Auctions Versus Negotiations Revisited*

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First draft, April 2002.
This draft, May 2004.

Abstract

Auctions are compared with different forms of negotiation and bargaining. The value of an extra customer is shown to exceed the value of being able to ration supply, if the seller is otherwise unconstrained. Thus, we generalize and provide a simple proof of a result due to Bulow and Klemperer (1996), in which auctions and negotiations are compared. We also compare auctions and seller-offer bargaining with many privately informed buyers. With discrete demand, bargaining yields higher revenue, in the limit, than the English auction. When demand is continuous, we argue that the English auction can be improved upon by pre-auction negotiations.

Keywords: Auctions, bargaining, incomplete information, rationing.
JEL classification: C78, D44, D82.

*I would like to thank Arieh Gavious, Per B. Overgaard and Gopal Das Varma for valuable comments.
1 Introduction

There is a multitude of ways to sell goods, among them several different kinds of auction and bargaining procedures. A large part of the theoretical literature on auctions has been preoccupied with deriving the optimal mechanism, or, if this cannot be implemented, finding the best auction from among a set of feasible auction formats. In the latter case, the auctions considered usually include the first and second price auctions, which are generally suboptimal. However, if the seller can choose only from suboptimal auctions, he has a clear incentive to consider other selling methods, including bargaining.

Likewise, in the bargaining literature, it is customary to focus on one bargaining protocol, and examine the set of equilibria hereof. However, if the seller has some choice in the matter of bargaining protocol, it is also relevant to compare different bargaining protocols, and compare bargaining with auctions.

Consequently, in this paper we examine different aspects of bargaining and auctions. We use mechanism design to unify the treatment of the two. First, we revisit the existing comparison between auctions and “negotiations”. Then, we go on to compare auctions with a specific bargaining protocol when demand is discrete (the type-space is finite). We conclude with a discussion of continuous demand, and how the English auction can be improved upon by what we informally will term pre-auction negotiations.

Bulow and Klemperer (1996) argue that optimally negotiating with a set of buyers is informationally demanding, and ask how much information is worth.\footnote{In Bulow and Klemperer’s (1996) terminology, optimal negotiation does not refer to a specific selling procedure, but rather to a mechanism which maximizes revenue. In this paper, I will also generally (with some exceptions) abstract from the details of the selling procedure, by referring to it simply as a mechanism.} Under certain assumptions, including buyer symmetry, they show that an optimal mechanism with $n$ buyers is revenue inferior to the English auction with $n + 1$ buyers. That is, it is preferable to invest in gathering more buyers than gathering information.

Allowing for asymmetric buyers, we will show that the ability to ration supply is inferior to an extra bidder, if the seller is otherwise unconstrained. As we argue in Section 2, this is one way of extending Bulow and Klemperer’s (1996) result to asymmetric buyers. For now, however, we state a simple proof of Bulow and Klemperer’s (1996) result.

Notice first that an optimal mechanism with $n$ buyers is inferior to the
following mechanism with \( n + 1 \) buyers. First, arrange an optimal mechanism with \( n \) buyers. If this mechanism results in no trade, the object is sold to the remaining buyer at a price equal to the lowest possible valuation that buyer can have. Hence, the object is sold with certainty and revenue is no lower than without the additional bidder. At the same time, the mechanism is inferior to a mechanism that maximizes revenue subject to the constraint that the object is sold with certainty. Now, given the symmetry and regularity assumptions imposed by Bulow and Klemperer (1996), the English auction is optimal among mechanisms in which the object is sold with probability one. Thus, an optimal mechanism with \( n \) buyers is inferior to the English auction with \( n + 1 \) buyers.

In Section 3, we assume buyers have discrete demand. We derive the highest possible revenue, and show that the seller can obtain it by committing to a sequence of offers. Of course, if the seller can commit, he is not constrained by “sequential rationality” or time consistency, i.e. by the fact that it may be profitable to deviate from the sequence after the first offer is made.

We therefore go on to examine a seller-offer bargaining game in which the seller can switch from buyer to buyer, and make an offer each period. Though the seller is now constrained by sequential rationality, we show that the seller prefers seller-offer bargaining to the English auction when agents are sufficiently patient. In some cases, there is no static mechanism which yields higher revenue than seller-offer bargaining.

In this part of the paper we contribute to the very small literature on non-cooperative bargaining with several privately informed buyers. In the only related paper we are aware of, De Fraja and Muthoo (2000) consider the same game but assume buyers are symmetric. In addition to generalizing their results, we show how mechanism design can be utilized to simplify some of the arguments and to supply intuition for the results.

As an aside, the English auction is not optimal among efficient mechanisms when buyers have discrete demand, and so the proof we gave of Bulow and Klemperer’s (1996) result does not hold. In fact, we show that the result is generally invalid with discrete demand.

Finally, in Section 4, we assume buyers have continuous demand. The English auction is only rarely optimal, but it can be improved upon by pre-auction negotiations. An example with continuous demand is provided in which Bulow and Klemperer’s (1996) result does not hold. In the example, the regularity assumption is not satisfied, implying that the English auction
is not constrained optimal.

2 Rationing, auctions and negotiations

Consider a seller with an indivisible object for sale. Assuming he is risk neutral and puts zero value on consuming the good himself, he aims to maximize revenue. Buyers are also assumed to be risk neutral, and to have independent and private valuations which are bounded below by the seller’s valuation (of zero).\(^2\)

Two fundamental lessons from monopoly theory are very useful in understanding the principles that underlies the revenue maximizing mechanism.\(^3\) First, it is well known that a monopolist generally wishes to ration supply, to avoid supplying customers who have negative marginal revenue. Similarly, a mechanism designer usually desires to design a mechanism where the good is withheld with positive probability. For this reason, reserve prices are often used in auctions. In the following, we will say than a mechanism is \textit{optimal} if it maximizes revenue, and \textit{constrained optimal} if it maximizes revenue subject to the constraint that the good is sold with probability one, i.e. that the seller is unable to ration supply.\(^4\)

Secondly, a monopolist who supplies different markets often has an incentive to price discriminate between the markets. Likewise, a mechanism designer has an incentive to discriminate between buyers, although this incentive is absent when buyers are symmetric.

Now, when analyzing the classical monopoly problem, it is natural to start by assuming that demand is continuous and marginal revenue curves are monotonic. The same assumption, which we will refer to as the \textit{regularity assumption}, is often imposed in auction theory. Bulow and Klemperer (1996) assume buyers are symmetric, implying there is no incentive to discriminate between them, and that demand is regular. Given these assump-

\(^2\)The assumption of private valuations is not important for our discussion of Bulow and Klemperer’s (1996) result. When signals are dependent, they obtain a less general result, namely that the English auction with \(n+1\) buyers is more profitable than any \(n\) buyer mechanism in a certain class. See Kirkegaard (2003) for a more detailed discussion.

\(^3\)The English auction and the first price auction are examples of mechanisms. In the following, the term auction will be used to refer to either of these mechanisms.

\(^4\)Contrary to the (constrained) optimal revenue, the (constrained) optimal mechanism is not unique.
tions, however, the English auction is constrained optimal.\textsuperscript{5} Consequently, their comparison between an optimal mechanism with \(n\) buyers and the English auction with \(n+1\) buyers is in effect a comparison between an optimal mechanism with \(n\) buyers and a constrained optimal mechanism with \(n+1\) buyers.

One possible way to extend Bulow and Klemperer’s (1996) result, then, is to show that an optimal mechanism with a given set of buyers is revenue inferior to a constrained optimal mechanism with a larger set of buyers. In Section 2.1, we provide a straightforward proof of this fact. Bulow and Klemperer’s (1996) result follows as a corollary, and Section 2.2 contains a short discussion relating to this point.

In this paper, we will consider both discrete and continuous demand. One important difference is that the Revenue Equivalence Theorem holds if demand is continuous, but not if it is discrete. In Section 2.3 we provide a short explanation of the reason behind this difference.

### 2.1 Rationing

We show that the seller would prefer an extra bidder to the ability to ration supply. That is, an optimal mechanism with a given set of buyers is revenue inferior to a constrained optimal mechanism with the same set of buyers and a newcomer. The proof is extremely simple, and does not presuppose any knowledge of auction theory. Furthermore, it is not assumed that buyers are symmetric, implying that any extra bidder is more valuable than the ability to ration supply. Likewise, it is not assumed that demand is regular.

To prove the result, notice that an optimal mechanism with the initial set of buyers yields the same revenue as the following mechanism with the initial set of buyers and a newcomer. Simply stage an optimal mechanism with the initial set of buyers, and if the object is not sold in this mechanism, give the object to the newcomer. However, since the good is sold with probability one in this mechanism, revenue is at most as high as revenue in a constrained optimal mechanism with the initial set of buyers and the newcomer.

**Proposition 1** Revenue in a constrained optimal mechanism with the initial set of buyers and a newcomer is at least as high as revenue in an optimal mechanism with the initial set of buyers.

\textsuperscript{5}This is a standard result in auction theory, and an explanation can be found in Section 3.1.
In an optimal mechanism with the initial set of buyers, the seller can credible threaten, or commit, to ration supply. This threat enables him to extract more rent from buyers. In the intermediate mechanism used in the proof, the seller uses the newcomer to threaten the initial set of buyers.

Generally, a constrained optimal mechanism with the initial set of buyers and a newcomer is strictly better than an optimal mechanism with the initial set of buyers. A sufficient condition is that the newcomer would have a positive probability of winning an efficient mechanism (e.g. an English auction). That is, the support of the newcomer’s valuation is not to the left of the support of the valuations of the initial buyers. Clearly, this is satisfied if the initial buyers and the newcomer are symmetric, as assumed by Bulow and Klemperer (1996). The explanation is postponed to Section 3.1.

### 2.2 Auctions vs. negotiations

As mentioned, Bulow and Klemperer (1996) assume that buyers are symmetric, and that demand is regular. Given these assumptions, the English auction is constrained optimal.

**Corollary 1** If buyers are symmetric and demand regular, an optimal mechanism with \( n \) buyers is revenue inferior to the English auction with \( n + 1 \) buyers.

Bulow and Klemperer (1996) consider the English auction because it is easily implemented, and does not require any information concerning the distribution from which valuations are drawn. An optimal mechanism, on the other hand, requires sufficient information to determine the optimal level of rationing (the optimal reserve price in an auction) when buyers are symmetric, and, additionally, how to best discriminate if buyers are asymmetric. They refer to an optimal mechanism as negotiation.

From this perspective, Corollary 1 establishes that “it will often be more worthwhile for a seller to devote resources to expanding the market than to collecting the information and making the calculations required to figure out the best mechanism”. Hence, Bulow and Klemperer (1996) recognize the difficulties in gathering information, but not the difficulty associated with committing to rationing supply. In the discussion above, we have recognized that it can be difficult to ration supply, but ignored the difficulties in gathering information.
In the special case of symmetric buyers with regular demand, the value of information (given the ability to ration supply) is the same as the value of being able to ration supply (given the information required to do so optimally). In either case, the comparison is between an optimal mechanism and the English auction with an additional buyer.

In principle, however, the seller need not know the distributions from which buyers’ valuations are drawn in order to implement an optimal mechanism. For instance, Caillaud and Robert (2003) suggest a straightforward mechanism which implements an optimal mechanism given the buyers know each others’ distributions. However, the ability to ration supply is necessary to implement an optimal mechanism.

2.3 Revenue equivalence

The Revenue Equivalence Theorem is probably the best known result in auction theory. One version of the Revenue Equivalence Theorem states that two mechanisms which yield the same winner also yield the same revenue. This version of the theorem is due to Myerson (1981) and Riley and Samuelson (1981).

To see why this result is plausible, it is useful to consider the monopoly problem again. First, assume the (inverse) demand curve is continuous and given by \( P(q) \), say. Then, if we are informed that the quantity sold (the “allocation”) is \( q' \), we know immediately, by a revealed preference argument, that the net price is \( p' = P(q') \). Obviously, if the net price had been lower, the quantity sold would have been higher than \( q' \). Likewise, a net price exceeding \( p' \) can be ruled out. Hence, revenue is \( q' P(q') \).

Now, it is clearly irrelevant how the net price is paid. Whether a one time lump-sum payment is made, or whether the good is paid for in 12 installments, say, does not change the fact that the net price must be exactly \( p' \). In other words, to determine revenue we need not know the rules of the selling mechanism as long as the allocation, in this case \( q' \), is known. All selling mechanisms that yield the same quantity must yield the same revenue.

Another version states that the first price auction and the English auction yield the same revenue when buyers are symmetric. This, however, follows from (1) the fact that the two auctions produce the same winner if buyers are symmetric and (2) an application of the Revenue Equivalence Theorem stated above. This version of Theorem is due to Vickrey (1961).
When buyers are symmetric and demand regular, any constrained optimal mechanism must be efficient. Since the English auction is efficient, it produces the constrained optimal allocation, and hence the constrained optimal revenue. If marginal revenue is non-monotonic, any constrained optimal mechanism must be inefficient, and the English auction is not constrained optimal. In Section 4 we propose a modified English auction which is optimal in these circumstances.

The Revenue Equivalence Theorem holds when demand is continuous, but not when demand is discrete.\(^7\) Again, it is useful to look at the monopoly problem to gain an understanding of why the Revenue Equivalence Theorem fails in this case. To illustrate, assume that

\[
P(q) = \begin{cases} \bar{\theta} & \text{if } 0 \leq q \leq \bar{q} \\ \theta & \text{if } \bar{q} < q \leq 1 \end{cases}
\]

and that \(\bar{\theta} > \bar{\theta} \geq 0\) (with \(P(q) = 0\) if \(q > 1\)). Then, knowing that \(q\) units were sold does not reveal the price, but only that it is at most \(\bar{\theta}\), and at least \(\bar{\theta}\). Thus, two selling mechanisms that both generate sales of \(q\) units could in principle yield different revenues.

Given the quantity sold is \(q\), the net price that maximizes revenue is clearly \(\bar{\theta}\). If the price is \(\bar{\theta}\), the high valuation buyer is willing to buy and does not get any rent. If the price is below \(\bar{\theta}\), on the other hand, the buyer is left with unnecessarily much rent, and the monopolist can do better.

Notice that if the price is indeed \(\bar{\theta}\), a high valuation buyer is indifferent between buying and not buying. Since a low valuation buyer does not buy, a buyer with high valuation is therefore indifferent between acting like the other high valuation buyers (and buy) and acting like a low valuation buyer (and not buying).

Similarly, for a mechanism designer designing an optimal mechanism, the rules of the mechanism must be constructed in such a way that a buyer with a high valuation is indifferent between following his equilibrium strategy

\(^7\)Of course, this ignores the possibility that the seller can perfectly price discriminate. To do so, however, requires that the seller knows the valuation of each individual. In this case, the analogue between mechanism design and monopoly pricing breaks down, because in the former it is explicitly assumed that valuations (or, more generally, signals) are private.

\(^8\)Maskin and Riley (1985) and Riley (1989) show that the other version of the theorem holds for discrete demand as well as continuous demand.
(acting as if his valuation is high) and using the equilibrium strategy of the low valuation buyer (acting as if his valuation is low). That is, the downwards incentive compatibility constraint must be binding (see Section 3.1).

When buyers are symmetric and valuations either high or low, the constrained optimal allocation is efficient (see Section 3). Although the English auction is efficient, it is not constrained optimal. Another mechanism can be constructed which moves more rent from the high valuation buyers to the seller. As we shall see, seller-offer bargaining is constrained optimal when the constrained optimal allocation is efficient.

3 Discrete demand

There are several important differences between discrete and continuous demand, as illustrated above. While the largest part of auction theory, including Bulow and Klemperer (1996), assume demand is continuous, we will focus on discrete demand in this section.

Bergemann and Pesendorfer (2001) consider a seller who faces the problem of (1) determining the accuracy, by releasing information about the object for sale, with which buyers learn their valuations and subsequently (2) designing an optimal selling mechanism, given the released information. If the seller chooses to reveal all his information to a given buyer, the buyer will learn his exact valuation, which is assumed drawn from a continuous distribution. If the seller withholds some information, the buyer may learn in which interval his valuation lies, but not his exact valuation. Now, the more accurate the estimate of a buyer’s valuation is, the more informational rent the seller will have to afford him when selling the object. Consequently, Bergemann and Pesendorfer (2001) are able to show that it is optimal for the seller to release only partial information such that each buyer will learn only that his valuation is in one of a finite set of intervals. In effect, demand is, endogenously, discrete.

