t\tau}(\alpha)/\partial\alpha = U'([(W_0+B_0-\alpha)(1+R_f)^t - z](1+R_f)^{(T-t)} + \alpha(P_T)/P_0)\{-(1+R_f)^T + P_T/P_0\}$$

if $(W_0-\alpha)(1+R_f)^t - z > 0$

$$\partial O_{t\tau}(\alpha)/\partial\alpha = U'([\alpha(\delta P_\tau)/P_0 - (z-(W_0+B_0-\alpha)(1+R_f)^t)](1+R_f)^{(T-\tau)})\{[(\delta P_\tau)/P_0 - (1+R_f)^t)](1+R_f)^{(T-\tau)}\}$$

otherwise. (A3)

Since the marginal utility of wealth is always positive, the sign of each expression in equation (A3) is determined by the sign of

$E(\{-(1+R_f)^T + P_T/P_0\})$ if $(W_0+B_0-\alpha)(1+R_f)^t - z > 0$

$E(\{[(\delta P_\tau)/P_0 - (1+R_f)^t)](1+R_f)^{(T-\tau)}\})$ if otherwise (A4)

Since $R_f > 0$, an increase in t makes the condition $(W_0+B_0-\alpha)(1+R_f)^t - z > 0$ more likely to be true for a given α. At the same time, $E(R_i) > R_f$ implies that $E(\{-(1+R_f)^T + P_T/P_0\}) > 0$. If $(W_0+B_0-\alpha)(1+R_f)^t - z > 0$ is false, then $E(R_i) > R_f$ implies that $E(\{[(\delta P_\tau)/P_0 - (1+R_f)^t)](1+R_f)^{(T-\tau)}\})$ is increasing in t and τ. We conclude that $E(\partial O_{t\tau}(\alpha)/\partial\alpha)$ increases with t and τ, and that $E(\partial L_t(\alpha)/\partial\alpha)$ increases with t.

An increase in λ (i.e. an increase in the probability of a liquidity shock) increases the weight given to the early time periods, and decreases $\partial EU_0/\partial\alpha$. Since $\partial^2 EU_0/\partial\alpha^2 < 0$, α* cannot rise and satisfy the optimum condition. Q.E.D.

Proposition 5

If $T \geq t^*$ then an increase in T increases α*.

Proof

The optimal choice of α balances the cost of experiencing a liquidity shock against the higher rate of return earned on the illiquid asset. An increase in T has no effect on λ but increases the total probability that the investor would experience a liquidity shock at some time before T. We consider two cases since the cost of that shock depends on whether the shock occurs before or after t*. Remember, t* is independent of T but varies with α.

If T> t*, then an increase in T has no effect on the probability of being forced to liquidate the illiquid asset. But the increase in T increases the time period over which the higher rate of return would be earned if a shock does not occur before t*: since $\mu > R_f$, an increase in T increases $(1+\mu)^{T-t^*} - (1+R_f)^{T-t^*}$. Therefore, an increase in T would increase α*.

If T= t* then $T = t^*(\alpha^*) = \ln(z/(W_0 + B_0 - \alpha^*))/(1+R_f)$. Therefore, $\partial\alpha^*/\partial T = (W_0 + B_0 - \alpha^*)(1+R_f) > 0$ if $\alpha^* < W_0 + B_0$. Q.E.D.

Appendix 2: Computing Cumulative Mean and Cumulative Variance for T= 2 and Proof of the Lemma 1 ii)

Deriving $CumMean_T$ and $CumVar_T$ based on the parameters of the one-period distribution is non-trivial. Even for the simple case of T= 2, computation shows that

$$E((1+ R_1)(1+ R_2))= 1+ E(R_1)+ E(R_2)+ E(R_1 R_2) \qquad (A5)$$
$$= 1+ 2 \mu+ \mu^2+ \rho \sigma^2$$
$$> (1+ E(R_1))^2 \qquad \text{if and only if } \rho \sigma^2> 0$$

Estimating the variance of wealth is more complex. Formally,

$$var((1+ R_1)(1+ R_2))= var(R_1+ R_2+ R_1R_2) \qquad (A6)$$
$$= var(R_1)+ var(R_2)+ var(R_1R_2)+ 2 cov(R_1, R_2)+ 2 cov(R_1, R_1R_2)+ 2 cov(R_2, R_1R_2).$$

