# A Poor Bidder's Perspective on All-Pay Auctions: More Competitors, Please*

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**Abstract**

A three-bidder all-pay auction with heterogenous and privately informed bidders is analyzed. In such environments it is often impossible to fully characterize equilibrium. However, the additional assumption that one of the weaker bidders face a binding budget constraint (or a bidding cap) makes the problem tractable and an equilibrium can sometimes be characterized. In a *disjoint equilibrium*, weaker bidders are never active on the same range of bids. The comparative statics of disjoint equilibria are examined. Compared to an all-pay auction against only the strong bidder, the constrained bidder is shown to be better off when a third, weaker, competitor joins the auction. Likewise, the constrained bidder may be better off if the third bidder becomes more competitive, for instance due to preferential treatment. In other words, increased competition or preferential treatment of one of his rivals may benefit the constrained bidder.

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# 1 Introduction

In games of conflict it is natural to assume that an economic agent is better off the fewer competitors he has. In contrast, the purpose of the current paper is to show that some agents may in fact benefit when an additional, albeit weak, competitor joins the fray. Similarly, an agent may become better off when a rival becomes more competitive. Here, the competition is modeled as an all-pay auction.

The all-pay auction is a useful model of contests. It describes any situation in which a fixed prize is contested by agents who expend resources in its pursuit, and where these expenditures are forfeited regardless of whether the prize is won or not. Examples include lobbying, litigation, campaign spending, war, R&D races, and so on. Formally, the all-pay auction is a deterministic contests; the winner is the contestant who spends the most.

The core of the paper is a characterization of an equilibrium in a non-trivial auction environment. It is the equilibrium characterization that subsequently allows the comparative statics mentioned above to be derived. The model and the logic behind its construction is described next. The challenges inherent in the analysis of all-pay auctions with multiple agents are emphasized, as is the "trick" used to make the problem tractable.

The all-pay auction in the current paper has three distinct features, all of which are important in their own way. First, agents (bidders) are privately informed about the value they place on winning the prize. The remaining two features describe different ways in which bidders are heterogeneous ex ante. It is assumed that the cost of bidding is discontinuous for precisely one bidder. Specifically, the bidder is budget constrained, meaning that there is a limit or cap to how much he can bid. Finally, bidders have different constant marginal costs (when the budget constraint is not met), although this is isomorphic to bidders drawing valuations from different distributions.

In comparison, Siegel (2009, 2010) considers complete information deterministic contests with continuous cost functions. Thus, his analysis does not accommodate a budget constraint, although it can perhaps be approximated with a very steep kink in the cost-function. In a generic contest, a bidder is unambiguously worse off if more competitors arrive or if an existing rival becomes more competitive. Che and Gale (1997) examine all-pay auctions in which the prize is the same for all competitors, but where different competitors face different caps. In their model, it is also the case that a bidder can never benefit from more severe competition. These results indicate that incomplete information is required to obtain the result mentioned in the opening paragraph, at least for all-pay auctions.

Consider now a model in which bidders are symmetric ex ante, with valuations that are identically and independently distributed. In this case, however, bidders are worse off, regardless of their type, when the number of bidders increases.[1] I therefore relax the assumption that bidders are symmetric ex ante. As mentioned, this is achieved by assuming that different bidders have different cost functions (or, equivalently, independently draw valuations from different distributions).[2]

However, all-pay auctions with heterogeneous, privately informed bidders present some analytical challenges, especially when there are multiple bidders. Amann and Leininger (1996) offer an (implicit) characterization of equilibrium in a model with exactly two bidders. Siegel (2011) constructs the unique equilibrium in a two-bidder all-pay auction with discrete signal distributions and potentially interdependent valuations. In contrast, Parreiras and Rubinchik (2010) consider many-bidder all-pay auctions. Their conclusion is that equilibrium in such auctions may be qualitatively different from equilibrium in two-bidder auctions. Although they do not fully characterize equilibrium, they show that equilibrium may, in principle, have the properties that (1) some bidders use discontinuous strategies (if they bid, they bid a lot), and/or (2) some bidders never bid enough to win with a probability close to one. It is precisely these features, neither of which occur in two-bidder auctions, that make it difficult to characterize equilibrium in general.