We begin by deriving the optimal revenue when demand is discrete (Section 3.1). It is shown that the optimal and the constrained optimal revenue can be obtained by a mechanism in which the seller makes a sequence of take-it-or-leave-it offers to buyers (Section 3.2), an observation made earlier by Bergemann and Pesendorfer (2001). That is, buyer 1, say, is offered the good initially. If he declines, buyer 2 is offered the good, and should he also decline, an offer is made to a third buyer, or another offer is made to buyer
1, and so on.

To prove the optimality of this mechanism, it is assumed that the seller can commit to the sequence of offers. Consequently, questions of “sequential rationality” on the part of the seller are ignored. After the seller has made the first offer, he updates his beliefs, and generally he would want to change the offers to follow. Thus we may want to ask whether the ability to commit to the sequence is important.

Hence, we consider a seller who does not have the ability to commit, but is constrained by “sequential rationality” (Section 3.3). That is, we consider a bargaining game in which the seller makes offers to buyers. We show that if the constrained optimal allocation is efficient, this bargaining protocol has an equilibrium in which revenue converges to the constrained optimal revenue as the discount factor converges to one. Thus, when the constrained optimal allocation is efficient, the ability to commit to the sequence of offers is not important. However, we explicitly allow only the seller to make offers.

As mentioned above, when buyers are symmetric but demand discrete, the English auction is not constrained optimal. Then, we will show that it is not necessarily the case that the English auction with \( n + 1 \) buyers dominates the optimal mechanism with \( n \) buyers (Section 3.4).

In Section 4, we conclude with a discussion of continuous demand. In cases where the English auction is not optimal, we show that it can be improved upon by pre-auction negotiations. Another example is provided in which Bulow and Klemperer’s (1996) result does not hold.

### 3.1 Mechanism design

Fudenberg and Tirole (1991) derive the optimal mechanism in the presence of two symmetric buyers with valuations that take one of two possible values.\(^9\) We will extend their method of solving incentive and individual rationality constraints to an arbitrary, finite number of valuations. Inspired by Myerson’s (1981) approach, the method of analysis employed here then depart from Fudenberg and Tirole’s, to allow us not only to extend to asymmetric buyers, but also to illustrate the similarities between mechanism design with finitely many types and mechanism design with uncountably many types.\(^{10}\)

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\(^9\) Armstrong (2000) considers \( n \) symmetric buyers.

\(^{10}\) Bergemann and Pesendorfer (2001) consider a more general environment (which encompasses discrete valuations). They derive the expected revenue from a given allocation (see their Theorem 1) while implicitly assuming that the downwards incentive compat-
Appendix A contains the proofs of the results in this subsection.

We consider a seller with one object to sell. The value that the seller attaches to consuming the good himself is assumed to be zero. There are \( n \) potential buyers, described by the set \( I = \{1, \ldots, n\} \). The type of buyer \( i \), \( \theta_i \), is his reservation price, or valuation, which falls in the interval \([\theta_i, \overline{\theta}_i]\). Buyer \( i \) has one of \( M_i \) possible types, \( 1 \leq M_i < \infty \), the set of types being denoted by \( \Theta_i = \{\theta_i^{M_i}, \theta_i^{M_i-1}, \ldots, \theta_i^2, \theta_i^1\} \), where \( \infty > \overline{\theta}_i = \theta_i^{M_i} > \theta_i^{M_i-1} > \ldots > \theta_i^2 > \theta_i^1 = \theta_i \geq 0 \). The probability that \( \theta_i = \theta_i^j \) is \( f_i(\theta_i^j) > 0 \), where \( \sum_{l=1}^{M_i} f_i(\theta_i^l) = 1 \).

We denote \( \sum_{l=1}^{M_i} f_i(\theta_i^l) \) by \( F_i(\theta_i^l) \).

The parameters are common knowledge, but only buyer \( i \) knows the realized value of \( \theta_i \). We let \( \theta = (\theta_1, \ldots, \theta_n) \) denote the ordered \( n \)-tuple of realized valuations, and define \( \theta_{-i} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \). Occasionally we will then use \( (\theta_i, \theta_{-i}) \) to mean \( \theta \). Let \( \overline{\theta} = \times_{i \in I} \Theta_i \). For convenience, and without loss of generality, we order the buyers such that \( \theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \).

We use the revelation principle to characterize the optimal mechanism. In short, the revelation principle states that any allocation that can be implemented by some arbitrary mechanism can also be implemented by a direct mechanism. A direct mechanism is a construct where buyers report a type to the seller, who then decides how to allocate the object based on some prespecified rules, the mechanism. Hence, when looking for the optimal mechanism, it is sufficient to search among the direct mechanisms.

Thus, buyer \( i \) reports a valuation from \( \Theta_i \). Given the vector of reports, \( \omega \) say, the (ex post) probability that buyer \( i \) wins the mechanism is \( X_i(\omega) \). Of course, when buyer \( i \) submits his own report he does not know what his competitors will report, and therefore can only form expectations about the outcome of the mechanism. Hence, we let \( X_i^j \) and \( T_i^j \) be the expected probability of winning and the expected payment, respectively, if buyer \( i \) reported \( \theta_i^j \).

To induce the buyer to participate in the mechanism he should in expectation be left at least as well off by participating as by not participating. Furthermore, the mechanism must be designed in such a way that it is in the buyer’s interest to report the true valuation. These are the individual rationality (IR) and incentive compatibility (IC) constraints, and they are
formalized by the following inequalities:

\[
\begin{align*}
\theta_j^i X^j_i - T^j_i &\geq 0 \quad (1) \\
\theta_j^i X^j_i - T^j_i &\geq \theta_l^i X^l_i - T^l_i \quad (2)
\end{align*}
\]

which we require to hold for all \(i, j\) and \(l\). We denote \(\theta_j^i\)'s individual rationality constraint by \(IR_j^i\). We use \(IC_{i,j}^{j,l}\) to denote the incentive compatibility constraint that \(\theta_j^i\) reports \(\theta_l^i\) rather than \(\theta_j^i\). Finally, \(IC_{i,j}^{j,l}\) is said to be a downward IC constraint if \(j > l\) and an upwards IC constraint if \(j < l\).

The optimal mechanism maximizes expected revenue subject to the \(IR\) and \(IC\) constraints. The expected revenue is

\[
ER = \sum_{i \in I} \sum_{j \in \{1, \ldots, M_i\}} T^j_i f_i(\theta_j^i). \tag{3}
\]

However, the payment \(T^j_i\) that the seller can extract from the buyer \(i\) of type \(\theta_j^i\) depends on how likely it is that this buyer will win the object when reporting truthfully. Thus, we need to start by taking a look at the probabilities of winning, \(X^j_i\).

**Lemma 1 (Monotonicity)** If the mechanism is IC then \(X^j_i \geq X^{j-1}_i\). Furthermore, \(T^j_i \geq T^{j-1}_i\).

Hence, it is necessary that \(X^j_i\) is monotonic in order for the mechanism to be IC.

**Lemma 2** If \(X^j_i \geq X^{j-1}_i\), the mechanism is IR and IC if, and only if, \(IR^1_i\) and \(IC_{i,j}^{j,i+1}\) are satisfied.

If the collection of winning probabilities is monotonic (which is necessary for \(IR\) and \(IC\)) it is sufficient to consider a subset of the constraints in order to determine whether the mechanism is \(IR\) and \(IC\). However, for a given monotonic collection of winning probabilities, there is no unique collection of transfers that satisfies these constraints.\(^1\) Even if \(IR^1_i\) is binding is it

\(^1\)Nevertheless, for a given monotonic collection of winning probabilities, it is always possible to find a collection of transfers such that the mechanism (i.e. the collection of probabilities and transfers) is \(IR\) and \(IC\). For a proof, see the second part of Proposition 2.
rarely the case that the collection of transfers satisfying the IC constraints is unique. Consequently, expected revenue (or the expected transfer) is not determined by the winning probabilities alone, and two mechanisms that produce the same allocation are not necessarily revenue equivalent even when buyers with the lowest possible valuations are indifferent between the two. In Section 2.3 it is explained why such revenue equivalence holds when demand is continuous rather than discrete.

Among the mechanisms that are IR and IC, the optimal mechanism extracts as much rent as possible from the buyers.

**Proposition 2** If the mechanism is IR and IC and if it maximizes expected revenue for given \( X_i^j \) then \( T_i^j \) ensures that \( IR_i^1 \) and \( IC_i^{j-1} \) are binding. If, for given \( X_i^j \) where \( X_i^j \geq X_i^{j-1} \), \( T_i^j \) ensures that \( IR_i^1 \) and \( IC_i^{j-1} \) are binding, then the mechanism is IR and IC and expected revenue is maximized given \( X_i^j \).

We now know how to determine the optimal payments, \( T_i^j \), given a mechanism with probabilities of winning \( X_i^j \). To summarize, \( IR_i^1 \) binding implies that

\[
T_i^1 = \theta_i^1 X_i^1,
\]

while \( IC_i^{j-1} \) binding yields

\[
T_i^j = \theta_i^j (X_i^j - X_i^{j-1}) + T_i^{j-1}.
\]

Using (3) we can calculate expected revenue, given \( X_i^j \), in mechanisms where \( IR_i^1 \) and \( IC_i^{j-1} \) are binding. For notational convenience, define \( \theta_i^{M_i+1} = \theta_i^{M_i} \).

**Corollary 2** In any IR and IC mechanism with probabilities of winning given by \( X_i^j \), the least upper bound on expected revenue is

\[
ER = \sum_{i \in I} \sum_{j \in \{1, \ldots, M_i\}} MR_i(\theta_i^j) X_i^j f_i(\theta_i^j)
\]

\[
= E_{\theta_i} \left[ \sum_{i \in I} MR_i(\theta_i) X_i(\theta) \right],
\]

where

\[
MR_i(\theta_i^j) = \theta_i^j - (\theta_i^{j+1} - \theta_i^j) \frac{1 - F_i(\theta_i^j)}{f_i(\theta_i^j)}
\]

and where \( X_i^j = E_{\theta_i \sim X_i(\theta_i^j, \theta_{-i})} \).

\[ER\] is attained if and only if \( IR_i^1 \) and

\[12MR_i(\theta_i^j) = \theta_i^j \] for \( j = M_i \), as the last term is zero.
Contrary to the case of continuous demand examined by Myerson (1981), we find that two mechanism with the same \( X_i \) need not yield the same revenue, although they will if all types of all buyers are indifferent among them. This observation is far from new, as it is well known that with symmetric, discrete demand the English auction is not constrained optimal, although it induces the constrained optimal allocation. Nevertheless, it is revenue equivalent with the first price auction.

The name \( MR_i(\theta_i) \) in (6) has been chosen because it is the marginal increase in expected revenue if there is a marginal increase in the ex ante probability that buyer \( i \) is of type \( \theta_i \) and wins, \( X_i f_i(\theta_i) \). When designing the optimal mechanism, the seller should thus make it more likely that the object is bought by types with high \( MR \). This is analogous to the way a capacity constrained monopolist operating on several markets should determine the price in each market. The prices should be chosen such that customers with high marginal revenue buy the limited number of objects. Bulow and Roberts (1989) expand on the analogy between mechanism design and monopoly.

**Theorem 1** The optimal expected revenue is given by the maximum of (5) over \( X_i(\theta) \), subject to the constraint that \( E_{\theta-i}X_i(\theta_i, \theta_{-i}) \geq E_{\theta-i}X_i(\theta_i^{j-1}, \theta_{-i}) \) and subject to feasibility, \( \sum_{i \in I} X_i(\theta) \in [0, 1] \) for all \( \theta \in \Theta \). In the optimal mechanism, \( T_i^1 = \theta_i^1 X_i^1 \) and \( T_i^j = \theta_i^j (X_i^j - X_i^{j-1}) + T_i^{j-1} \) for \( j \geq 2 \).

Notice that \( MR_i(\theta_i^M) = \theta_i^M > M_i > MR_i(\theta_i^j) \) when \( M_i > 1 \). Assume for now that this property is global, i.e. that \( MR_i(\theta_i^j) > MR_i(\theta_i^{j-1}) \) for all \( i \) and \( j \). It is then simple to maximize (5), because in this case it is permissible to maximize inside the expectations operator. That is, for every realization of \( \theta \) the optimal mechanism awards the object to the buyer with the highest marginal revenue, if this is positive. Otherwise, the object is not sold, i.e. supply is rationed. Since \( MR_i(\theta_i^j) > MR_i(\theta_i^{j-1}) \), buyer \( i \) is more likely to win contingent on reporting \( \theta_i^j \) than \( \theta_i^{j-1} \), and it follows that \( E_{\theta-i}X_i(\theta_i^j, \theta_{-i}) \geq E_{\theta-i}X_i(\theta_i^{j-1}, \theta_{-i}) \).

The constrained optimal revenue is obtained by maximizing (5) under the constraint that \( \sum_{i \in I} X_i(\theta) = 1 \) for all \( \theta \in \Theta \). If \( MR_i(\cdot) \) is monotonic, the buyer with the highest marginal revenue wins the object in a constrained optimal mechanism.

In Section 2.1, we claimed that the constrained optimal mechanism with the initial set of buyers and a newcomer is strictly better than the optimal
mechanism with the initial set of buyers, when the support of the newcomer’s valuation is not to the left of the support of the valuations of the initial buyers. The reason is that the constrained optimal mechanism is strictly better than the intermediate mechanism used in the proof of Proposition 1. To see this, notice that in the constrained optimal mechanism, the newcomer wins more often than in the intermediate mechanism.\textsuperscript{13}

First, notice that if the newcomer, buyer $n + 1$, has his highest possible valuation, $\theta_{n+1}$, his marginal revenue, also $\theta_{n+1}$, strictly exceeds the marginal revenue of the rivals with positive probability, because $MR_i(\theta_i) < \theta_i \leq \theta_{n+1}$ for all $i = 1, 2, ..., n$, where the last inequality is by assumption. Second, in the intermediate mechanism, he only wins if everybody else has negative marginal revenue. However, it is now easy to see that the newcomer wins more often (at least when he has valuation $\theta_{n+1}$) in the constrained optimal mechanism, implying that the intermediate mechanism cannot be constrained optimal because the allocation is not the same.

If buyers are symmetric and $MR_i(\theta_i^j) > MR_i(\theta_i^{j-1})$, the constrained optimal mechanism is efficient since the winner will always be found among those with the highest willingness to pay. Although the English auction and the first price auction are efficient, they are not constrained optimal when demand is discrete. The reason is that the downwards IC is not binding.\textsuperscript{14,15} See Maskin and Riley (1985) for a way to adjust the first price sealed bid auction to make it constrained optimal, and Fudenberg and Tirole (1991) for an adjustment of the English auction, when $M_i = 2$. Next, we show that the seller can implement the optimal (and the constrained optimal) mechanism by making a sequence of offers to the buyers.

\textsuperscript{13}The argument also holds if demand is continuous.

\textsuperscript{14}If buyer $i$ has valuation $\theta_i^j$ it is \textit{strictly} better for him to stay in the auction until the price reaches $\theta_i^j$ rather than $\theta_i^{j-1}$. If he stays until the price reaches $\theta_i^j$, he is sure to win if the highest rival valuation is $\theta_i^{j-1}$, which is not the case if he drops out the when the price reaches $\theta_i^{j-1}$.

\textsuperscript{15}However, the English auction and the first price auction are constrained optimal when demand is regular and buyers are symmetric. The winner is the buyer with the highest marginal revenue (and valuation) and the downwards incentive compatibility constraint is binding when demand is continuous.
### 3.2 Sequence of offers

Since there are finitely many buyers, each with a finite number of possible valuations, it is possible to order marginal revenue in descending order. Then, by committing to a sequence of offers, the seller can set the prices to target the buyer with the highest marginal revenue first, proceed to the buyer with the second highest marginal revenue if the first declines, and so on. Next, we verify that such a mechanism can be implemented (i.e. it is incentive compatible and individual rational) and maximizes revenue.\(^\text{16}\)

If marginal revenue is non-monotonic, it would be tempting to favor types with high marginal revenue.\(^\text{17}\) However, such a mechanism would violate incentive compatibility. Instead, it is optimal to pool some types together and treat them as if they were one and the same, i.e. for \(j, k\) in this pool, let \(X_i^j = X_i^k\) and \(T_i^j = T_i^k\). When marginal revenue is non-monotonic for a set of types, a strictly increasing \(X_i^j\) would imply that more weight would be put on terms with low MR in (5). By letting \(X_i^j\) be constant over the interval, this is avoided. For more on how to determine the optimal interval over which \(X_i^j\) is flat, see section 4 and the references given there.