These computations rely on two implications of the Normal distribution and one implication of independence. First, the correlation between R_2 and R_1 can be summarized by

$$R_2= \mu + \rho(R_1- \mu)+ \varepsilon \qquad (A7)$$

where $\varepsilon \sim N(0, (1- \rho^2) \sigma^2)$ is independent of R_1. Second, the Normal(0, 1) distribution is distinguished by the fact that its skewness is 0 and its kurtosis is 3. Independence implies that

$E(R_1^c \varepsilon)= E(R_1^c) E(\varepsilon)= 0$

for any power c and that $E(R_1^c \varepsilon^2)= E(R_1^c) E(\varepsilon^2)$.

Computation shows that

$$1+ CumMean_2= E((1+ R_1)(1+ R_2)) = (1+ \mu)^2 + \rho \sigma^2 \qquad (A8)$$

Computation also shows that

$$CumVar_2= var((1+ R_1)(1+ R_2)) \qquad (A9)$$
$$= var(R_1)+ var(R_2)+ var(R_1R_2)+ 2 cov(R_1, R_2)+ 2 cov (R_1, R_1R_2)+ 2 cov(R_2, R_1R_2).$$

This expression can be expressed in terms of the fundamental parameters after some algebraic manipulation. By definition, $var(R_1)= var(R_2)= \sigma^2$ and $cov(R_1, R_2)= \rho \sigma^2$. Since $E(x^2)= var(x)+ (E(x))^2$ for any random variable x, $var(R_1R_2)= E(R_1^2 R_2^2)- (E(R_1 R_2))^2$ where

$$R_1^2 R_2^2= R_1^2 (\mu + \rho(R_1- \mu)+ \varepsilon)^2$$
$$= R_1^2 (\mu^2 + 2 \mu \rho(R_1- \mu)+ 2 \mu \varepsilon+ \rho^2 (R_1- \mu)^2+ 2 \rho(R_1- \mu) \varepsilon + \varepsilon^2)$$

Therefore

$E(R_1^2 R_2^2)= E(R_1^2 \mu^2)+ E(R_1^2 2 \mu \rho(R_1- \mu))+ E(R_1^2 2 \mu \varepsilon)+ E(R_1^2 \rho^2 (R_1- \mu)^2)+$
$E(R_1^2 2 \rho(R_1- \mu) \varepsilon) + E(R_1^2 \varepsilon^2).$ (A10)

Since ε is independent of R_1,

$E(R_1^2 2 \mu \varepsilon)= 2 \mu E(R_1^2) E(\varepsilon)= 0$

$E(R_1^2 2 \rho(R_1- \mu) \varepsilon)= E(R_1^2 2 \rho(R_1- \mu)) E(\varepsilon)= 0$

and

$E(R_1^2 \varepsilon^2)= E(R_1^2) E(\varepsilon^2)= (\sigma^2+ \mu^2) (1- \rho^2) \sigma^2$

Thus, we can simplify equation (A20):

$E(R_1^2 R_2^2)= (\sigma^2+ \mu^2) \mu^2+ E(R_1^2 2 \mu \rho(R_1- \mu)) + E(R_1^2 \rho^2 (R_1- \mu)^2)+ (\sigma^2+ \mu^2) (1- \rho^2) \sigma^2$ (A11)

Using the expansion $R_1= \mu+ (R_1- \mu)$ and the fact that the third central moment, "skewness", of the Normal distribution is 0,

$E(R_1^2 2 \mu \rho(R_1- \mu))= E((\mu+ (R_1- \mu))^2 2 \mu \rho(R_1- \mu))$ (A12)
$= 2 \mu \rho E((R_1- \mu)\mu^2+ 2 \mu (R_1- \mu)^2 + (R_1- \mu)^3)$
$= 2 \mu \rho (0+ 2 \mu \sigma^2 + 0)$

Similarly,

$E(R_1^2 \rho^2 (R_1- \mu)^2)= E((\mu+ (R_1- \mu))^2 \rho^2 (R_1- \mu)^2)$ (A13)
$= \rho^2 E((R_1- \mu)^2\mu^2+ 2 \mu (R_1- \mu)^3 + (R_1- \mu)^4)$
$= \rho^2 (\sigma^2 \mu^2 + 2 \mu 0+ 3 \sigma^4).$