The third modeling assumption – that one bidder faces a budget constraint – is interesting in its own right, but also serves the crucial function of making the analysis tractable.[3] With this assumption, an equilibrium can in fact be characterized for some parameter constellations. While the constrained bidder necessarily bids below his cap, in a *disjoint equilibrium* the other weak bidder bids above his rival's cap, if he bids at all. The remaining, strong, bidder may submit bids below or above the cap, depending on his valuation. Thus, precisely two bidders are "active" at any given bid. As a consequence, the equilibrium can be characterized by "piecing together" two of Amann and Leininger's (1996) two-bidder equilibria. Note that the disjoint equilibrium has both of the features identified by Parreiras and Rubinchik (2010).[4]

[1] Any standard auction, such as the all-pay auction, is efficient in the independent private values model when bidders are symmetric. Thus, the addition of a competitor reduces the winning probability of an incumbent. Following a result in Myerson (1981), the incumbents are therefore made worse off.

[2] Alternatively, the independence assumption could be relaxed, but that is not pursued here. The addition of more bidders is not necessarily straightforward in such models. Pinske and Tan (2005) show that bidders may bid less aggressively in first-price auctions the more competitors they have. Menicucci (2009) shows that this effect may cause expected revenue to decrease in the number of bidders. Finally, Fang and Parreiras (2003) demonstrate that the linkage principle fails when bidders are financially constrained.

[3] Other papers have also considered caps in all-pay auctions with private information. Gavious et al (2002) consider a model where bidders are symmetric and the bidding cap is the same for all bidders. Sahuguet (2006) assume there are two asymmetric bidders with identical caps. Che and Gale (1998) analyze auctions with symmetric bidders who are privately informed about both their valuations and financial constraints.

[4] See Siegel (2010, Example 1 and Figure 2) for what amounts to a disjoint equilibrium in a complete information setting. However, as mentioned, the comparative statics are different with complete information.

When the constrained bidder and the strong bidder are alone in the auction, the latter may simply bid at the former's cap to ensure a win, if his valuation is sufficiently high. Thus, the constrained bidder wins only if the strong bidder's valuation happens to be quite low. Now, when the third bidder enters the auction, bids below the cap are less likely to be successful. Therefore, the strong bidder bids more timidly if his valuation is low. As a consequence, the constrained bidder's bidding cap effectively becomes less of a constraint in his fight against the strong bidder. In a disjoint equilibrium, this positive effect outweighs the negative effect of having an extra competitor. Thus, with the extra rival, the constrained bidder participates more often and is better off whenever he participates. Similar effects come into play if the third bidder becomes stronger. Hence, the constrained bidder may become better off if the third bidder becomes more competitive or is given preferential treatment.

The budget constraint can be viewed as a non-linearity in the cost function, although it more precisely introduces a discontinuity. The intuition behind the comparative statics result appears to survive if the cost function is continuous but very steep after a certain point. However, equilibrium in the three-bidder auction cannot be analytically characterized in the latter case.[5] In a companion paper, Kirkegaard (2011), I consider all-pay auctions with non-linear, but continuous, cost functions. There, the focus is on the effect that preferential treatment has on a diverse group of bidders. Although equilibrium is not characterized, it is nevertheless possible in that paper to prove that a bidder who receives preferential treatment may be hurt by it.

The paper is organized as follows. Section 2 describes the model. As a stepping stone, Section 3 considers two-bidder all-pay auctions with and without a budget constraint. Section 4 analyzes the three-bidder auction and characterizes the disjoint equilibrium. Comparative statics are also described, and possible extensions are briefly discussed. Section 5 concludes. Lengthier proofs are in the Appendix.

## 2 Model

Three risk neutral bidders are competing in an all-pay auction for a single prize. Bidder $i$, $i = 1, 2, 3$, has a privately known type, $v$, which captures how much he values winning the prize. Types are identically and independently distributed according to some strictly increasing and twice continuously differentiable distribution function, $F(v)$, with no mass points and support $[\underline{v}, \overline{v}]$, where $\overline{v} > \underline{v} > 0$. The density, $f$, is bounded above and below, away from zero.

[5] Parreiras and Rubinchik (2010) assume costs are linear but allow bidders to be risk averse. Risk aversion also makes payoff non-linear.

The "average probability", $F(v)/v$, is assumed to be strictly increasing in $v$. This assumption is satisfied if $F(v)$ is convex. In particular, the property holds if $F(v)$ is the uniform distribution with support $[\underline{v}, \overline{v}]$, $\underline{v} > 0$.