For our purposes it is sufficient to note that buyer \(i\)'s types, \(\theta_{1i}, \theta_{2i}, \ldots, \theta_{Mi}\), will be broken into ordered, disjoint sets, where each set contains only consecutive types.\(^\text{18}\) The types in each set will be treated the same, i.e. \(X_i^j = X_i^k\) and \(T_i^j = T_i^k\) for all \(\theta_{1i}, \theta_{2i}\) in the set. Let these sets be numbered consecutively and let them be denoted by \(\Phi_i^l\). Assuming there are \(L_i\) sets, buyer \(i\) will be treated as a customer with \(L_i\) potential types. To determine IR and IC constraints, the type of the set \(\Phi_i^l\) is \(\varphi_i^l = \min_{\theta_j \in \Phi_i^l} \theta_j\), the type of its smallest member. For future use, let \(\varphi\) be the ordered vector of realizations of types of all buyers. The marginal revenue of the \(\varphi_i^l\) type,

\[
MR_i(\varphi_i^l) = \sum_{\theta_j \in \Phi_i^l} \frac{MR_i(\theta_j)}{\sum_{\theta_j \in \Phi_i} f_i(\theta_j)}.
\]

\(^{16}\)The method also appears in Bergemann and Pesendorfer (2001), but we prove its usefulness here for more general environments.

\(^{17}\)In Sections 3.3 and 3.4 we will assume that \(M_i = 2\) for all \(i\). In this case, however, marginal revenue is monotonic since \(MR_i(\overline{\theta}_i) = \overline{\theta}_i > \overline{\theta}_k > MR_i(\underline{\theta}_k)\).

\(^{18}\)For example, when \(M_i = 3\), the types can be ordered as \(\{\theta_1^1\}, \{\theta_1^2\}, \{\theta_1^3\}\) or as \(\{\theta_1^1\}, \{\theta_1^2, \theta_1^3\}\), but not as \(\{\theta_1^1, \theta_1^2\}, \{\theta_1^3\}\).
is a weighted average of marginal revenues. In order for this pooling of types to be optimal, it must be the case that $MR_i(\varphi^l_i)$ is monotonic.

**Proposition 3** Assume that $MR_i(\cdot)$ has been “ironed” optimally, such that it is monotonic. Arrange the $L = \sum_i L_i$ values of marginal revenues in descending order in a set, $\Omega$. It does not matter how ties are broken. The optimal revenue is then attained if, for all realizations of $\varphi$, the object is awarded to the buyer whose realized marginal revenue appears first in $\Omega$. Let the induced, optimal probabilities of winning be given by $X^j_i$, and let $T^1_i = \varphi^1_i X^1_i$ and $T^j_i = \varphi^j_i (X^j_i - X^{j-1}_i) + T^{j-1}_i$ for $j \geq 2$.

Then, the optimal mechanism can be implemented as follows. Let $l = 1$ and approach the buyer who has the $l$'th first element in $\Omega$, buyer $i$ of type $j$, say. Offer him the object at a price of $b^j_i = T^j_i / X^j_i$. If he accepts, the auction ends, and buyer $i$ purchases the object at a price of $b^j_i$. If he declines, iterate over $l = 2, \ldots, l^*$, where $l^*$ is such that the $l^*$'th element of $\Omega$ is non-negative, and the $(l^* + 1)$'th element is negative.

**Proof.** It has been established previously that revenue is maximized if the object is awarded to the buyer with highest marginal revenue, when marginal revenue is monotonic. Assume for now that the mechanism is $IC$ and $IR$, such that buyer $i$ of type $\varphi^l_i$ accepts the offer aimed at precisely that type. Then, $X^j_i$ is indeed the ex ante probability that buyer $i$ of type $\varphi^l_i$ wins. The expected payment of type $\varphi^l_i$ is then $X^j_i b^j_i = T^j_i$, the optimal payment. Hence, the mechanism is optimal if it is incentive compatible, which we now verify.

We thus assume that it is an equilibrium for buyers to accept the $l$'th offer if and only if they have type no smaller than the $l$'th type in $\Omega$, and show that no deviation is profitable. When the $l$'th offer is made, it is common

---

19 The reason is that when we want to calculate the expected contribution to revenue of $\varphi^l_i$ using Theorem 1, it is

\[
\left( \sum_{\theta^j_i \in \Phi^l_i} MR_i(\theta^j_i) \frac{f_i(\theta^j_i)}{\sum_{\theta^j_i \in \Phi^l_i} f_i(\theta^j_i)} \right) \left[ X^l_i \sum_{\theta^j_i \in \Phi^l_i} f_i(\theta^j_i) \right],
\]

where the term in square brackets is the ex ante probability that agent $i$ is of type $\varphi^l_i$ and wins the mechanism.

20 The resulting marginal revenue curve is said to be “ironed”, because agents in the set $\Phi^l_i$ are treated as if they all had the same marginal revenue, $MR_i(\varphi^l_i)$. That is, the marginal revenue curve has been smoothed out.
knowledge that the previous offers were turned down. By accepting the offer aimed at $\phi^k_i$, buyer $i$ of type $\phi^j_i$ wins with probability one, and realizes a payoff of $\phi^j_i - b^k_i$. Assume for now that $j \geq k$, implying that the buyer should accept this and all future offers, in equilibrium. By declining the offer, positive payoff is obtained only if other buyers decline in the future, until buyer $i$ is asked again and at that time accepts the offer (by the one deviation property). This happens with probability $X_{k-1}^i / X_k^i$. Expected payoff from accepting immediately should thus exceed the payoff from waiting,

$$\phi^j_i - b^k_i \geq (\phi^j_i - b^{k-1}_i)X_{k-1}^i / X_k^i,$$

or by definition of $b^k_i$

$$\phi^j_i X_k^i - T_k^i \geq \phi^j_i X_{k-1}^i - T_{k-1}^i.$$

Finally, by definition of $T^j_i$,

$$\phi^j_i (X_k^i - X_{k-1}^i) \geq T_k^i - T_{k-1}^i = \phi^k_i (X_k^i - X_{k-1}^i),$$

which is satisfied since, by assumption, $j \geq k$. Thus, it is a best response for $j \geq k$ to accept offers aimed at $\phi^k_i$. It is straightforward to show that it is inoptimal to deviate and accept offers that were meant for types higher than the realized type.

For brevity, Proposition 3 is stated in a way that allows a given buyer to be made several offers in quick succession, with no other buyer being asked in between. In this case, all the offers to the buyer in question are the same, and so he is indifferent between which one to accept.

The proposition shows that buyers with finitely many possible valuations can give rise to a plethora of observations. When the seller optimizes we could observe that the good is attempted sold in sequence to different buyers at prices that may or may not be decreasing. Some buyers might never be approached again if they were uncooperative the first time, but we could also observe that the seller returns to a buyer he visited a long time ago, and closes a deal with this buyer. Thus, the model can explain many different “negotiation” processes as the result of optimizing behavior.

Clearly, the constrained optimal revenue can be obtained by a similar process. The only difference is that the seller does not stop giving offers once marginal revenue becomes negative.

Thus far, we have assumed the seller is able to commit to the sequence of offers. We now relax this assumption.
3.3 Seller-offer bargaining

In most of the bargaining literature with incomplete information, it is assumed that there is a single buyer and a single seller. For a recent review of this literature, see Ausubel, Cramton and Deneckere (2001). Among the bargaining protocols that have been studied is the seller-offer game, in which the seller makes an offer each period, until the buyer accepts. This game was first studied by Fudenberg, Levine and Tirole (1985) and Gul, Sonnenschein and Wilson (1986). A discussion of the model when the buyer has discrete demand can be found in Fudenberg and Levine (1991) and Ausubel, Cramton and Deneckere (2001).

In the preceding subsection we saw that a seller with several buyers can implement the optimal mechanism if he can commit to a sequence of offers. Consequently, a bargaining procedure in which the seller makes the offers holds some promise when it comes to generating revenue, although concerns of sequential rationality arise. Thus, in this section we consider a seller-offer bargaining game with several privately informed buyers.

For simplicity, we assume there are two buyers, and that buyer $i$ has one of two possible valuations, $\theta_i$ and $\bar{\theta}_i$, with $\bar{\theta}_i > \theta_i$, $i = 1, 2$. We assume that $\overline{\theta}_i > 0$, implying we are in what is known as the “gap” case in bargaining, i.e. the lowest valuation is strictly higher than the seller’s valuation. The probability that buyer $i$ has valuation $\theta_i$ is $\bar{q}_i$, and the probability that the valuation is $\bar{\theta}_i$ is then $q_i = 1 - \bar{q}_i$.

Thus far, De Fraja and Muthoo (2000) is the only other paper which allows for more buyers in the seller-offer game.\footnote{Fudenberg, Levine and Tirole (1987) consider the game with infinitely many buyers. In this case, however, the seller never returns to a buyer once he has switched to another. In this respect, the game (or at least the equilibrium) with infinitely many buyers is qualitatively different from the game with infinitely many buyers.} They assume discrete demand, but they also assume that buyers are symmetric.\footnote{Furthermore, De Fraja and Muthoo (2000) consider the possibility that the seller incurs a cost if he switches from one buyer to another, but we ignore this consideration. In their model, any strictly positive switching cost will lead the seller to make offers only to one buyer when agents are sufficiently patient. This stark result, however, relies on the assumption of symmetry. See the next footnote for details.} Here, we will allow for asymmetry, thus permitting us to extend some of their result. Furthermore, we believe the approach taken here of stressing the connections with mechanism design provides a new and more intuitive way of understanding these results.\footnote{Fudenberg, Levine and Tirole (1987) consider the game with infinitely many buyers. In this case, however, the seller never returns to a buyer once he has switched to another. In this respect, the game (or at least the equilibrium) with infinitely many buyers is qualitatively different from the game with infinitely many buyers.}
Before analyzing the seller-offer game with many buyers, we review the game with only one buyer. For completeness, Appendix B contains a formal description of this game and the equilibrium (when demand is discrete). This may be skipped on a first reading, but we suggest the reader familiarize himself with the details of the one-buyer game before studying the proofs of the results of the two-buyer game.

3.3.1 Seller-offer bargaining with one buyer

Each period, the seller makes a take-it-or-leave-it offer to the buyer. If the buyer declines the current offer, the seller makes a new offer next period. Should the buyer accept the offer, the seller has to sell at the advertised price. There is a common discount factor, $\delta$, and an infinite horizon.

On the (generically) unique equilibrium path, the seller offers a sequence of decreasing prices. The higher is the valuation, the earlier an offer is accepted (this is the “skimming” property). After a finite number of periods, the seller makes an offer equal to the buyer’s lowest valuation, at which time the buyer accepts with probability one.

As the discount factor converges to one, the first offer converges to the buyer’s lowest valuation, $\theta_1$. It follows that revenue converges (from above) to the lowest valuation as well. This is the Coase Conjecture.$^{23,24}$

In the gap case, the seller cannot credibly threaten to ration supply. The reason is that because the buyer’s lowest valuation, $\theta_1$, by assumption exceeds the seller’s valuation, the seller is better off selling at a price of $\theta_1$ than withholding the object and consuming it himself. But it is easily seen

---

$^{23}$The Coase Conjecture is the driving force behind De Fraja and Muthoo’s (2000) result concerning switching costs (see the previous footnote). If buyers are symmetric, the seller earns $\theta_1$ (in the limit) if he is committed to bargaining only with one buyer. Hence, a threat by the seller of switching from the current buyer, buyer 1 say, to buyer 2 is not credible, because (regardless of beliefs) he can ensure himself $\theta_1$ by offering $\theta_1$ to buyer 1, whereas if he switches to buyer 2 he will earn $\theta_1$ (by the Coase Conjecture) but will at the same time incur the switching cost, $c$. Hence, the seller bargains with only one buyer, and revenue converges to $\theta_1$. However, if $\theta_1 < \theta_2$ and $c < \theta_2 - \theta_1$ the threat of switching from buyer 1 to buyer 2 is credible when agents are sufficiently patient. Hence, with asymmetric buyers, switching costs do not necessarily rule out switching (if $\theta_2 \geq \theta_1$, the seller will bargain only with buyer 2 but not because of concerns about switching costs).

$^{24}$However, when there is no gap between the seller’s valuation and the buyer’s lowest valuation, there are equilibria in which the Coase Conjecture does not hold. See Ausubel and Deneckere (1989).
that in a static mechanism in which the seller cannot ration supply, the maximal revenue is precisely equal to the buyer’s lowest valuation.\textsuperscript{25} Hence, although revenue may seem low in the seller-offer bargaining game as $\delta \to 1$, there exist no static mechanism in which the good is sold with probability one and which yields higher revenue. This fundamental observation is very important for the following analysis. In fact, when allowing for more buyers we obtain a general result of this type.

\subsection{Seller-offer bargaining with two buyers}

With more buyers, the seller-offer bargaining game can be described as follows. At the beginning of each period, the seller chooses a buyer. Then, the seller makes a take-it-or-leave-it offer to the chosen buyer, and the buyer responds. If the buyer accepts, the seller has to sell at the advertised price. A new period begins if the buyer declines.

As we turn to the seller-offer game with two buyers, we look for an equilibrium in which the buyer with the highest valuation wins the object (efficiency). When $\delta \to 1$, such an equilibrium will be shown to exist, and \textit{revenue converges to the revenue of the static mechanism which maximizes revenue subject to the constraint that the allocation is efficient.}\textsuperscript{26} Hence, bargaining is at least as profitable as the English auction in this model, since the English auction is not optimal among efficient mechanism. Furthermore, when the constrained optimal allocation is efficient, seller-offer bargaining yields the constrained optimal revenue in the limit.

To illustrate, assume for now that buyers are symmetric. In this case, De Fraja and Muthoo (2000) establish that in any equilibrium of the game, revenue is bounded above by

\begin{equation}
\bar{q}_1 \bar{\theta}_1 + q_1 \omega_1,
\end{equation}

as $\delta \to 1$. This, however, is unsurprising when we consider the lessons from mechanisms design. Specifically, from Section 3.1 we know that there is no static mechanism (corresponding to $\delta = 1$) which generates higher revenue than (7) when the good is sold with probability one.\textsuperscript{27}

\textsuperscript{25}In the following, we will use the term \textbf{static mechanism} to refer to a mechanism of the type described in Section 3.1, i.e. a mechanism in which there is no discounting ($\delta \equiv 1$). By this definition, the mechanism in Section 3.2 is static.

\textsuperscript{26}Since we are in the “gap” case it is efficient to sell the object with probability one.

\textsuperscript{27}In a constrained optimal mechanism, the good is sold to a high valuation buyer if such
De Fraja and Muthoo (2000) go on to show that there is an equilibrium which yields revenue of (7) in the limit. In this equilibrium, the seller first makes one offer (equal to $\theta_1$) to buyer 1, and thereafter switches to buyer 2, never to return to buyer 1. From the Coase Conjecture, the offers (and revenue) converge to $\theta_1$ when the seller focuses on just one buyer. Hence, it is easily confirmed that revenue converges to (7).\textsuperscript{28}

The similarities between this equilibrium (or the equilibrium path) and the mechanism examined in Section 3.2 are self evident. If the seller can commit to a sequence of offers, he could do no better than offer the good at a price of $\theta_1$ to one buyer and, if unsuccessful, turn to the other buyer and sell at price $\theta_1$. This explains why seller-offer bargaining yields the constrained optimal revenue in the limit.

Obviously, when buyers are symmetric and marginal revenue monotonic (which is always the case when there are only two possible valuations), any constrained optimal mechanism must be efficient. Hence, another way of stating De Fraja and Muthoo’s (2000) result is that there is an equilibrium of the seller-offer bargaining game in which revenue converges to the highest possible revenue in an efficient static mechanism.

In the following it will be assumed that \textit{the efficient allocation is unique}. That is, there are no “ties” in valuations, $\{\theta_1, \theta_1\} \cap \{\theta_2, \theta_2\} = \emptyset$.\textsuperscript{29}

**Proposition 4** There exists a $\delta \in (0, 1)$, such that if $\delta \in (\delta, 1)$, an efficient equilibrium of the seller-offer bargaining game exists.\textsuperscript{30} As $\delta \to 1$, expected revenue converges to the expected revenue from a static mechanism that maximizes revenue subject to efficiency.

**Proof.** See Appendix C. ■

Observe that a mechanism similar to the one in Section 3.2 could be used to implement the static mechanism that maximizes revenue subject to efficiency. The only change needed is that buyers are approached based on

\textsuperscript{28}The offer to buyer 1, $\theta_1$, is accepted with probability $q_1$. With probability $q_1$, the offer is rejected, in which case the offers in the remainder of the game converge to $\theta_1$.