Thus, we conclude that

$E(R_1^2 R_2^2)= (\sigma^2+ \mu^2) \mu^2+ 4 \mu^2 \rho \sigma^2+ \rho^2 (\mu^2 \sigma^2 + 3 \sigma^4) + (\sigma^2+ \mu^2) (1- \rho^2) \sigma^2$
$= \mu^4+ \sigma^2(\mu^2+ 4 \mu^2 \rho+ \rho^2 (\mu^2+ 3) + \mu^2 (1- \rho^2) \mu^2 \sigma^2 + (1+ 2 \rho^2) \sigma^4.$ (A14)

and that, since part of equation (A19) shows that $E(R_1 R_2)= \mu^2 + \rho \sigma^2$,

$var(R_1R_2)= \mu^4+ \sigma^2(\mu^2+ 4 \mu^2 \rho+ \rho^2 (\mu^2+ 3 \sigma^2)+ \mu^2 (1- \rho^2) \mu^2 \sigma^2 + (1+ 2 \rho^2) \sigma^4 – (\mu^2 + \rho \sigma^2)^2$
$= 2 \mu^2 \sigma^2 (1+ \rho)+ (1+\rho^2) \sigma^4.$ (A15)

By substituting equation (A18), and using the fact that $E(R_1^2 \varepsilon)= 0$,

$cov(R_1, R_1R_2)= E([R_1- \mu][R_1 (\mu + \rho(R_1- \mu)+ \varepsilon)- (\mu^2 + \rho \sigma^2)])= \mu (1+ \rho) \sigma^2$ (A16)

Similar steps show that $cov(R_2, R_1R_2)= \mu (1+ \rho) \sigma^2$. Combining all of these expressions and simplifying shows that

$CumVar_2= 2 (1+ \rho)(1+ \mu)^2 \sigma^2+ (1+ \rho^2) \sigma^4.$ (A17)

Consequently, the closed form expression of the optimal solution for the optimal initial investment in Case 2 when T= 2 is

$a^{0*} = [(1+\mu)^2 + \rho\, \sigma^2 - (1+R_f)^2]/[2k\, \sigma^2\, (2\,(1+\rho)(1+\mu)^2 + (1+\rho^2)\, \sigma^2)]$. (A18)

This appendix shows that the mean of final wealth varies with σ^2 for T= 2. Though this result uses some elementary statistical algebra that can be easily extended to T> 2, the algebra used to prove that the level of risk varies with the expected return per period, is sufficiently complicated that our argument cannot be extended beyond T> 2. The following proposition offers two simple conditions that are sufficient to provide the same implications. x and y may be interpreted as the cumulative returns over many periods.

Proposition A1

Consider two random variables, x and y. Suppose that cov(x+ y, x y)> 0 and, given a correlation coefficient of ρ between x and y, an increase in ρ increases cov(x+ y, x y). Consider two related random variables, x* and y* which are distributed equally except for a shift in the mean: i.e. x*= μ+ x and y*= μ+ y. Then an increase in μ increases var((1+ x*)(1+ y*)) and increases ∂var((1+ x*)(1+ y*))/∂ρ.

Proof

By definition,

$\text{var}((1+\mu+x)(1+\mu+y)) = \text{var}((1+\mu)^2 + (1+\mu)(x+y) + x\,y)$ (A19)

$= \text{var}((1+\mu)\,(x+y) + x\,y)$

$= (1+\mu)^2\, \text{var}(x+y) + \text{var}(x\,y) + 2\,(1+\mu)\,\text{cov}((x+y), (x\,y))$.

Since cov(x+ y, x y)> 0, this equation implies that an increase in μ increases var((1+ x*)(1+ y*)). Since var(x+ y) is independent of μ and since an increase in ρ increases cov(x+ y, x y), an increase in μ increases ∂var((1+ x*)(1+ y*))/∂ρ. Q.E.D.

Acknowledgement

We thank Charles Leung, Len Lin, Jane Londerville, Francis Longstaff, James Shilling, Yuntong Wang, Kuan Xu, participants at the Asian Real Estate Society and AREUEA Conferences and, especially, Shi Fu for comments on an earlier draft. We also appreciate the insights and comments offered by the referees and editor which improved the quality of this paper. We remain responsible for any errors.