By submitting a bid $b \geq 0$, bidder $i$ incurs a cost of $\alpha_i c_i(b)$, $\alpha_i > 0$. If $c_i(b) = b$ then bidder $i$ is said to have linear costs. In this paper, it will be assumed that costs are either linear or that $c_i(b)$ takes the form

$$c_i(b) = \begin{cases} b & \text{if } b \in [0, m_i] \\ \infty & \text{otherwise} \end{cases}$$

for some $m_i \in \mathbb{R}_{++}$. Here, $m_i$ can be thought of as a bidding cap or a resource constraint. The cap is exogenous to the bidder and may or may not be imposed by some regulator or designer. Linear costs correspond to $m_i = \infty$.

If bidder $i$ wins the auction with probability $q_i(b)$ when bidding $b$, such a bid would yield expected payoff of

$$vq_i(b) - \alpha_i c_i(b)$$

if bidder $i$'s type is $v$. The win probability $q_i(b)$ is determined by the strategies used by bidder $i$'s rivals. Given these strategies, bidder $i$'s objective is to maximize his expected payoff. Clearly, expected payoff is maximized where

$$\frac{v}{\alpha_i} q_i(b) - c_i(b)$$

is maximized. Thus, an alternative interpretation of the model is that bidders have different type distributions but the same marginal costs (below $m_i$). In this interpretation, bidder $i$'s type is in the interval $[\underline{v}/\alpha_i, \overline{v}/\alpha_i]$, distributed according to $F(\alpha_i v)$. In either interpretation, lower values of $\alpha_i$ means that bidder $i$ is stronger (he either has lower costs or is more likely to value the prize highly).

Amann and Leininger (1996) characterize equilibrium in all-pay auctions with two heterogeneous bidders. Parreiras and Rubinchik (2010) study all-pay auctions with more, potentially risk averse, bidders and linear costs. In that case, however, it is typically impossible to characterize equilibrium.

In contrast, in the environment being studied in this paper, two of the three bidders have linear costs (or the cap is so high as to be irrelevant) while the third faces a binding cap. In some cases, this is sufficient to make the problem tractable and permit an equilibrium to be characterized. The trick is that the cap eliminates some of the complications that arise in many-bidder all-pay auctions and in fact makes techniques from the two-bidder case sufficient to tackle the model. Hence, I begin with a discussion of two-bidder auctions, the first part of which is based on Amann and Leininger (1996).

# 3 Two-bidder all-pay auctions

Consider an all-pay auction with only two bidders, $i$ and $j$. A slight generalization will be examined, for future use, in which the prize is awarded to the bidder with the highest bid with some exogenous probability $p \in (0, 1]$. The prize is withheld with probability $1 - p$.[6]

## 3.1 A two-bidder all-pay auction without caps

Assume, for now, that costs are linear and $\alpha_i \geq \alpha_j$. Hence, bidder $i$ is the weaker bidder. Amann and Leininger (1996) prove that an equilibrium exists in which at most one bidder stays out for a mass of types. Indeed, it turns out that bidder $i$ does so whenever $\alpha_i > \alpha_j$.[7] I will follow Amann and Leininger (1996) in searching for an equilibrium in which bidding strategies are differentiable and strictly increasing among the set of types that participate.

For the types that participate, bids are strictly increasing in types. Let $\varphi_i(b)$ and $\varphi_j(b)$ denote the inverse strategy of bidder $i$ and bidder $j$, respectively. Then, the probability that bidder $i$ bids $b$ or below is $F(\varphi_i(b))$. If bidder $j$ bids $b$, his winning probability is thus $pF(\varphi_i(b))$. His objective is to maximize $vpF(\varphi_i(b)) - \alpha_j b$ if his type is $v$. Assuming the optimal bid is in the interior, it must solve the first order condition,

$$vp\frac{dF(\varphi_i(b))}{db} - \alpha_j = 0.$$

In equilibrium, bidder $j$ with type $\varphi_j(b)$ is supposed to bid $b$. Thus, the first order condition should be satisfied at $v = \varphi_j(b)$, implying

$$\frac{dF(\varphi_i(b))}{db} = \frac{\alpha_i}{p\varphi_j(b)} \tag{1}$$

or

$$\varphi_i'(b) = \frac{\alpha_j}{pf(\varphi_i(b))\varphi_j(b)}. \tag{2}$$

[6] Alternatively, $p$ can be interpreted as a measure of the quality of the prize, in which case $p$ need not be constrained to be below one.