\textsuperscript{29}This assumption simplifies the proofs and the exposition of the results. We will return to the latter point momentarily. Notice that the assumption is violated if buyers are symmetric.

\textsuperscript{30}If $\delta$ is low, the seller may have an incentive to deviate.
valuations rather than marginal revenues. This is the starting point in proving the proposition. Specifically, an equilibrium is constructed in which the offers are designed to target buyers with higher valuations first, implying efficiency. In this equilibrium, the buyer is indifferent between accepting the offer targeted at him, and waiting (this is also the case in the mechanism in Section 3.2). This indifference condition in essence implies that the downwards incentive compatibility constraint is binding. Consequently, as \( \delta \to 1 \) expected revenue converges to revenue of a mechanism which maximizes revenue subject to efficiency.

When \( \theta_1 > \theta_2 \), it is efficient to sell the object to buyer 1. This allocation can easily be implemented in seller-offer bargaining. In equilibrium, the seller simply bargains with buyer 1 as in the one-buyer game, and ignores buyer 2, who does not contribute to competition due to his low valuation. Expected revenue is at least \( \theta_1 \).\(^{31}\) An English auction, on the other hand, yields revenue of at most \( \theta_2 \). However, an English auction with a non-prohibitive reserve price (one that is low enough to ensure that the good is sold with probability one) of \( \theta_1 \) also produces revenue of \( \theta_1 \).

On the other hand, when \( \theta_1 < \theta_2 < \theta_1 \) the seller must switch at least once to implement the efficient allocation. In this case, the English auction does not maximize revenue among static and efficient mechanisms, even with a non-prohibitive reserve price. The reason is that buyer 1's (and possibly buyer 2's) downwards incentive compatibility constraint is not binding.

**Corollary 3** If \( \delta \) is sufficiently large, there is an equilibrium of the seller-offer game which yields at least as much expected revenue as an English auction with any non-prohibitive reserve price.

In some cases, the constrained optimal allocation is efficient.\(^{32}\) In this case seller-offer bargaining is optimal among all mechanism in which the good is sold with certainty. In particular, it is at least as profitable as the English auction and the first price auction with a non-prohibitive reserve price.

\(^{31}\)If \( \theta_1 = \theta_2 \) the same equilibrium exists, and revenue again converges to \( \theta_1 \). However, \( \theta_1 \) is no longer the maximal revenue in a static and efficient mechanism. The reason is that buyer 2 should win the object when he has valuation \( \theta_2 \) and buyer 1 has valuation \( \theta_1 \), because \( MR_2(\theta_2) = \theta_2 = \theta_1 > MR_1(\theta_1) \). Consequently, Proposition 4 does not always hold when the efficient allocation is not unique.

\(^{32}\)This is the case if the buyer with the highest valuation also has the highest marginal revenue, that is if the ranking of marginal revenues coincides with the ranking of valuations.
Corollary 4 If the constrained optimal allocation is efficient, there is an equilibrium of the seller-offer bargaining game in which expected revenue converges to the constrained optimal revenue.

While we have established that seller-offer bargaining outperforms the English auction, it is impossible to rank seller-offer bargaining and the first price auction when the constrained optimal mechanism is inefficient. The reason is that the first price auction is generally inefficient, though not necessarily in the way that maximizes revenue.\(^{33}\)

Finally, we remark on a property of the equilibrium of the seller-offer bargaining game when \(\theta_2 < \theta_1 < \overline{\theta}_2 < \overline{\theta}_1\). In this case, the seller starts with buyer 1, switches to buyer 2 (and makes a one-time offer of \(\overline{\theta}_2\)) and finally returns to buyer 1 (and offers \(\theta_1\)). It is possible that the initial offer to buyer 1 is lower than \(\overline{\theta}_2\).

Corollary 5 The sequence of offers in the seller-offer game may be non-monotonic.

It is of interest to consider an example from Bergemann and Pesendorfer (2001). Recall that in their model, the seller has the power to determine the accuracy by which buyers learn their valuations. In the example, there are 2 ex ante symmetric buyers, with valuations drawn from the uniform distribution. It is shown that it is optimal for the seller not to release any information whatsoever to buyer 1, implying that his estimate of his valuation is the expected value of the uniform distribution, or \(.5\). In terms of our model, \(\overline{\theta}_1 = .5\) and \(\overline{q}_1 = 1\). When it comes to buyer 2, the seller should only release information sufficient to allow the buyer to determine whether his valuation is above or below \(.5\). It follows that \(\overline{\theta}_2 = .75\), \(\theta_2 = .25\), and \(\overline{q}_2 = .5\).

When selling the object it is optimal to first offer the object at a price of \(.75\) to buyer 2 and, if the offer is declined, to switch to buyer 1 and offer him the object at a price of \(.5\). Notice that the object is sold with probability one, implying that the seller does not need the ability to ration supply. Furthermore, the optimal allocation is efficient.\(^{34}\) Hence, while Bergemann

\(^{33}\)See Maskin and Riley (2000) for a comparison of the English auction and the first price auction when buyers are asymmetric.

\(^{34}\)The allocation is efficient in the sense that the buyer who has the highest expected valuation wins, where the expectation is contingent on the information released by the seller. However, since valuations are not known accurately in that model, it may be the case that the winner does not have the highest realized valuation.
and Pesendorfer (2001) point out that the optimal revenue can be achieved if the seller can commit to a sequence of offers, the results in this section prove that the ability to commit to offers is not necessary in this particular example.

In this subsection we have considered a specific bargaining game. It was shown that the seller is better off using seller-offer bargaining than using an English auction. At times, seller-offer bargaining is at least as profitable as any static mechanism in which the good is sold with probability one.

McAfee and Vincent (1997) consider another bargaining protocol. In each period, the object is attempted sold in an English auction with a reserve price that is determined by the seller. If the good is not sold, the seller can adjust the reserve price in the next period. With symmetric buyers expected revenue in this game converges to the expected revenue of the English auction with no reserve price. Since McAfee and Vincent (1997) assume demand is continuous, it follows that expected revenue converges to the revenue of a static mechanism which maximizes revenue subject to efficiency (of which the English auction is one).

3.4 Auctions vs. negotiations revisited

As we have seen, the English auction is not optimal among efficient mechanisms when demand is discrete. Therefore, the argument we gave to prove Bulow and Klemperer’s (1996) result is not applicable. We show that an optimal mechanism with \( n \) symmetric buyers may or may not be more profitable than an English auction with \( n + 1 \) buyers. Again, we assume that valuations take one of two possible values. By symmetry \( \overline{\theta}_i = \overline{\theta}_1, \overline{\theta}_i = \overline{\theta}_1 \).

Notice that given these assumptions, the optimal sequential mechanism is to offer the first \( n - 1 \) buyers the object at a price of \( \overline{\theta}_1 \). If the object has not been sold after these buyers have been consulted, the good should be offered to buyer \( n \) at a price of \( \overline{\theta}_1 \) if \( MR_1(\overline{\theta}_1) \geq 0 \) and at a price of \( \overline{\theta}_1 \) otherwise.

**Proposition 5** Assume buyers are symmetric. The optimal mechanism with \( n \) buyers yields higher expected revenue than the English auction with \( n + 1 \) buyers if, and only if, either

\[
q_i \leq \min \left\{ \frac{\overline{\theta}_i - \theta_1}{\overline{\theta}_1}, 1 - \frac{1}{n} \frac{\theta_1}{\overline{\theta}_1 - \theta_1} \right\},
\]
or

\[ q_1 \geq \max\left\{ \frac{\theta_1 - \theta_1}{\theta_1}, \frac{1}{n} \right\}. \]

**Proof.** Consider first the possibility that \( q_1 \leq (\theta_1 - \theta_1)/\theta_1 \), or that \( MR_1(\theta_1) \leq 0 \). Expected revenue in the optimal mechanism is

\[ ER^O(n) = (1 - q^n_1)\theta_1, \]

since revenue is \( \theta_1 \) unless every player has type \( \theta_1 \), in which case revenue is zero. In the English auction with \( n + 1 \) bidders, on the other hand, expected revenue is

\[ ER^E(n + 1) = [q_1^{n+1} + (n + 1)q^n_1\theta_1] + [1 - (q_1^{n+1} + (n + 1)q^n_1\theta_1)]\theta_1. \]

It follows that \( ER^O(n) \geq ER^E(n + 1) \) if, and only if, \( q_1 \leq 1 - \theta_1/(\theta_1 - \theta_1) \).

However, if \( MR_1(\theta_1) \geq 0 \), expected revenue in the optimal mechanism is

\[ ER^O(n) = q_1^{n-1}\theta_1 + (1 - q_1^{n-1})\theta_1. \]

Then, \( ER^O(n) \geq ER^E(n + 1) \) if, and only if, \( q_1 \geq 1/n \).

The English auction is not constrained optimal when demand is irregular. Therefore, an optimal mechanism can be superior to an English auction with an additional bidder. Nevertheless, as shown in Section 2, a constrained optimal mechanism with \( n + 1 \) bidders dominates an optimal mechanism with \( n \) bidders.\(^{35}\)

If \( MR_1(\theta_1) \leq 0 \), or \( q_1 \leq (\theta_1 - \theta_1)/\theta_1 \), implementing the optimal mechanism entails a risk of not selling the object. If there are two or more buyers of type \( \theta_1 \) among the \( n \) buyers initially in the market, both mechanisms will yield a revenue of \( \theta_1 \). However, if there is only one such buyer, revenue will decrease by \( \theta_1 - \theta_1 \) with probability \( q_1 \), when we go from the optimal mechanism to the English auction with an additional bidder. The probability of there being precisely one buyer of high type is \( nq_1q^n_1 \). If there are no buyer of type \( \theta_1 \) initially, a probability \( q^n_1 \) event, revenue will increase by \( \theta_1 \) with

\(^{35}\)The constrained optimal mechanism with \( n + 1 \) bidders can be implemented by first attempting to sell the item at a price of \( \theta_1 \) to one of the first \( n \) agents and, if this is unsuccessful, to sell the object to agent \( n + 1 \) at a price of \( \theta_1 \). Clearly, this yields more revenue than the optimal mechanism with \( n \) agents.
probability one, as the object is sold at price $\theta_1$ rather than withheld. If the condition in the Theorem is met, the expected loss more than offsets the gain. Clearly, if $\theta_1 = 0$, there is no advantage whatsoever in the extra bidder.

On the other hand, if $MR_1(\theta_1) \geq 0$ the two mechanisms will yield the same actual revenue unless precisely one of the $n$ first buyers have a high valuation. If the only buyer with a high valuation is buyer $i = 1, ..., n - 1$, revenue can decrease but not increase by introducing an extra buyer and holding an English auction. The decrease in revenue happens with probability $q_1$. However, if buyer $n$ is the only buyer with a high valuation among buyers $i = 1, ..., n$, inviting another player to take part in an English auction can increase but not decrease actual revenue. Contingent on there being exactly one buyer with high valuation, the probability that this player is buyer $n$ is obviously $1/n$. Hence, if there is only one buyer with high valuation the probability that revenue decreases is $q_1 (n - 1)/n$ while the probability that revenue increases is $\bar{q}_1 / n$. Optimal negotiation is superior to a larger English auction if the first probability is larger than the second probability, which reduces to the condition that $q_1 \geq 1/n$ as stated in the Theorem.

When $MR_1(\theta_1) \geq 0$ the extra bidder in the English auction dominates the ability to construct the optimal mechanism when $n = 1$. The reason is that the seller wishes to sell to both types, and can therefore not ask for more than $\theta_1$. Clearly, competition among bidders in a two person English auction will yield strictly more expected revenue. Notice that this result generalizes to continuous valuations or discrete valuations with $M_i > 2$. If marginal revenue is positive for all types, the best mechanism with one buyer is a fixed price equal to the lowest possible valuation. Competition among bidders is therefore preferable.

4 Continuous demand

We have considered the possibility that valuations take one of a finite number of values. In this section we assume that the distribution of types is continuous, where a type is denoted by $\theta_i \in [\theta_i, \bar{\theta}_i]$. The distribution function, $F_i(\theta_i)$, is strictly increasing on this interval and has no mass points. The density is denoted by $f_i(\theta_i)$, and it is assumed to be strictly positive on $[\theta_i, \bar{\theta}_i]$. It is well known that expected revenue in the optimal mechanism can be written

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36In fact, it is sufficient that ironed marginal revenue is positive. This is implied by positive marginal revenue for all types, but the opposite is not true in general.
as (5) in this case as well, see for instance Bulow and Roberts (1989). With continuous valuations, \( MR_i(\theta_i) \) is

\[
MR_i(\theta_i) = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}.
\] (8)

Usually, marginal revenue is thought to be monotonic. Observe that the argument in (8) is \( \theta_i \), the valuation or price, contrary to quantity as is usual. Given the inverse relationship between price and quantity, we expect the function in (8) to be strictly increasing. It is with this assumption, the regularity assumption, that Bulow and Klemperer (1996) derive their results.

For now, let us maintain the regularity assumption and assume that buyers are symmetric. Assuming that \( MR_1(\theta_1^*) < 0 \), we let \( \theta_1^* \) be the unique value of \( \theta_1 \) such that \( MR_1(\theta_1^*) = 0 \). It is then straightforward to design a mechanism which is optimal and still has a flavor of sequential negotiation. For instance, start by offering the object to buyer 1 at a price equal to \( \theta_1^* \). If he declines, give the same offer to buyer 2, and so on. If buyer \( i \) accepts the offer, allow the other buyers to challenge buyer \( i \), in which case an English auction with reserve price \( \theta_1^* \) is held.

It is easy to establish that this mechanism yields the same revenue as the English auction with reserve price \( \theta_1^* \), and it follows that the mechanism is optimal. We can extend this mechanism to involve an arbitrary number of possible offers to each buyer. This can be done by targeting \( m \) types, \( \theta_1^1 > \theta_1^2 > ... > \theta_1^m \geq \theta_1^* \) such that the \( j \)’th time buyer \( i \) is offered the good, the price is designed in such a way that types higher than \( \theta_1^j \) will accept. The probability that all buyers will be asked a number of times is increasing in \( m \), whereas the probability that we will observe an English auction is decreasing in \( m \).  

In the following, we will consider situations in which demand is either not regular, or buyers are asymmetric. We will show that the seller can improve upon the English auction by conducting pre-auction “negotiations”. Again, the resulting mechanisms are sequential in nature.

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37The mechanism is similar to the Dutch auction when \( m \) is large. The Dutch auction is a symmetric auction in which a display shows the continuously decreasing price. At any moment an agent can purchase the object by telling the server that he is willing to buy the object at the price currently displayed. This is not unlike a bargaining process where, in each period, all agents are offered the same price.
4.1 Non-monotonic marginal revenue

Consider a distribution with negligible density except in two small neighborhoods around $\theta_i$ and $\bar{\theta}_i$. The last term in (8) will then be quite small close to $\theta_i$, but very large outside the two neighborhoods. Hence, marginal revenue is not monotonic. Observe that the distribution in this example is a continuous approximation of the discrete distribution used earlier. Therefore, we might reasonably expect a relationship between results given discrete distributions, and results given irregular continuous distributions.

Figure 1 depicts such an irregular marginal revenue curve. Considering (5), the highest possible expected revenue would supposedly be derived from a mechanism that always allocates the good to the buyer with the highest marginal revenue. However, such a mechanism is not feasible, since it violates incentive compatibility. This follows from the fact that the probability of winning would not be increasing in valuation.

Instead, the optimal mechanism takes as its starting point the “ironed” marginal revenue curve, also indicated in Figure 1. Then, in the optimal mechanism, if a buyer reports a valuation in the interval $[x_i, \bar{x}_i]$, his chance of winning is exactly the same as if he had reported another valuation in this interval. In this way the probability that the good is sold to a buyer with a high marginal valuation is maximized, subject to the incentive compatibility condition. If it is interior, the optimal choice of the interval $[x_i, \bar{x}_i]$ satisfies

$$MR_i(x_i) = MR_i(\bar{x}_i) = \int_{x_i}^{\bar{x}_i} MR_i(\theta_i) \frac{f_i(\theta_i)}{F_i(\bar{x}_i) - F_i(x_i)} d\theta_i,$$

such that the expected value of the marginal revenue conditional on the report being in the interval $[x_i, \bar{x}_i]$ is identical to the marginal revenue of the endpoints.\(^{38}\) For more on ironing in auctions, see Myerson (1981), Bulow and Roberts (1989) or Maskin and Riley (1989).