Bibliography

Anglin, P., 2006. Value and liquidity effects of a change in market conditions, *Journal of Housing Economics*, 15 (4), 293- 304.

Anglin, P., R. Rutherford and T. Springer, 2003. The trade-off between the selling price of residential properties and time-on-the-market: The impact of price setting, *Journal of Real Estate Finance and Economics*, 26 (1), 95- 111.

Bangia, A., F. Diebold, T. Schuermann, J. Stroughair, 1999. Modelling liquidity risk, with implications for traditional market risk measurement and management, Wharton School working paper 99-06.

Baroni, M., F. Barthelemy and M. Mokrane, 2007. Optimal holding period for a real estate portfolio, working paper, ESSEC Business School.

Benmelech, E., M.J. Garmaise and T. J. Moskowitz, 2005. Do liquidation values affect financial contracts? Evidence from commercial loan contracts and zoning regulation, *Quarterly Journal of Economics*, 120 (3), 1121- 1154.

Bond, S., S. Hwang, Z. Lin and K. Vandell, 2007. Marketing period risk in a portfolio context: Theory and empirical estimates from the UK commercial real estate market, *Journal of Real Estate Finance and Economics*, 34 (4), 447- 462.

Brown, B., B.A. Ciochetti, and T.J. Riddiough, 2006. Theory and evidence on the resolution of financial distress, *Review of Financial Studies*, 19(4), 1357- 1397.

Brown, R., and T. Guerts, 2005. Private investor holding period, *Journal of Real Estate Portfolio Management*, 11 (2), 93- 104.

Brunnermeier, M. K., 2008. Deciphering the Liquidity and Credit Crunch 2007-08, *Journal of Economic Perspectives*, 23 (1), 77- 100.

Buckles, B. 2008. Liquidity dynamics in commercial real estate, *Journal of Real Estate Portfolio Management*, 14 (4), 307- 324.

Capozza, D., P. Hendershott and C. Mack, 2004. An anatomy of price dynamics in illiquid markets: Analysis and evidence from local housing markets, *Real Estate Economics,* 32 (1), 1-32.

Capozza, D., and R. Israelsen, 2007. Predictability in equilibrium: The price dynamics of real estate investment trusts, *Real Estate Economics*, 35 (4), 541- 568.

Case, K., and R. Shiller, 1989. The efficiency of the market for single family homes, *American Economic Review*, 79 (1), 125-137.

Cheng, P., Z. Lin, and Y. Liu, 2010. Illiquidity and portfolio risk of thinly traded assets, *Journal of Portfolio Management*, 36 (2), 126- 138.

Chinloy, P., 1999. Housing, Illiquidity, and Wealth, *Journal of Real Estate Finance and Economics*, 19 (1), 69-83.

Clayton, J., G. MacKinnon and L. Peng, 2008. Time variation of liquidity in the private real estate market: Causes and consequences, *Journal of Real Estate Research*, 30 (2), 125-60.

Clayton, J., and G. MacKinnon, 2007. Rational hot markets: Price and volume in the housing market, working paper, St. Mary's University.

Collett, D., C. Lizieri and C. Ward, 2003. Timing and the holding periods of institutional real estate, *Real Estate Economics*, 31 (2), 205- 222.

Duffie, D., N. Garleanu, and L.H. Pedersen, 2005. Over-the-counter markets, *Econometrica*, 73 (6), 1815- 1848.

Fama, E., 1970. Multi-period consumption-investment decisions, *American Economic Review*, 163- 174.

Fisher, J., D. Gatzlaff, D. Geltner, and D. Haurin, 2003. Controlling for the impact of variable liquidity in commercial real estate price indices, *Real Estate Economics*, 31 (2), 269- 303.

Fu, Y. 1995. Uncertainty, liquidity, and housing choices, *Regional Science and Urban Economics*, 25 (2), 223-36.

Fugazza, C., M. Guidolin and G. Nicodano, 2009. Time and risk diversification in real estate investments: Assessing the ex post economic value, *Real Estate Economics*, 37 (3), 341- 382.