[7] In particular, neither bidder bids zero with positive probability. Otherwise, the rival with valuation $\underline{v} > 0$ should not bid zero, but rather marginally above zero. Amann and Leininger (1996), in contrast, assume $\underline{v} = 0$, in which case one of bidders may bid zero with positive probability. Amann and Leininger (1996) also assume $p = 1$, but $p \in (0, 1)$ is easily incorporated into their analysis.

Applying the same logic to bidder $i$ produces a similar expression for $\varphi_j'(b)$. Thus, the two first order conditions yield a system of differential equations,

$$\varphi_i'(b) = \frac{\alpha_j}{pf(\varphi_i(b))\varphi_j(b)}, \quad \varphi_j'(b) = \frac{\alpha_i}{pf(\varphi_j(b))\varphi_i(b)}. \tag{3}$$

A boundary condition is necessary to solve the system of differential equations. Note that there must be some endogenous maximal bid, $\bar{b}$, that is common to both bidders. Otherwise, it would pay for one bidder to lower his bid and still win with probability one. Thus, the boundary condition is $\varphi_i(\bar{b}) = \varphi_j(\bar{b}) = \overline{v}$, where $\bar{b}$ is yet to be determined.[8]

Amann and Leininger (1996) observe that (3) is an autonomous system. It can thus be solved for a relationship between $\varphi_i$ and $\varphi_j$. More specifically, they define a tying function, $k_i(v)$, where $k_i$ is the type of bidder $i$ who submits the same bid as bidder $j$ with type $v$. Since the bidders share the same maximal bid, the boundary condition is that $k_i(\overline{v}) = \overline{v}$. Following Amann and Leininger (1996), the idea is to use this boundary condition and the system of differential equations to "shoot backwards" in order to complete describe $k_i(v)$. Adapting Amann and Leininger (1996) to the environment considered here, the tying function is implicitly (and uniquely) characterized by

$$\alpha_i \int_{k_i(v)}^{\overline{v}} \frac{f(x)}{x} dx = \alpha_j \int_{v}^{\overline{v}} \frac{f(x)}{x} dx, \tag{4}$$

for any $v \in [\underline{v}, \overline{v}]$.[9] Note that at $v = \overline{v}$, $k_i(\overline{v}) = \overline{v}$ solves the equation. It is also immediate that $k_i(\underline{v}) > \underline{v}$ whenever $\alpha_i > \alpha_j$, meaning that bidder $i$ has a mass of types that stay out of the auction.

Let $b_j(v)$ denote bidder $j$'s bidding strategy. Once $k_i(v)$ has been obtained, the counterpart to (2) for bidder $j$ can be used to derive $b_j(v)$ since it implies that

$$db = \frac{1}{\alpha_i} pf(\varphi_j(b))\varphi_i(b)d\varphi_j.$$

[8] From (2), strategies are monotonic, or $\varphi_i'(b) > 0$. Then, $F_i(\varphi_i(b))$ is concave, from (1). Thus, bidder $j$'s expected payoff is concave in $b$ and the first order conditions are necessary and sufficient for a maximum.

[9] Divide $\varphi_i'(b) = \frac{d\varphi_i}{db}$ by $\varphi_j'(b) = \frac{d\varphi_j}{db}$ to obtain

$$\frac{d\varphi_i}{d\varphi_j} = \frac{\alpha_j f(\varphi_j)\varphi_i}{\alpha_i f(\varphi_i)\varphi_j}.$$

Since types $\varphi_i$ and $\varphi_j$ submit the same bid, the would tie in the auction. Letting $v = \varphi_j$ and $\varphi_i = k_i(v)$, the previous expression can be written

$$k_i'(v) = \frac{\alpha_j f(v) k_i(v)}{\alpha_i f(k_i(v))v}.$$

Using the boundary condition $k_i(\overline{v}) = \overline{v}$ produces (4).

Recalling that $b_j(\underline{v}) = 0$ and that bidder $j$ with type $\varphi_j$ ties with bidder $i$ with type $\varphi_i = k_i(\varphi_j)$ then yields

$$b_j(v) = \frac{1}{\alpha_i} p \int_{\underline{v}}^{v} k_i(x) f(x) dx, \; v \in [\underline{v}, \overline{v}] \tag{5}$$

upon integration. Likewise, bidder $i$ stays out if $v \in [\underline{v}, k_i(\underline{v}))$, but otherwise he uses the bidding strategy

$$b_i(v) = \frac{1}{\alpha_j} \int_{k_i(\underline{v})}^{v} p k_i^{-1}(x) f(x) dx, \; v \in [k_i(\underline{v}), \overline{v}]. \tag{6}$$

It can be verified that $b_j(v) = b_i(k_i(v))$.