In the following, we will present a mechanism which implements the optimal mechanism when demand is not regular. As mentioned, it is sequential in nature.\(^{39}\)

\(^{38}\)The optimal value of $x_i$ may equal $\theta_i$, in which case only $MR_i(\bar{x}_i)$ equals the conditional expected marginal revenue in the interval.

\(^{39}\)Kirkegaard and Overgaard (2003) presents an alternative revenue maximizing mechanism with two stages. First, an English auction with a high reserve price is held. If the object is not sold, another English auction is held, with a lower reserve price as well as a maximum price.
For simplicity, we assume that buyers are symmetric and that there is only one interval in which ironing is necessary. We also assume that there exists a $\theta_1^*$ such that $MR_1(\theta_1) \leq 0$, $\forall \theta_1 \in [\theta_1, \theta_1^*]$, and such that the ironed marginal revenue curve is strictly positive to the right of $\theta_1^*$. That is, we consider situations like that illustrated in Figure 1.

It is useful to define

$$H_i(\theta_1) = \frac{(F(\theta_1))^{i-1}}{(F(x_1))^{n-1}},$$

for $\theta_1 \in [\theta_1, x_1]$. Let $h_i(\theta_1)$ denote the derivative of $H_i(\theta_1)$.

**Proposition 6** Let

$$b_i = x_1 - \int_{\theta_1}^{x_1} (x_1 - \max\{\theta_1^*, x\})h_i(x)dx,$$

and consider the following mechanism:

(i) Let $i = 1$. Approach buyer $i$ and ask whether he would be willing to purchase the good at a price of $b_i$. If the answer is yes, ask the other buyers if they wish to challenge buyer $i$, in which case an English auction with a reserve price of $x_1$ is held. If no one wishes to challenge buyer $i$ the good is sold to buyer $i$ at a price of $b_i$. If buyer $i$ is unwilling to purchase the good at price $b_i$, iterate over $i = 2, ..., n$.

(ii) If the good is unsold after buyer $n$ has been asked to purchase the good, an English auction with a reserve price of $\theta_1^*$ is held.

This mechanism has an equilibrium in which (1) buyer $i$ accepts the offer if and only if $\theta_i \geq x_1$, (2) a challenge is made by buyer $j$ if and only if $\theta_j \geq x_1$, and (3) buyers bid up to their true valuation if an English auction is held. This equilibrium of the mechanism maximizes expected revenue.

**Proof.** Vickrey (1961) showed that it is a dominating strategy for buyer $i$ to bid up to his true valuation in an English auction. Assuming that other buyers follow the equilibrium strategy, we know that if buyer $i$ is offered the good, buyers $1, ..., i - 1$ must have type below $x_1$. Hence, define $G_i(x)$ as the distribution function of the highest type among the competitors given current information, and let $g_i(x)$ be its derivative, where
\[ G_i(x) = \begin{cases} 
\frac{F_1(x)^{n-1}}{F_1(\bar{x})^{n-1}}, & \text{if } x \in [\theta_i, \bar{x}_1] \\
\frac{1}{F_1(x)^{n-1}}, & \text{if } x \in (\bar{x}_1, \theta_i]. 
\end{cases} \]

Note also that buyer \( i \) will challenge another buyer if \( \theta_i \geq \bar{x}_1 \), and has no incentive to do so otherwise.

If offered the good at price \( b_i \), buyer \( i \) can either accept, \( a \), or reject, \( r \). We observe that given the other buyers follow the equilibrium strategy and buyer \( i \) bids up to \( \theta_i \) if an English auction is held, expected payoff by accepting is

\[ E\pi_i(a, \theta_i) = (\theta_i - b_i)G_i(\bar{x}_1) + I_{\bar{x}_1} \int_{\theta_i}^{\bar{x}_1} (\theta_i - x)g_i(x)dx, \]

where \( I_{\bar{x}_1} \) is one if \( \theta_i \geq \bar{x}_1 \) and zero otherwise.

Expected payoff by rejecting the offer is

\[ E\pi_i(r, \theta_i) = I_{\theta_i} \int_{\theta_i}^{\min(\bar{x}_1, \theta_i)} (\theta_i - \max\{\theta_i, x\})g_i(x)dx + I_{\bar{x}_1} \int_{\theta_i}^{\bar{x}_1} (\theta_i - \max\{\bar{x}_1, x\})g_i(x)dx, \]

where \( I_{\theta_i} \) is one if \( \theta_i \geq \theta_1^* \) and zero otherwise. The last term follows from the fact that if buyer \( i \) rejects the offer and has type above \( \bar{x}_1 \), then he will challenge a successor who accepts the offer. When \( b_i \) is as specified in the Proposition, the buyer is indifferent between accepting and rejecting the offer when \( \theta_i = \bar{x}_1 \), and we will thus assume he accepts the offer. It is also straightforward to show that accepting dominates rejecting for all \( \theta_i > \bar{x}_1 \), and vice versa for \( \theta_i < \bar{x}_1 \).

Finally, observe that given this equilibrium, the good is always allocated to the player with highest marginal revenue if there is at least one \( i \) such that \( \theta_i \geq \bar{x}_1 \) or if there is no \( i \) for which \( \theta_i \geq \bar{x}_1 \). Otherwise, the good is allocated to a buyer with valuation in the interval \([x_1, \bar{x}_1] \), but such that the exact valuation does not affect the probability of winning. Hence, expected revenue is maximized.\(^{40}\)
Observing that $h_i(\cdot)$ is increasing in $i$, it follows that $b_i$ is decreasing in $i$. Furthermore,

$$b_n = \int_{\varepsilon_1}^{\varepsilon_2} \max\{\theta_1^*, x\} h_n(x) dx \geq \theta_1^*.$$ 

Hence, if the seller has met with only uninterested buyers, he keeps lowering his price. If all $n$ bidders are initially reluctant to trade, the seller lowers the price further, and in a last attempt to sell invites all buyers to participate in an auction. On the other hand, if buyer $i$ is willing to buy at the price of $b_i$, rival bidders will have to be willing to bid significantly more in order to challenge the incumbent and have a chance of winning.

The proposed mechanism is optimal, and it is thus assumed that the seller can ration supply. If he is unable to do so, a similar mechanism is constrained optimal. Should all buyers decline the initial offers, the reserve price in the ensuing auction should simply be set to $\theta_1^*$ rather than $\theta_1^*$.

4.2 Asymmetry

In this section it is assumed that demand is regular, but buyers are allowed to be asymmetric. We consider sequential mechanisms similar to that described above, where there is a positive probability of an English auction taking place. Since the winner of the English auction is the participant with the highest valuation, not the highest marginal revenue, it is clear that the procedures are inoptimal when buyers are asymmetric. Nevertheless, the mechanisms improve upon the English auction.

First define

$$x^* = \max\{MR_1(\theta_1), MR_2(\theta_2), ..., MR_n(\theta_n), 0\}.$$ 

If $x^* > 0$, there is a buyer who always has positive marginal revenue, implying that in the optimal mechanism the good is always sold, and that buyers with marginal revenue below $x^*$ has probability zero of winning the mechanism. If $x^* = 0$, all buyers have negative marginal revenue with positive probability, and the good might not be sold in the optimal mechanism. The procedures we consider in this section improve upon the English auction with a common reserve price by ensuring that the good is never sold to a buyer with marginal revenue below $x^*$, something a common reserve price does not rule out.
For each buyer $i$ with $MR_i(\theta_i) = \bar{\theta_i} > x^*$, let $\theta^*_i = \theta_i$ if $MR_i(\theta_i) = x^*$ and let $\theta^*_i$ solve $MR_i(\theta^*_i) = x^*$ if $MR_i(\theta_i) < x^*$. If $MR_i(\bar{\theta_i}) = \bar{\theta_i} \leq x^*$, let $\theta^*_i = \bar{\theta_i}$. Order the buyers such that

$$\theta^*_1 \leq \theta^*_2 \leq \ldots \leq \theta^*_n.$$

We now specify two sequential mechanisms.

**Definition 1 (Ascending Offer Negotiation (AON))** Let $i = 1$. Ask buyer $i$ whether he is willing to purchase the good at a price of $\theta^*_i$. If the answer is no, continue to buyer $i + 1$. However, if buyer $i$ answers in the affirmative, he becomes the “favored buyer”. If buyer $i$ is the favored buyer, buyer $i + 1$ can challenge him, in which case buyer $i + 1$ becomes the favored buyer. If buyer $i + 1$ does not challenge, buyer $i + 2$ is allowed to challenge, and so on until buyer $n$ is reached. When all buyers have been asked and buyer $i$ is the favored buyer, an English auction with reserve price $\theta^*_i$ is held, in which buyers who at one time have been favored buyers participate. If there is no favored buyer at the end, the good is not sold.

**Definition 2 (Descending Offer Negotiation (DON))** Let $i = n$. Ask buyer $i$ if he is willing to pay $\theta^*_i$. If yes, buyers in the set $\{1, \ldots, i - 1\}$ can challenge the buyer to an English auction with reserve $\theta^*_i$, in which the challengers and buyer $i$ participate. If there is no challenge, buyer $i$ gets the good at a price of $\theta^*_i$. If buyer $i$ is not willing to pay $\theta^*_i$, he is excluded from the rest of the negotiations, and buyer $i - 1$ is approached.

If buyer $i$ is not interested in the object, the seller increases the asking price before going to the next buyer in AON, but lowers the asking price in DON. In AON, it is a dominant strategy for buyer $i$ to accept the offer, or to challenge the favored buyer, if $\theta_i \geq \theta^*_i$, and refrain from doing so otherwise. Hence, AON culminates in an English auction with a reserve price of $\theta^*_k$, where $k = \max\{i|\theta_i \geq \theta^*_i\}$. Buyer $i$ is included in the English auction if $\theta_i \geq \theta^*_i$. If there has been only one favored buyer (buyer $k$), the good is sold at the reserve price, $\theta^*_k$.

In DON, it is again a dominant strategy to accept the offer if $\theta_i \geq \theta^*_i$. It is a dominant strategy for buyer $i$ to challenge buyer $j$, $i < j$, if $\theta_i \geq \theta^*_j \geq \theta^*_i$. If a buyer accepts the offer, and no one challenges, the good is sold at a price of $\theta^*_k$, where $k = \max\{i|\theta_i \geq \theta^*_i\}$. If the buyer is challenged, an English auction with a reserve price of $\theta^*_k$ ensues.
It is easily seen that, in either mechanism, the set of buyers who participates in the English auction, and who has valuations exceeding $\theta^*_k$ is the same. Hence, the winner is the same in both mechanisms, as is the revenue (by the Revenue Equivalence Theorem).

**Proposition 7** Ascending Offer Negotiation and Descending Offer Negotiation are revenue equivalent, and revenue superior to the English auction with a common reserve price.

**Proof.** The equivalence of the different forms of negotiation is proven in the text. Here we prove the revenue superiority of negotiation.

In AON/DON buyer $i$ has a positive probability of winning if, and only if, $\theta_i \geq \theta^*_i$, or $MR_i(\theta_i) \geq x^* \geq 0$. Hence, if the good is sold, it is sold to a buyer with marginal revenue exceeding $x^*$. If $x^* > 0$, the good is sold with probability one in AON/DON.

If the winner of the English auction with a common reserve price is not the same as the winner of AON/DON, the former must have marginal revenue below $x^*$. Otherwise, he would also have been included in the final auction of AON/DON, and he would have won, since, by assumption, he has the highest valuation. This contradicts the assumption that the identity of the winner differs.

Furthermore, if the good is not sold in the English auction with a common reserve price, it might have been sold in AON/DON, in which case it would have been sold to a buyer with positive marginal revenue.

Consequently, if the allocation is not the same in the two different types of mechanisms, marginal revenue of the winner is below $x^*$ in the English auction with a common reserve (or zero if there was no winner), and marginal revenue of the winner in AON/DON is above $x^*$ (or zero if there was no winner, which necessitates $x^* = 0$). It follows that negotiation dominates an English auction with a common reserve price. ■

In the mechanisms discussed above, the object is not necessarily sold. If the seller is constrained to having to sell the object, we can modify AON and DON simply by redefining $x^*$ as

$$x^* = \max\{MR_1(\theta_1), MR_2(\theta_2), ..., MR_n(\theta_n)\}.$$ 

In this case, the object is sold with certainty, but never to anyone with marginal valuation below $x^*$. 

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Of course, it is not surprising that adding a stage of pre-auction negotiation before an English auction can improve revenue. After all, the seller could just ignore this stage. Thus, we have merely given two examples of how pre-auction negotiation can be designed to improve upon an English auction.

4.3 Auctions vs. negotiations revisited

We conclude with an example in which Bulow and Klemperer’s (1996) result does not hold, despite valuations being drawn from a continuous distribution function. Specifically, consider the power distribution, \( F(v) = \theta^a, \theta \in [0, 1], \ a > 0 \). When \( a \geq 1 \), the distribution function is convex, and it follows that \( MR(\cdot) \) is monotonic. In that case, we know that Bulow and Klemperer’s result holds. Thus, in the following we assume that \( a \in (0, 1) \).

First, notice that

\[
MR(\theta) = \theta - \frac{1 - \theta^a}{a\theta^a - 1} = \frac{(1 + a)\theta^a - 1}{a\theta^a - 1},
\]

and that \( MR(0) = 0 \). Hence, \( MR(\cdot) \) is U shaped. That is, \( MR(\cdot) \) starts at zero, becomes negative and subsequently starts to increase. Once \( MR(\cdot) \) becomes positive it remains positive and is in fact strictly increasing. This implies that the English auction with a reserve price of

\[
r(a) = \left( \frac{1}{1 + a} \right)^{\frac{1}{a}}
\]

is optimal, since this mechanism ensures that the object is sold to the bidder with the highest marginal revenue, if this is positive. However, because \( MR(\cdot) \) is non-monotonic, the English auction without a reserve price is not constrained optimal.

To proceed, let \( D(a, n) \) denote the difference in expected revenue between the English auction with reserve \( r(a) \) (i.e. the optimal mechanism) and \( n \) bidders, and the English auction with \( n + 1 \) bidders. It can be shown that

\[
D(a, n) = na^{\left(\frac{na + a + 1}{na + a + 1}\right)^{\frac{1}{a} - (n-1)} - 2a - (na + a + 1)}.
\]

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the sign of which is determined by the numerator. Observe that
\[
\lim_{a \to 0} \left( (na + a + 1)(1 + a)^{-\frac{3}{8} - (n-1)} - 2a \right) = \lim_{a \to 0} (1 + a)^{-\frac{3}{8}} = e^{-1} > 0.
\]

Consequently, \(D(a, n)\) is positive when \(a\) is sufficiently small. In this case, the seller is better off with an optimal mechanism, than with an English auction with an additional buyer.

To give an idea of how \(D(a, n)\) depends on \((a, n)\), Figure 2 graphs the level curve, \(D(a, n) = 0\). Above the curve, Bulow and Klemperer’s result holds. Below it, however, the opposite result obtains. That is, the ability to negotiate optimally is more valuable than an extra bidder.

5 Conclusion

In this paper we compared auctions with different forms of negotiation and bargaining. First, we showed that the seller prefers getting an extra buyer to the ability to ration supply, if he is otherwise unconstrained. The reason is that the initial buyers react in the same way to the threat of giving the object to a newcomer as to the threat of not selling the object at all. Hence, the seller can do no worse with an extra bidder than with the ability to ration supply.

When buyers are symmetric and demand regular, the English auction is constrained optimal. In this case, and as shown by Bulow and Klemperer (1996), an optimal mechanism with \(n\) buyers is revenue inferior to an English auction with \(n + 1\) buyers.

Secondly, seller-offer bargaining was compared to the English auction. Assuming demand is discrete, the former was shown to dominate the latter in terms of revenue when agents are sufficiently patient. However, it is not possible to obtain an unambiguous ranking of the first price auction and seller-offer bargaining.