Garleanu, N., and L.H. Pedersen, 2007. Liquidity and risk management, *Papers and Proceedings, American Economic Review*, 97 (2), 193- 197.

Garleanu, N., 2009. Portfolio choice and pricing in illiquid markets, *Journal of Economic Theory*, 144, 532–564.

Getmansky, M., A. Lo and I. Makarov, 2004. An econometric model of serial correlation and illiquidity in hedge fund returns, *Journal of Financial Economics*, 74 (3), 529-609.

Giambona, E., J. Harding and C.F. Sirmans, 2008. Explaining the variation in REIT capital structure: The role of asset liquidation value, *Real Estate Economics*, 36(1), 111- 138.

Gray, M.R., and M. Parkin, 1973. Portfolio diversification as optimal precautionary behaviour, in M. Morishima, et al, *Theory of Demand: Real and Monetary*, Clarendon Press, Oxford, pp. 301-315; reprinted in *Foundations of Monetary Economics Vol. I*, 2000, ed. by D. Laidler, Edward Elgar, Northampton, MA.

Gu, A.Y., 2002. The Predictability of House Price *Journal of Real Estate Research*, 24 (3), 213-

32.

Haurin, D. and H.L. Gill, 2002. The impact of transaction costs and the expected length of stay on homeownership, *Journal of Urban Economics*, 51, 563- 584.

Holmstrom, B., and J. Tirole, 2001. LAPM: A Liquidity-Based Asset Pricing Model, *Journal of Finance*, 56 (5) 1837- 1867.

Huang, M., 2003. Liquidity shocks and equilibrium liquidity premium, *Journal of Economic Theory*, 109, 104- 129.

Johnson, K., J. Wiley, and Z. Wu, 2007. On the relationship between commercial property price and its selling time, *Journal of Real Estate Portfolio Management*, 13(4), 389- 400.

Jud, G.D., T. Wingler and D. Winkler, 2006. Single family housing and wealth portfolios, *Journal of Real Estate Portfolio Management*, 12 (1), 13- 22.

Leung, C.K.Y., G.C.K. Lau and Y.C.F. Leong, 2002. Testing Alternative Theories of the Property Price-Trading Volume Correlation, *Journal of Real Estate Research*, 23 (3), 253-63.

Lin, Z., and K. Vandell, 2007. Illiquidity and pricing biases in the real estate market, *Real Estate Economics*, 35 (3), 291- 330.

Lin, Z., Y. Liu and K. Vandell, 2009. Marketing period risk in a portfolio context: Comment and extension, *Journal of Real Estate Finance and Economics*, 38, 183- 191.

Longstaff, F., 2001. Optimal portfolio choice and the valuation of illiquid securities, *Review of Financial Studies*, 14 (1), 407- 431.

Kahl, M., J. Liu and F. Longstaff, 2003. Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it?, *Journal of Financial Economics*, 67, 385- 410.

Longstaff, F., 2009. Portfolio claustrophobia: Asset pricing in markets with illiquid assets, *American Economic Review*, 99(4): 1119–44.

MacKinnon, G., and A.A. Zaman, 2009. Real estate for the long term: The effect of return predictability on long-horizon allocations, *Real Estate Economics*, 37 (1), 117- 153.

Montes- Negret, F., 2009. The heavenly liquidity twin: The increasing importance of liquidity risk, Policy Research Working paper 5139, World Bank.

Novy-Marx, R., 2009. Hot and cold markets, *Real Estate Economics*, 37 (1), 1- 22.

Ortalo-Magné, F. and S. Rady, 2002. Tenure choice and the riskiness of non-housing consumption, *Journal of Housing Economics*, 11, 266-279.

Plantin, G., H. Sapra and H.S. Shin, 2008. Marking-to-Market: Panacea or Pandora's Box, *Journal of Accounting Research*, 46 (2), 435-60.

Rabin, M., 2000. Risk aversion and expected utility theory: A calibration theorem, *Econometrica*, 68 (5), 1281- 1292.

Stulz R., 2007. Hedge Funds: Past, Present, and Future, *Journal of Economic Perspectives*, 21 (2), 175-194.

Weill, P.-O., 2008. Liquidity premia in dynamic bargaining markets, *Journal of Economic Theory*, 140, 66 – 96.