Note that the allocation, summarized by $k_i(v)$, is independent of $p$. The magnitude of the bids do, however, depend on $p$. In particular, the highest possible equilibrium bid is

$$\overline{b} = b_j(\overline{v}) = \frac{1}{\alpha_i} p \int_{\underline{v}}^{\overline{v}} k_i(x) f(x) dx, \tag{7}$$

which depends on both $\alpha_i$ and $\alpha_j$, since $k_i(x)$ is a function of both.

**Proposition 1** *There is a unique equilibrium with differentiable and strictly increasing bidding strategies (among the set of participating types). Bidding strategies are given by (5) and (6), with $k_i(v)$ uniquely described by (4).*

**Proof.** Recall that the system of differential equations produces

$$k_i'(v) = \frac{\alpha_j f(v) k_i(v)}{\alpha_i f(k_i(v)) v},$$

which is differentiable in $k$ and $v$ and Lipschitz since, by assumption, $f(v), v \in \mathbb{R}_{++}$. Given the boundary condition $k_i(\overline{v}) = \overline{v}$, these properties imply that $k_i(v)$ is uniquely determined; see e.g. Hirsch and Smale (1974, Theorem 1, p. 297) for a formal proof.

Assuming bidder $i$ follows his equilibrium strategy, bidder $j$'s problem can be written

$$\max_z vpF(k_i(z)) - \alpha_j b_j(z)$$

with first derivative

$$\frac{\alpha_j}{\alpha_i} p k_i(z) f(z) \left( \frac{v}{z} - 1 \right).$$

Since this derivative is positive for $z < v$ and negative for $z > v$, the solution to the maximization problem is at $z = v$, as required. A similar argument proves that bidder $i$ has no incentive to deviate either. ■

### 3.2 A two-bidder all-pay auction with a cap

Assume now that bidder $i$, the weaker bidder, faces a budget constraint, $m_i$.[10] The constraint is not binding if $m_i \geq \bar{b}$. Hence, assume that $0 < m_i < \bar{b}$. In this case, it is natural to hypothesize that bidder $j$ will bid $m_i$ or slightly above $m_i$ if his type is sufficiently high, in order to outbid bidder $i$ with probability one. Suppose bidder $j$ bids at least $m_i$ if his type is $\widetilde{v}_j$ or higher, where $\widetilde{v}_j \in (\underline{v}, \overline{v})$. Of course, for this to be part of an equilibrium, bidder $i$ with type $\overline{v}$ must exhaust his budget and bid $m_i$. Otherwise, bidder $j$ could lower his bid and still outbid bidder $i$ for sure. Therefore, I will search for an equilibrium in which bidding strategies are strictly increasing and differentiable among the types that submit bids in the interval $(0, m_i)$.

First, however, it cannot be the case that bidder $i$ bids $m_i$ for a set of types of strictly positive mass. If such behavior formed part of the equilibrium, bidder $j$ with a very high type would have no best response (no equilibrium would exist) unless the tie-breaking rule is such that he wins a tie. With such a tie-breaking rule, however, bidder $j$ wins significantly more often by bidding $m_i$ than slightly below $m_i$. Thus, bidder $j$ would never submit bids in an interval just below $m_i$. This means, in turn, that bidder $i$ should not bid $m_i$ after all, since he could lower his bid slightly without affecting his probability of winning.

Assume from now on that the tie-breaking rule is such that bidder $j$ wins any tie, as above. Note that this rule also discourages bidder $i$ with types below $\overline{v}$ to jump from a bid below $m_i$ to a bid of precisely $m_i$. Even though bidder $j$ bids $m_i$ with positive probability, bidder $i$ can never outbid this mass of types, given the tie-breaking rule. As will become evident as the analysis unfolds, the tie-breaking rule thus guarantees existence of an equilibrium. In the (unique) equilibrium, bidder $i$ bids below $m_i$ if his type is below $\overline{v}$. Ties occur with probability zero, in equilibrium.