When demand is continuous, pre-auction negotiations were shown to improve the English auction.
References


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A Proofs, Section 3.1

Proof of Lemma 1. Assume the mechanism is $IC$. Then $IC_{i}^{k,l}$ implies that

$$T_{i}^{l} - T_{i}^{k} \geq \theta_{i}^{k}(X_{i}^{l} - X_{i}^{k}).$$

Letting $k = j - 1, l = j,$

$$T_{i}^{j} - T_{i}^{j-1} \geq \theta_{i}^{j-1}(X_{i}^{j} - X_{i}^{j-1}), \quad (10)$$

while $k = j, l = j - 1$ yields

$$T_{i}^{j-1} - T_{i}^{j} \geq \theta_{i}^{j}(X_{i}^{j-1} - X_{i}^{j}). \quad (11)$$

Combining the two,

$$\theta_{i}^{j}(X_{i}^{j} - X_{i}^{j-1}) \geq T_{i}^{j} - T_{i}^{j-1} \geq \theta_{i}^{j-1}(X_{i}^{j} - X_{i}^{j-1}),$$

implying that $X_{i}^{j} - X_{i}^{j-1} \geq 0$. From (10) it follows that $T_{i}^{j} \geq T_{i}^{j-1}$.

Proof of Lemma 2. The “only if” part is trivial.

To prove “if”, assume that the condition is satisfied. $IC_{i}^{j,j-1}$ and $IR_{i}^{j,j-1}$ implies

$$\theta_{i}^{j}X_{i}^{j} - T_{i}^{j} \geq \theta_{i}^{j}X_{i}^{j-1} - T_{i}^{j-1} \geq \theta_{i}^{j-1}X_{i}^{j-1} - T_{i}^{j-1} \geq 0.$$ 

Hence, if $IR_{i}^{1}$ is satisfied, so is the remaining $IR_{i}$ constraints. Observe that the higher $j$, the higher is the expected payoff to the buyer.

Consider now the downwards $IC$ conditions. Assuming that $IC_{i}^{j,j-1}$ is satisfied for all $j$

$$\theta_{i}^{j}(X_{i}^{j} - X_{i}^{j-1}) - (T_{i}^{j} - T_{i}^{j-1}) \geq 0,$$

implying that

$$\theta_{i}^{k}(X_{i}^{j} - X_{i}^{j-1}) - (T_{i}^{j} - T_{i}^{j-1}) \geq 0, \quad (12)$$

for all $k \geq j$ since $X_{i}^{j} \geq X_{i}^{j-1}$. Rewrite $IC_{i}^{j,j-1},$

$$\theta_{i}^{j}X_{i}^{l} - T_{i}^{l} \geq \theta_{i}^{j}X_{i}^{l-1} - T_{i}^{l-1}$$

$$= \theta_{i}^{j}X_{i}^{l-2} - T_{i}^{l-2} + \theta_{i}^{j}(X_{i}^{l-1} - X_{i}^{l-2}) - (T_{i}^{l-1} - T_{i}^{l-2})$$

$$\geq \theta_{i}^{j}X_{i}^{l-2} - T_{i}^{l-2},$$

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where the last inequality follows from (12) with \( j = l - 1 \) and \( k = j + 1 \). Hence, \( IC_{i}^{l,l-2} \) is satisfied as well, for all \( l \). Thus,

\[
\theta_i^l X_i^l - T_i^l \geq \theta_i^l X_i^{l-2} - T_i^{l-2} \\
= \theta_i^l X_i^{l-3} - T_i^{l-3} + \theta_i^l (X_i^{l-2} - X_i^{l-3}) - (T_i^{l-2} - T_i^{l-3}) \\
\geq \theta_i^l X_i^{l-3} - T_i^{l-3},
\]

where (12) have been used again, this time with \( j = l - 2 \) and \( k = j + 2 \). We can continue in this way to prove that all downwards \( IC \) constraints are satisfied.

Consider next the upwards \( IC \) constraints, assuming that \( IC_{i}^{j,j+1} \) is satisfied for all \( j \). Then,

\[
\theta_i^j (X_i^{j+1} - X_i^j) - (T_i^{j+1} - T_i^j) \leq 0
\]

and it follows that

\[
\theta_i^k (X_i^{j+1} - X_i^j) - (T_i^{j+1} - T_i^j) \leq 0, \tag{13}
\]

for all \( k \leq j \). Rewriting \( IC_{i}^{j,j+1} \),

\[
\theta_i^j X_i^j - T_i^j \geq \theta_i^j X_i^{j+1} - T_i^{j+1} \\
= \theta_i^j X_i^{j+2} - T_i^{j+2} - [\theta_i^j (X_i^{j+2} - X_i^{j+1}) - (T_i^{j+2} - T_i^{j+1})] \\
\geq \theta_i^j X_i^{j+2} - T_i^{j+2},
\]

using (13) with \( j = l + 1 \) and \( k = l - 1 \). Continuing as before proves that all upwards \( IC \) are satisfied as well. \( \blacksquare \)

**Proof of Proposition 2.** Assume that the mechanism is \( IR \) and \( IC \) and the mechanism is optimal but \( IR_i^1 \) is slack. From \( IC_{i}^{j,j+1} \) it follows that

\[
\theta_i^j X_i^j - T_i^j \geq \theta_i^j X_i^1 - T_i^1 \geq \theta_i^1 X_i^1 - T_i^1 > 0,
\]

i.e. all \( IR_i \) are slack. For all \( j \), increase \( T_i^j \) by the constant \( \varepsilon > 0 \), where \( \varepsilon \leq \theta_i^1 X_i^1 - T_i^1 \). This increase in \( T_i^j \) increases expected revenue. All the \( IC \) constraints are unaffected, since \( \varepsilon \) is subtracted on both sides of the inequality. Furthermore, all the \( IR \) constraints remains satisfied. Hence, \( IR_i^1 \) slack cannot be optimal.
Assume that the mechanism is IR and IC and the mechanism is optimal but $IC_i^{j,j-1}$ is slack for some $j$. For any $l \geq j$,

$$\theta_i^j X_i^j - T_i^j > \theta_i^j X_i^j - T_i^j > \theta_i^j X_i^{j-1} - T_i^{j-1} > \theta_i^{j-1} X_i^{j-1} - T_i^{j-1} \geq 0,$$

implying that IR is slack. Increase $T_i^l$ for all $l \geq j$ by the same amount $\epsilon > 0$, where $\epsilon \leq \theta_i^l (X_i^j - X_i^{j-1}) - (T_i^j - T_i^{j-1})$. All IC’s involving types no smaller than $j$ are unaffected, as are all IC’s involving types strictly smaller than $j$. Consider then $IC_i^{m,k}$. If it is an upwards IC, $m < j \leq k$, it will remain satisfied since the increase in $T_i^k$ creates an even larger disincentive for $\theta_i^j$ to report $\theta_i^k$. If it is an upwards IC, $m \geq j > k$, the fact that the initial mechanism is IC implies that

$$\theta_i^m X_i^m - T_i^m \geq \theta_i^m X_i^j - T_i^j \geq \theta_i^m X_i^{j-1} - T_i^{j-1} + \epsilon \geq \theta_i^m X_i^k - T_i^k + \epsilon,$$

which in turn means that

$$\theta_i^m X_i^m - (T_i^m + \epsilon) \geq \theta_i^m X_i^k - T_i^k \geq 0.$$

Hence, the new mechanism is IR and IC and yields higher expected revenue than the original one.

To prove the second part, observe that IR$_i^1$ binding means that $T_i^1 = \theta_i^1 X_i^1$, while IC$_i^{j,j-1}$ binding implies that $T_i^j = \theta_i^j (X_i^j - X_i^{j-1}) + T_i^{j-1}$ (it is easily shown that IC$_i^{j,j+1}$ is satisfied, implying by Lemma 2 that the mechanism is IR and IC). Thus, there is a unique value of each $T_i^j$ which ensures that IR$_i^1$ and IC$_i^{j,j-1}$ are binding. That is, there is only one way in which IR$_i^1$ and IC$_i^{j,j-1}$ can be binding, and as we have just shown that the optimal mechanism has this feature, it follows that if IR$_i^1$ and IC$_i^{j,j-1}$ are binding the mechanism must be optimal.

**Proof of Corollary 2.** In the optimal mechanism, $T_i^1 = \theta_i^1 X_i^1$, while for $j \geq 2$,

$$T_i^j = \theta_i^j (X_i^j - X_i^{j-1}) + T_i^{j-1} \equiv \theta_i^j X_i^j - \sum_{l=2}^{j} (\theta_i^l - \theta_i^{l-1}) X_i^{l-1}.$$
Inserting this in (3) yields
\[
ER = \sum_{i \in I} \left[ \sum_{j=1}^{M_i} \theta_i^j X_i^j f_i(\theta_i^j) - \sum_{j=2}^{M_i} \left( \sum_{l=1}^{j-1} (\theta_i^{l+1} - \theta_i^l) X_i^l \right) f_i(\theta_i^j) \right]
\]
\[
= \sum_{i \in I} \left[ \sum_{j=1}^{M_i} \theta_i^j X_i^j f_i(\theta_i^j) - \sum_{l=1}^{M_i} \left( (\theta_i^{l+1} - \theta_i^l) X_i^l \sum_{j=l+1}^{M_i} f_i(\theta_i^j) \right) \right],
\]
where we have used \( \theta_i^{M_i+1} = \theta_i^{M_i} \). Rearranging and changing the names of the subscripts,
\[
ER = \sum_{i \in I} \left[ \sum_{j=1}^{M_i} \left( \theta_i^j - (\theta_i^{l+1} - \theta_i^l) \frac{1 - F(\theta_i^j)}{f(\theta_i^j)} \right) X_i^j f_i(\theta_i^j) \right].
\]
Since reports are truthful, \( X_i^j = E_{\theta \sim \theta_i} X_i(\theta_i^j, \theta_{-i}) \), and \( X_i^j f_i(\theta_i^j) \) is the expected probability that buyer \( i \) is of type \( \theta_i^j \) and wins the mechanism. Furthermore, as types are independent across buyers,
\[
\sum_j MR_i(\theta_i^j) X_i^j f_i(\theta_i^j) = \sum_j MR_i(\theta_i^j) E_{\theta \sim \theta_i} X_i(\theta_i^j, \theta_{-i}) f_i(\theta_i^j) = E_\theta [MR_i(\theta_i) X_i(\theta)],
\]
and (5) follows. That \( ER \) is attained if and only if the mechanism is optimal follows directly from Proposition 2.

**Proof of Theorem 1.** If (5) is maximized with respect to \( X_i(\theta) \), then there exists \( T_i^1 = \theta_i^j X_i^1 \) and \( T_i^j = \theta_i^j (X_i^j - X_i^{j-1}) + T_i^{j-1} \) for \( j \geq 2 \), such that \( IR_i^1 \) and \( IC_i^{j-1} \) are binding. The expected revenue is maximized by Proposition 2.

**B The one-buyer game: Review**

We use notation similar to that used in Ausubel, Cramton and Deneckere (2001). Instead of saying that the buyer has valuation \( \overline{\theta} \) with probability \( \overline{\theta} \), and valuation \( \theta \) with the complementary probability, we say that the buyer has type \( q \), which is drawn from the uniform distribution on \([0, 1]\), and where the type translates into a valuation in the following way
\[
v(q) = \begin{cases} 
\overline{\theta} & \text{if } 0 \leq q \leq \overline{\theta} \\
\theta & \text{if } \overline{\theta} < q \leq 1
\end{cases}
\]
where $\bar{\theta} > \theta > 0$, and $\bar{q} \in (0, 1)$. Hence, the probability, ex ante, that the buyer has the high valuation is $\bar{q}$. The agents are assumed to have the same discount factor, $\delta \in (0, 1)$. The seller’s valuation is zero.

If $\bar{\theta} \geq \bar{q} \bar{\theta}$, the seller can do no better than to sell immediately at the price $\theta$ (we verify this momentarily). Hence, in the following we focus on the case that

$$\bar{q} \bar{\theta} > \theta$$

such that bargaining lasts more than 1 period. We start by specifying a sequence of prices and a sequence of acceptance probabilities.

**Prices.** The prices are set in such a way that the high valuation buyer is indifferent between accepting the current price, or waiting and accepting some future price. Let $p_t$ be the price when $t$ periods remains before $\theta$ is offered, and bargaining concludes with certainty. In particular, $p_0 = \bar{\theta}$. Then,

$$\bar{\theta} - p_t = \delta^t (\bar{\theta} - \theta) \iff p_t = \bar{\theta} - \delta^t (\bar{\theta} - \theta),$$

in order for the buyer to be indifferent between $p_t$ and the price $\theta$ in $t$ periods. It follows that

$$\bar{\theta} - p_t = \delta^t (\bar{\theta} - \theta) = \delta^{t-i} (\bar{\theta} - (\bar{\theta} - \delta^i (\bar{\theta} - \theta))) = \delta^{t-i} (\bar{\theta} - p_i),$$

or that the buyer is indifferent between $p_t$ today, and $p_i$ in $t - i$ periods. Therefore, the high valuation buyer is willing to randomize between acceptance dates.

**Probabilities.** Next, given the sequence of prices, acceptance probabilities are determined in such a way that the seller is indifferent between charging $p_t$ and $p_{t-1}$ on the equilibrium path. Let $m_t$ be the ex ante probability that $p_t$ is accepted (recall that the buyer is willing to randomize). That is, before the game commences, the probability that bargaining concludes $t$ stages before a price of $\bar{\theta}$ is offered, is $m_t$.

Now, the idea is that when 0 rounds remain, the buyer will already have purchased the good if he has high valuation. Hence, $p_0 = \bar{\theta}$ is accepted by
the low valuation buyer, such that \( m_0 = 1 - \bar{q} \). Defining \( q_{-1} = 1 \) and \( q_0 = \bar{q} \), we can write \( m_0 = q_{-1} - q_0 \). The price \( p_1 \) is then accepted by a measure \( m_1 \) of types. We assume that these are located immediately to the left of \( q_0 \), i.e. if \( q \in (q_1, q_0] \) the buyer waits until one round remains of effective bargaining, and then accepts \( p_1 \), where \( m_1 = q_0 - q_1 \) defines \( q_1 \). The price \( p_2 \) is accepted by a measure \( m_2 \), which we again assume is to the left of \( q_1 \), such that \( m_2 = q_1 - q_2 \). In this manner we can define

\[
m_t = q_{t-1} - q_t,
\]

(18)

for all \( t \), since \( p_t \) is assumed accepted by the buyer if he has type \( q \in (q_t, q_{t-1}] \) (this corresponds to a high valuation buyer randomizing between prices). As \( t \) increases, however, the fact that \( q \geq 0 \) eventually becomes binding, i.e. \( q_t \) will eventually become negative. Let \( T \) be defined by \( q_T \leq 0 < q_{T-1} \). Then, bargaining lasts at most \( T \) periods. In the first round the price is \( p_{T-1} \), which is accepted by \( q \leq q_{T-2} \).\(^{41}\) In the second round, the price is \( p_{T-2} \), which is accepted by \( q \in (q_{T-2}, q_{T-3}] \) and so on. Observe that when \( t \) rounds remains of bargaining, beliefs are essentially a left truncation of \([0, 1]\), i.e. the belief is that the buyer has a type in \((q_t, 1]\).

To determine \( m_t \), or \( q_t \), we need to consider the aforementioned indifference condition. To start, equilibrium payoff if beliefs are described by \( q_t \), \( t \geq 1 \), is

\[
W(q_t) = \frac{V(q_t)}{1 - q_t},
\]

where

\[
V(q_t) = m_t p_t + \delta m_{t-1} p_{t-1} + \delta^2 m_{t-2} p_{t-2} + \ldots + \delta^t m_0 p_0.
\]

This can also be expressed as

\[
V(q_t) = m_t p_t + \delta V(q_{t-1}).
\]

(19)

If the seller chooses to jump one step forward in the sequence of offers, offering the price \( p_{t-1} \) instead of \( p_t \), his expected payoff will be a fraction \((1 - q_t)\) of

\[
(m_t + m_{t-1}) p_{t-1} + \delta m_{t-2} p_{t-2} + \ldots + \delta^{t-1} m_0 p_0 = m_t p_{t-1} + V(q_{t-1}).
\]

\(^{41}\)Bargaining lasts at most \( T \) rounds, so at the very first round, there is at most \( T - 1 \) rounds remaining (recall that \( p_0 \) is charged when 0 rounds remain).
Hence, in order for the seller to be indifferent between \( p_t \) and \( p_{t-1} \) it must be the case that
\[
V(q_t) = m_t p_{t-1} + V(q_{t-1}). \tag{20}
\]
Observe that
\[
V(q_t) = m_t p_{t-1} + V(q_{t-1}) = m_t p_{t-1} + m_{t-1} p_{t-2} + V(q_{t-1}) = \ldots = m_t p_{t-1} + \ldots + m_{t-i+1} p_{t-i} + V(q_{t-i}) = m_t p_{t-1} + \ldots + m_{t-i+1} p_{t-i} + m_{t-i} p_{t-i} + \delta V(q_{t-i-1}) > (m_t + m_{t-1} + \ldots + m_{t-i}) p_{t-i} + \delta V(q_{t-i-1}), \tag{21}
\]
for any \( i > 1 \). Consequently, the seller is willing to randomize between \( p_t \) and \( p_{t-1} \) but strongly prefers either of these to any \( p_{t-i}, i > 1 \).

For \( t = 1 \) the indifference condition is
\[
m_1 p_1 + \delta m_0 p_0 = (m_1 + m_0) p_0. \tag{22}
\]
After inserting \( p_1 \) and \( p_0 \), this implies that
\[
m_1 = \frac{\theta}{\theta - \tilde{\theta}} m_0 \tag{23}
\]
Since \( m_0 \) is independent of \( \delta \), it follows that \( m_1 \) is independent as well.\(^{42}\) For

\(^{42}\)Now, we can also see why it is an equilibrium to always offer \( \frac{\theta}{\theta - \tilde{\theta}} \) if \( \frac{\theta}{\theta - \tilde{\theta}} \geq \frac{\theta}{\theta - \bar{\theta}} \). Assume it is an equilibrium to always offer \( \frac{\theta}{\theta - \bar{\theta}} \). The high valuation buyer then accepts any offer that does not exceed \( p_1 \). Hence, the seller’s best strategy is either to offer \( p_1 \) or \( \frac{\theta}{\theta - \bar{\theta}} \). If he offers \( p_1 \), payoff is
\[
\tilde{\theta}(1 - \delta) \tilde{\theta} + \delta (1 - \bar{\theta}) \bar{\theta} = (1 - \delta) \tilde{\theta} \tilde{\theta} + \delta \bar{\theta} \leq \bar{\theta},
\]
by assumption. Hence, it is an equilibrium for the seller to always offer \( \frac{\theta}{\theta - \bar{\theta}} \) for the low valuation buyer to accept an offer of \( \frac{\theta}{\theta - \bar{\theta}} \) or lower, and for the high valuation buyer to accept any offer not exceeding \( p_1 \).

Alternatively, solving (23) for \( q_1 \) yields
\[
q_1 = \frac{\tilde{\theta} \theta - \theta}{\tilde{\theta} - \bar{\theta}}
\]
which is non-positive when \( \frac{\theta}{\theta - \tilde{\theta}} \geq \frac{\theta}{\theta - \bar{\theta}} \). Hence, at a price of \( p_1 \), the acceptance probability \( \frac{\theta}{\theta - \bar{\theta}} \) is not sufficiently high to make it worthwhile for the seller to bargain over several rounds (for this, an acceptance probability of \( \frac{\theta}{\theta - q_1} \geq \frac{\theta}{\theta - \bar{\theta}} \) is needed).
$t > 1$, we manipulate (20),
\begin{align*}
m_t p_t + \delta V(q_{t-1}) &= m_t p_{t-1} + V(q_{t-1}) \\
m_t(p_t - p_{t-1}) &= (1 - \delta)V(q_{t-1}) \\
m_t \delta^{t-1}(\overline{\theta} - \underline{\theta}) &= V(q_{t-1})
\end{align*}

(24)

From this we get
\begin{align*}
m_t \delta^{t-1}(\overline{\theta} - \underline{\theta}) &= m_{t-1} p_{t-1} + \delta V(q_{t-2}) \\
&= m_{t-1} p_{t-1} + \delta(m_{t-1} \delta^{t-2}(\overline{\theta} - \underline{\theta})) \\
&= m_{t-1}(\overline{\theta} - \delta^{t-1}(\overline{\theta} - \underline{\theta}) + \delta^{t-1}(\overline{\theta} - \underline{\theta})) \\
&= m_{t-1} \overline{\theta},
\end{align*}

(25)

such that
\begin{align*}
m_t = \delta^{-(t-1)} \frac{\overline{\theta}}{\overline{\theta} - \underline{\theta}} m_{t-1}
\end{align*}

(26)

for $t > 1$. Observe that $m_t > m_{t-1}$. From (24),
\begin{align*}
m_t \delta^t(\overline{\theta} - \underline{\theta}) &= \delta V(q_{t-1}) \\
m_t \overline{\theta} &= m_t(\overline{\theta} - \delta^t(\overline{\theta} - \underline{\theta})) + \delta V(q_{t-1}) \\
&= m_t p_t + \delta V(q_{t-1}),
\end{align*}

or
\begin{align*}
V(q_t) = m_t \overline{\theta}.
\end{align*}

(27)

Hence, given at most $t$ stages remains, expected payoff to the seller is
\begin{align*}
W(q_t) = \frac{m_t \overline{\theta}}{1 - q_t}
\end{align*}

(28)

Concerning the first stage, the offer is $p_{T-1}$ rather than $p_T$ because the higher price would be accepted with probability $q_{T-1} \leq q_T - q_T$ and so the seller is not indifferent between $p_T$ and $p_{T-1}$, but prefers the latter. Likewise $p_{T-1}$ is preferred to $p_t$, $t = 0, 1, ..., T - 2$ by a straightforward extension of (21).
Recap. We have covered the quantity-price space with a grid. A buyer of type \( q \in (q_t, q_{t-1}] \) accepts any price below \( p_t \), and the seller offers a decreasing sequence of prices. The prices are such that the high valuation buyer is willing to randomize, and the quantities are such that the seller cannot improve by jumping to another price in the sequence.

Strategies and beliefs. We now specify equilibrium strategies and beliefs.

The buyer of type \( q \in (q_t, q_{t-1}] \). If the current offer is the smallest yet, accept it if, and only if, it is at most \( p_t \). If the current offer is not the smallest offer in the history of the game, reject it if it exceeds \( p_1 (p_0) \) if the buyer has the high (low) valuation, and accept it otherwise.

The seller. Let \( p_0 \) be the smallest price offered in the past (excluding today, as that is the price we wish to determine). If \( p_0 > p_T \) or if no price has been offered yet, then offer the price \( p_T - 1 \). If \( p_0 = p_t \) for some \( t \in \{1, 2, ..., T\} \), then offer price \( p_{t-1} \). If \( p \in (p_t, p_{t+1}) \) for some \( t \in \{1, 2, ..., T-1\} \), then randomize between \( p_t \) and \( p_{t-1} \) with probabilities \( s \) and \( 1 - s \) respectively, such that

\[
\bar{\theta} - \bar{p} = \delta(s(\bar{\theta} - p_t) + (1 - s)(\bar{\theta} - p_{t-1}))
\]

Offer \( p_0 \) if \( p < p_1 \).

Beliefs. In the first period, or if \( p > p_T \) the belief is that the buyer has high valuation with probability \( q \). If \( p \in (p_{t-1}, p_t] \), \( t \in \{1, 2, ..., T\} \), beliefs are described by \( q_{t-1} \), meaning that the buyer is perceived to be high type with probability \( (q - q_{t-1})/(1 - q_{t-1}) \), or that the beliefs about the buyer’s type is a truncation of \([0, 1]\) to \([q_{t-1}, 1]\). If \( p \leq p_0 \) the belief is that the buyer has high valuation with probability zero.

Equilibrium. We now show that the beliefs and strategies described above form a PBE, using the one deviation property. First, it is easily seen that beliefs are updated using Bayes’ rule whenever \( p > p_0 \). When \( p \leq p_0 \), Bayes’ rule is not applicable. Next, we consider the incentives of the two agents.

Seller. If the seller is supposed to offer \( p_t \) with some probability \( s \), and \( p_{t-1} \) with probability \( 1 - s \), \( s \in (0, 1] \), beliefs must be described by \( q_t \). However, in this case, we have already established that the seller is indifferent between \( p_t \) and \( p_{t-1} \). Clearly, a price exceeding \( p_t \) is inoptimal, as it never results in a sale, and simply postpones revenue (next period, when the seller reverts back to the equilibrium strategy, the seller again randomizes between \( p_t \) and \( p_{t-1} \)). Regarding bids lower than \( p_{t-1} \), we have already shown that lower bids
in the sequence is strictly inferior to \( p_t \) or \( p_{t-1} \). It remains only to show that the seller cannot improve by offering a price not in the sequence, \( p \), say. By following the equilibrium strategy, expected payoff is \( W(q_t) \). By deviating to the price \( p \), with \( p \in (p_{t-i-1}, p_{t-i}) \) for some \( i \geq 0 \), and reverting to the equilibrium strategy next period, payoff is

\[
\frac{q_{t-i-1} - q_t}{1 - q_t} p + \frac{1 - q_{t-i-1}}{1 - q_t} \delta W(q_{t-i-1}) < \frac{1}{1 - q_t} ((q_{t-i-1} - q_t)p_{t-i} + \delta V(q_{t-i-1})).
\]

Since the right hand side is smaller than \( W(q_t) \) if \( i > 1 \), by (21), it is not profitable to offer a price below \( p_{t-2} \). If \( i = 1 \), we know from (20) that the right hand side equal \( W(q_t) \), implying that any price below \( p_{t-1} \) is unprofitable as well. Clearly, any price between \( p_{t-1} \) and \( p_t \) is also unprofitable (for \( i = 0 \), the right hand side also equals \( W(q_t) \), by (19)).

**Buyer.** If the current offer is the lowest yet, the high valuation buyer is indifferent between accepting and rejecting the offer or waiting (by (17) or (29), whichever applies). Hence, he is willing to follow the equilibrium strategy. If the current offer exceeds \( p_1 \) and is not the lowest in the history of the game, it is preferable to wait, by (29). If the current offer is lower than \( p_1 \) but a lower offer has been made, the seller believes the buyer has the low valuation, and offers \( p_0 \) next period. Hence, the high valuation buyer is better off accepting the current offer (by (17)). Clearly, the low valuation buyer is willing to follow the equilibrium strategy, since an offer below \( p_0 \) will never be made.

**Concluding remarks.** As \( \delta \) increases, \( m_t \) decreases. That is, there are more rounds of bargaining. However, \( m_t > 0 \), implying that there must be a finite number of rounds. Furthermore, \( p_t \to \theta \) as \( \delta \to 1 \), implying that the seller’s revenue converges to \( \theta \). Note, however, that for \( \delta = 1 \), this is the very best a mechanism designer can achieve if he has to sell the object with probability one. Observe that expected payoff to the seller of the beginning of the game, \( W(0) = V(0) \), is

\[
V(0) \geq \bar{\theta}p_1 + \delta(1 - \bar{\theta})p_0 \\
= \bar{\theta}(1 - \delta)\bar{\theta} + \bar{\theta}\delta\theta + \delta(1 - \bar{\theta})\theta \\
= (1 - \delta)\bar{\theta} + \delta\theta \\
> \theta.
\]
since $q\theta > \theta$, by assumption. The first inequality follows by the fact that the seller prefers to offer $p_{T-1}$ (yielding equilibrium revenue) to $p_1$. It follows from the inequality that for any $\delta \in (0, 1)$, the seller is strictly better off than in any static mechanism which allocates the good with probability one.

Recall, however, that if $q\theta \leq \theta$, the seller offers $\theta$ immediately, i.e. $W(0) = \theta$. Thus, regardless of $q\theta$, $W(0) \to \theta$ for $\delta \to 1$.

## C Proof of Proposition 4

There are 3 different cases to consider, depending on whether it is efficient to not switch between buyers, to switch once, or to switch twice. We construct an efficient equilibrium for each possibility in turn. Thereafter, we show that revenue converges to the revenue in a static mechanism which maximizes revenue subject to efficiency. Without loss of generality it is assumed that $\theta_1 > \theta_2$.

### C.1 No switching

Assuming that $\theta_1 > \theta_2$, it is efficient for buyer 1 to always win. This is easily accomplished in seller-offer bargaining, and we now outline strategies that ensure this outcome. Let $\mu_i$ denote the belief concerning buyer $i$ before the seller makes his offer. That is, buyer $i$ is thought to have a type drawn from the uniform distribution on $[\mu_i, 1]$. At the beginning of the game $\mu_i = 0$, and $\mu_i \leq q \theta_i$ before the final offer is made.

**Seller.** Pick buyer 1. If buyer 1 has been picked, follow the equilibrium strategy from the one-buyer game in which the seller faces buyer 1. If buyer 2 has been picked, offer the price $\theta_2$ if

$$\frac{q_2 - \mu_2 \theta_2}{1 - \mu_2} + \delta \frac{1 - q_2}{1 - \mu_2} W_1(\mu_1) \geq \max\{\theta_2, \delta W_1(\mu_1)\},$$

offer $\theta_2$ if

$$\theta_2 > \max\{\frac{q_2 - \mu_2 \theta_2}{1 - \mu_2} + \delta \frac{1 - q_2}{1 - \mu_2} W_1(\mu_1), \delta W_1(\mu_1)\},$$

and offer some price exceeding $\theta_2$ otherwise.

**Buyer 1.** Follow the strategy from the one-buyer game.
**Buyer 2.** Accept any price that does not exceed the valuation of the buyer.

**Beliefs.** Beliefs concerning buyer 1 are as in the one-buyer game between buyer 1 and the seller. For buyer 2, $\mu_2 = 0$ if he has not been offered the good at a price of $\theta_2$ or less. Otherwise, $\mu_2 = \theta_2$.

To show that this is an equilibrium, we need only establish that the seller has no incentive to pick buyer 2 over buyer 1. Consider the different possible offers the seller can make given he has picked buyer 2.

If he picks buyer 2 and then offers a price exceeding $\theta_2$, his payoff in the game is lower than if he had picked buyer 1. The reason is that the sale is simply postponed.

Next, assume that the seller’s best strategy is to offer $\theta_2$ if buyer 2 has been picked. To make it unprofitable to pick buyer 2, we then require that

$$W_1(\mu_1) \geq \frac{\theta_2 - \mu_2 \theta_2}{1 - \mu_2} + \delta \frac{1 - \theta_2}{1 - \mu_2} W_1(\mu_1).$$

However, this is trivially satisfied since $W_1(\mu_1) \geq \theta_1 \geq \theta_2$.

Finally, if the optimal offer to buyer 2 is $\theta_2$, it is inoptimal to pick buyer 2 if

$$W_1(\mu_1) \geq \theta_2,$$

which we also know to be satisfied.

## C.2 Switching once

Assume that $\theta_1 > \theta_2 > \theta_2 > \theta_1$. In the following equilibrium, the seller first offers buyer 1 the good at a price of $\theta_1$. If buyer 1 declines, the seller switches to buyer 2, with whom he plays the one-buyer game.

**Buyer 1.** Accept the offer if and only if it does not exceed the valuation.

**Buyer 2.** If the buyer’s valuation is $\theta_2$, he accepts an offer if, and only if, it does not exceed $\theta_2$. If the buyer’s valuation is $\theta_2$ and he was not offered the good before buyer 1 was made a serious offer (an offer not exceeding $\theta_1$), then follow the equilibrium strategy from the one-buyer game (between the seller and buyer 2).
To describe the strategy if the valuation is \( \theta_2 \) and buyer 2 was offered the good before buyer 1 was made a serious offer, we first define the price \( P_t \) as the price which satisfies

\[
\theta_2 - P_t = \delta^2 (1 - q_1)(\theta_2 - p_{t-1}) \iff P_t = \theta_2 - \delta^2 (1 - q_1)(\theta_2 - p_{t-1}),
\]

where \( p_{t-1} \) is a price in the sequence of offers in the one-buyer game. Then, if the current offer to buyer 2 is not preceded by a serious offer to buyer 1, buyer 2 uses the same strategy as in the one-buyer game, with \( P_t \) in place of \( p_t \). If the current offer is preceded by a serious offer to buyer 1, define \( P \) as the smallest offer to buyer 2 (if any) before the serious offer, and let \( p \) be the minimum of the smallest offer made since (excluding the current offer) and the following transformation\(^{43} \) of \( P \):

\[
\hat{P} = \theta_2 - \frac{\theta_2 - P}{\delta (1 - q_1)}. \tag{30}
\]

Then, if the current offer is higher than \( p \), reject it if it is higher than \( p_1 \), and accept it otherwise. If the current offer is no higher than \( p \) the buyer of type \( q \in (q_t, q_{t-1}] \) accepts if and only if it is at most \( p_t \).

**Seller.** Pick buyer 1 if he has not been made a serious offer in the past. Pick buyer 2 otherwise.

Given buyer 1 has been picked, offer \( \theta_1 \) if buyer 1 has not been given a serious offer before. Otherwise, offer \( \theta_1 \) if

\[
\theta_1 \geq \delta W_2(\mu_2),
\]

and offer the price \( \theta_1 \) (or a higher price) if \( \theta_1 < \delta W_2(\mu_2) \).

Given buyer 2 has been picked, offer the price \( \theta_1 \) (such that it is not accepted) if buyer 1 has not been made an offer of \( \theta_1 \) or lower. If such an offer has been made to buyer 1, use the following strategy. If buyer 2 was not approached before buyer 1 was made a serious offer, then follow the one-buyer strategy. If buyer 2 was offered the good before buyer 1 was made a serious offer but the current offer is the first since, then randomize between \( p_t \) and \( p_{t-1} \) with the probabilities \( s \) and \( 1 - s \), respectively, where

\[
\theta_2 - P = \delta^2 (1 - q_1) \left[ s(\theta_2 - p_t) + (1 - s)(\theta_2 - p_{t-1}) \right], \tag{31}
\]

\(^{43}\)Notice that if \( P = p_t \), \( \hat{P} = \theta_2 - \delta(\theta_2 - p_{t-1}) = \theta_2 - \delta(\theta_2 - \theta_2) = p_t \).
when $P \in [P_t, P_{t+1})$, $t \in \{1, 2, ..., T - 1\}$, and offer $p_0$ if $P \leq P_1$ and $p_{T-1}$ if $P \geq p_T$. Finally, if buyer 2 was offered the good both before and after the serious offer to buyer 1, then randomize between $p_t$ and $p_{t-1}$ with probabilities $z$ and $1 - z$ respectively, such that
\[
\theta_2 - p = \delta(z(\theta_2 - p_t) + (1 - z)(\theta_2 - p_{t-1})),
\]
when $p \in (p_t, p_{t+1})$ for some $t \in \{1, 2, ..., T - 1\}$. Offer $p_0$ if $p < p_1$, and $p_{T-1}$ if $p \geq p_T$.

**Beliefs.** $\mu_1 = 0$ if buyer 1 has not been made a serious offer, and $\mu_1 = \overline{\theta}_1$ otherwise. Beliefs about buyer 2 are the same as in the one-buyer game.

**Equilibrium.** We show that the strategies and beliefs are an equilibrium for sufficiently high $\delta$. The need for a high discount factor is solely to stop the seller from deviating.

**Buyer 1.** Since buyer 1 is never made an offer below $\overline{\theta}_1$ on the equilibrium path, he is willing to follow the proposed strategy.

**Buyer 2.** If the buyer has the low valuation, he is clearly willing to follow the strategy. If he has the high valuation, and was not made an offer before buyer 1 was made a serious offer, the remainder of the game is just as in the one-buyer game, and he is therefore willing to follow the proposed strategy.

Assume now that the buyer has valuation $\overline{\theta}_2$ and that he was made an offer before buyer 1 was given a serious offer. If the current offer is before a serious offer to buyer 1, buyer 2 knows that the seller switches to buyer 1 next period, to return the period after that with a lottery between two consecutive prices. Using (31), essentially the same argument as in the one-buyer game establishes that the buyer is willing to follow the equilibrium strategy.

Finally, assume the current offer is preceded by a serious offer to buyer 1, and that buyer 2 was also made an offer before the serious offer to buyer 1. If the current offer is the first after the (first) serious offer to buyer 1, the same argument as in the one-buyer game implies that the buyer is indifferent between accepting and rejecting the offer if it is in the sequence of offers $\{p_t, p_{t-1}, ..., p_1\}$ when beliefs are described by $q_t$, or $P \in (P_t, P_{t+1}]$. If the current offer coincides with $\hat{P}$, the payoff from accepting is, by (30),
\[
\overline{\theta}_2 - \hat{P} = \frac{\overline{\theta}_2 - P}{\delta(1 - \overline{\theta}_1)} = \delta \left[ s(\overline{\theta}_2 - p_t) + (1 - s)(\overline{\theta}_2 - p_{t-1}) \right],
\]
where the last equality follows from (31). If the buyer rejects the offer, the seller responds next period with the lottery on the right hand side of (32),
with \( z \) determined by \( p = \hat{P} \). Hence, \( z = s \), and the buyer is therefore indifferent between accepting \( \hat{P} \) today or waiting for the lottery tomorrow. If the current offer is higher than \( \hat{P} \), however, the buyer prefers rejecting the offer. If the current offer is lower than \( \hat{P} \), the same argument as in the one-buyer game implies that the buyer is willing to follow the equilibrium strategy. It is straightforward to show that the buyer is also willing to follow the equilibrium strategy if the current offer is not the first after the serious offer to buyer 1.

**Seller.** If buyer 1 has been picked, but not previously been made a serious offer, it is optimal to offer either \( \theta_1 \) or \( \theta_2 \). To show the former dominates, observe that it yields a higher payoff than the latter when \( \delta \) is large, since

\[
\quad \begin{align*}
\bar{q}_1 \bar{\theta}_1 + \delta (1 - \bar{q}_1) W_2(\mu_2) & \quad \rightarrow \quad \bar{q}_1 \bar{\theta}_1 + (1 - \bar{q}_1) \theta_2 \\
& \quad \geq \quad \bar{q}_1 \bar{\theta}_1 + (1 - \bar{q}_1) \theta_1 \\
& \quad > \quad \theta_1.
\end{align*}
\]

Once buyer 1 has rejected a serious offer, we require that it is better to pick buyer 2 and play the one-buyer strategy, than to pick buyer 1 and offer \( \theta_1 \), or

\[
W_2(\mu_2) \geq \theta_1.
\]

However, this is satisfied since \( W_2(\mu_2) \geq \theta_2 \).

If buyer 2 has been picked, but buyer 1 has not yet been made a serious offer, we require that it is optimal for the seller to make a prohibitively high offer to buyer 2 (forcing him to reject) and then switch to buyer 1 next period. That is,

\[
\delta \bar{q}_1 \bar{\theta}_1 + \delta^2 (1 - \bar{q}_1) W_2(\mu_2) \geq \frac{q_{t-1} - \mu_2}{1 - \mu_2} P_t + \frac{1 - q_{t-1}}{1 - \mu_2} \delta \left( \bar{q}_1 \bar{\theta}_1 + \delta (1 - \bar{q}_1) W_2(\mu_2) \right),
\]

for any \( P_t \) that the seller may choose to offer buyer 2 in the current period. As \( \delta \to 1 \), we see that this is satisfied since

\[
\bar{q}_1 \bar{\theta}_1 + (1 - \bar{q}_1) \theta_2 > \frac{q_{t-1} - \mu_2}{1 - \mu_2} (\bar{q}_1 \bar{\theta}_2 + (1 - \bar{q}_1) \theta_2) + \frac{1 - q_{t-1}}{1 - \mu_2} (\bar{q}_1 \bar{\theta}_1 + (1 - \bar{q}_1) \theta_2)
\]

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as $\theta_1 > \theta_2$.\footnote{De Fraja and Muthoo (2000) show that a one-switching equilibrium exists when buyers are symmetric (implying $\theta_1 = \theta_2$). Because of symmetry, however, it does not matter if a mistake was made and buyer 2 was approached before buyer 1 was made a serious offer. In this case, the seller simply makes a serious offer to buyer 2, and then switches to buyer 1, and stays there.}

Finally, for the same reasons as in the one-buyer game, the seller is willing to randomize between two subsequent offers following a deviation.

### C.3 Switching twice

Assume that $\theta_1 > \theta_2 > \theta_1 > \theta_2$. In the following equilibrium, the seller starts by making a sequence of offers to buyer 1, switches to buyer 2 and makes him a single offer of $\theta_2$, after which he returns to buyer 1 and makes an offer of $\theta_1$. By the time the seller switches from buyer 1 to buyer 2, buyer 1 is revealed to have the low valuation. Notice that the sequence of offers is not necessarily monotonic, as the offer to buyer 2 may exceed the offers made to buyer 1.

The sequence of offers to buyer 1 is determined in the same fashion as in the one-buyer game. If the (decreasing) sequence of offers before the switch to buyer 2 is given by $\{P_{T-1}, P_{T-2}, ..., P_1\}$, for some $T \geq 2$, the high valuation buyer 1 must be indifferent between $P_t$ today and waiting until the last period of bargaining,

$$\bar{\theta}_1 - P_t = \delta^{t+1}(1 - q_2)(\bar{\theta}_1 - \theta_1).$$

Similarly, the acceptance probabilities ensure that the seller is indifferent between two consecutive offers (except the last two). For instance, the mass of agents, $M_1 = q_1 - q_1$, that accepts $P_1$ satisfies

$$P_1 \frac{\bar{q}_1 - q_1}{1 - q_1} + \delta \frac{1 - \bar{q}_1}{1 - q_1} (q_2 \bar{q}_2 + \delta (1 - q_2) \theta_1) = q_2 \bar{q}_2 + \delta (1 - q_2) W_1(q_1), \quad (34)$$

and the acceptance probabilities of offers preceding $P_1$ are determined in the same way as in the one-buyer game (i.e. as in (20)). Now, assume that $M_1$ is so small that it is optimal for the seller to offer $\theta_1$ in a one-buyer game if beliefs are $\mu_1 = q_1$. That is, $W_1(q_1) = \theta_1$. Inserting this in (34) and rewriting yields

$$P_1 \frac{\bar{q}_1 - q_1}{1 - q_1} = (q_2 \bar{q}_2 + \delta (1 - q_2) \theta_1) \left(1 - \delta \frac{1 - q_1}{1 - q_1}\right). \quad (35)$$
Clearly, \( q_1 < \bar{q}_1 \) \((M_1 > 0)\). Notice that as \( \delta \to 1 \), \( q_1 \to \bar{q}_1 \) \((M_1 \to 0)\).\(^{45}\) However, as \( M_1 \to 0 \), it eventually becomes optimal for the seller to offer \( \theta_1 \) in a one-buyer game with beliefs \( \mu_1 = q_1 \), as was assumed in (35). So, when \( \delta \) is sufficiently large, it is optimal for the seller to offer the price \( \theta_1 \) in the (off equilibrium) subgame where beliefs are \( \mu_1 = q_1, \mu_2 = \bar{q}_2 \), i.e. when the seller has deviated and, instead of offering \( P_1 \) to buyer 1, has picked buyer 2, offered him \( \theta_2 \), and then returned to buyer 1.

We can now suggest strategies and beliefs. Later, these are shown to form an equilibrium when \( \delta \) is sufficiently large. Again, the need for a high discount factor is solely to stop the seller from deviating.

**Buyer 1.** Buyer 1’s strategy is the same as in the one-buyer game, but with \( P_t \) in place of \( p_t \), \( t \geq 1 \).

**Buyer 2.** Accept if, and only if, the price offered does not exceed the valuation.

**Seller.** Pick buyer 1 in the first period. Pick buyer 1 if buyer 2 has been made a serious offer (one not exceeding \( \bar{q}_2 \)), or if buyer 2 has not been made a serious offer, but the lowest offer to buyer 1 in the past was higher than \( P_2 \). Pick buyer 2 if he has not been made a serious offer, and the smallest offer to buyer 1 in the past was below \( P_1 \). However, if buyer 2 has not been made a serious offer, and the smallest bid to buyer 1 in the past is \( P \in (P_1, P_2) \), then randomize between buyer 1 and buyer 2 with probabilities \( s \) and \( 1 - s \), respectively, such that

\[
\bar{\theta}_1 - P = s\delta(\bar{\theta}_1 - P_1) + (1 - s)\delta^2(\bar{\theta}_1 - \theta_1).
\]

If buyer 1 has been picked before a serious offer has been given to buyer 2 \((\mu_2 = 0)\), the offer is determined as in the one buyer game (with \( P_t \) in place of \( p_t \)) as long as \( \mu_1 < \bar{q}_1 \), and an offer of \( \bar{q}_1 \) is made if \( \mu_1 = \bar{q}_1 \).\(^{46}\) If buyer 1 has been picked and buyer 2 has been given a serious offer in the past \((\mu_2 = \bar{q}_2)\), follow the one-buyer strategy.

If buyer 2 has been picked when \( \mu_1 < q_1 \), make him an unserious offer (strictly exceeding \( \bar{\theta}_2 \)). If buyer 2 has been picked and \( \mu_1 \geq q_1, \mu_2 = 0 \) then

---

\(^{45}\)However, it is easily verified that \( M_t, t > 1 \), does not converge to zero. Consequently, bargaining lasts a finite number of periods. The fact that \( q_1 \to \bar{q}_1 \) also implies that, for \( \delta \) large, \( q_1 > 0 \) or that buyer 1 is offered the good at least once before buyer 2 is.

\(^{46}\)This is optimal because \( \theta_1 < \delta \bar{q}_2 \bar{\theta}_2 + \delta^2(1 - \bar{q}_2)\bar{\theta}_1 \) for large \( \delta \).
offer \( \overline{\theta}_2 \).\(^{47}\) If buyer 2 has been picked when \((\mu_1, \mu_2) = (\overline{q}_1, \overline{q}_2)\), the seller offers \( \theta_2 \) if \( \theta_2 \geq \delta \overline{\theta}_1 \) and offers \( \overline{\theta}_2 \) otherwise.

**Beliefs.** \( \mu_2 = 0 \) if buyer 2 has not been made a serious offer, and \( \mu_2 = \overline{q}_2 \) otherwise. Beliefs about buyer 1 are similar to the ones in the one-buyer game.

**Equilibrium.** To establish that the strategies and beliefs form an equilibrium, we first notice that the buyers have no incentive to deviate, and that beliefs are consistent. To show the seller has no incentive to deviate, observe first that the seller is willing to randomize when supposed to do so.

Next, if buyer 2 has been picked when \( \mu_1 < q_1 \) the seller is willing to make an unserious offer. First, the offer \( \overline{\theta}_2 \) is better than \( \theta_2 \) since \( \overline{\theta}_2 < \overline{q}_2 \theta_2 + \delta (1 - \overline{q}_2) \theta_1 \) for large values of \( \delta \) (the seller can ensure himself at least the right hand side by offering \( \overline{\theta}_2 \)). Then, we need only show that an unserious offer is preferable to \( \theta_2 \) for large \( \delta \). Since \( \mu_1 < q_1 \) it is sufficient that

\[
\delta \frac{\overline{q}_1 - \mu_1}{1 - \mu_1} P_1 + \delta^2 \frac{1 - \overline{q}_1}{1 - \mu_1} \left( \overline{q}_2 \overline{\theta}_2 + \delta (1 - \overline{q}_2) \theta_1 \right) \geq \overline{q}_2 \overline{\theta}_2 + \delta (1 - \overline{q}_2) W_1(\mu_1),
\]

as the seller can guarantee himself the left hand side by making an unserious offer and switching to buyer 1 next period. As \( \delta \to 1 \), this is satisfied because

\[
\frac{\overline{q}_1 - \mu_1}{1 - \mu_1} \left( \overline{q}_2 \overline{\theta}_1 + (1 - \overline{q}_2) \theta_1 \right) + \frac{1 - \overline{q}_1}{1 - \mu_1} \left( \overline{q}_2 \overline{\theta}_2 + (1 - \overline{q}_2) \theta_1 \right) > \overline{q}_2 \overline{\theta}_2 + (1 - \overline{q}_2) \theta_1,
\]

since \( \overline{q}_1 > \overline{q}_2 \). It is not difficult to verify that the seller has no incentive to deviate in any of the other subgames.

**C.4 Expected revenue**

As \( \delta \to 1 \) expected revenue converges to the revenue of a static mechanism that maximizes revenue subject to efficiency. The easiest way to see this is...\(^{47}\) First, the offer \( \overline{\theta}_2 \) is better than \( \theta_2 \) since \( \theta_2 < \overline{q}_2 \theta_2 + \delta (1 - \overline{q}_2) \theta_1 \) for large values of \( \delta \). The offer \( \overline{\theta}_2 \) is also better than an unserious offer, for the following reason. Beliefs about buyer 1 are either given by \( \mu_1 = q_1 \) or by \( \mu_1 = \overline{q}_1 \). In the former case, the seller is by definition of \( q_1 \) better off offering \( \overline{\theta}_2 \) than waiting and switching to buyer 1 next period. In the latter case an unserious offer simply postpones revenue (as buyer 2 must be picked next period).
to observe that the high valuation buyers are indifferent between accepting and rejecting the offers meant for them. That is, the downwards incentive compatibility constraints are binding (and so are the individual rationality constraints for the low valuation buyers). ¥
Figure 1: An irregular (non-monotonic) marginal revenue curve

Figure 2: Auctions vs. negotiations, non-monotonic marginal revenue.