Bidder $j$ "engages" bidder $i$ by submitting lower bids if his type is below $\widetilde{v}_j$. For both bidders, the utility maximization problem is as described in Section 3.1 at bids below $m_i$. The only difference is the boundary conditions, which now become $\varphi_i(m_i) = \overline{v}$, $\varphi_j(m_i) = \widetilde{v}_j$. In this case, then, the maximum bid is known, but $\widetilde{v}_j$ remains to be identified. To this end, consider the tying function, $k_i(v|\widetilde{v}_j)$, defined on $v \in [\underline{v}, \widetilde{v}_j]$. The boundary condition now changes to $k_i(\widetilde{v}_j|\widetilde{v}_j) = \overline{v}$; bidder $j$ with type $\widetilde{v}_j$ and bidder $i$ with type $\overline{v}$ both bid $m_i$. Using this boundary condition, the tying function can now be characterized by shooting backwards from $\widetilde{v}_j$ to obtain

$$\alpha_i \int_{k_i(v|\widetilde{v})}^{\overline{v}} \frac{f(x)}{x} dx = \alpha_j \int_{v}^{\widetilde{v}_j} \frac{f(x)}{x} dx, \; v \in [\underline{v}, \widetilde{v}_j]. \tag{8}$$

[10] Sahuguet (2006) analyzes the case in which both bidders face the same budget constraint.

As before, the tying function is unique, for given $\widetilde{v}_j$. Note that $k_i(\cdot|\widetilde{v}_j)$ is decreasing in $\widetilde{v}_j$. The function $k_i(\cdot)$ derived in Section 3.1 obviously coincides with $k_i(\cdot|\overline{v})$.

To determine $\widetilde{v}_j$, recall that bidder $j$ with type $\widetilde{v}_j$ and bidder $i$ with type $\overline{v}$ is supposed to bid $m_i$ in equilibrium. For those types that bid below the cap, bidding strategies are

$$b_j(v|\widetilde{v}_j, p) = \frac{1}{\alpha_i} p \int_{\underline{v}}^{v} k_i(x|\widetilde{v}_j) f(x) dx, \; v \in [\underline{v}, \widetilde{v}_j] \tag{9}$$

and

$$b_i(v|\widetilde{v}_j, p) = \frac{1}{\alpha_j} \int_{k_i(\underline{v}|\widetilde{v}_j)}^{v} p k_i^{-1}(x|\widetilde{v}_j) f(x) dx, \; v \in [k_i(\underline{v}|\widetilde{v}_j), \overline{v}], \tag{10}$$

respectively. It is required that $b_i(\overline{v}|\widetilde{v}_j, p) = m_i$, or

$$m_i = \frac{1}{\alpha_j} \int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} p k_i^{-1}(x|\widetilde{v}_j) f(x) dx. \tag{11}$$

To summarize, starting from any $\widetilde{v}_j$, the system of differential equations is used to shoot backwards to obtain the function $k_i(v|\widetilde{v}_j)$. Bids are then obtained by using $b_i(k_i(\underline{v}|\widetilde{v}_j)) = 0$ and the tying function to shoot forward (integrate up); if $\widetilde{v}_j$ is an equilibrium then the resulting maximum bid should be precisely $m_i$.

Since $k_i(\underline{v}|\widetilde{v}_j)$ decreases with $\widetilde{v}_j$ and $k_i^{-1}(v|\widetilde{v}_j)$ increases with $\widetilde{v}_j$, $b_i(\overline{v}|\widetilde{v}_j, p)$ is strictly monotonic in $\widetilde{v}_j$. By assumption, $m_i < \overline{b} = b_i(\overline{v}|\overline{v}, p)$. It is also the case that $b_i(\overline{v}|\underline{v}, p) = 0$, since $k_i(\underline{v}|\underline{v}) = \overline{v}$. Hence, there is one and only one $\widetilde{v}_j \in (\underline{v}, \overline{v})$ such that $b_i(\overline{v}|\widetilde{v}_j, p) = m_i$. Given $\widetilde{v}_j$, the equilibrium allocation, summarized by $k_i(\underline{v}|\widetilde{v}_j)$, can be described. With these in hand, bidding strategies can then be derived, as above. This describes the unique equilibrium.

**Proposition 2** *Assume $0 < m_i < \overline{b}$. There is a unique equilibrium with differentiable and strictly increasing bidding strategies among the set of types bidding in $(0, m_i)$, with $\widetilde{v}_1$ and $k_i(v|\widetilde{v}_1)$ uniquely determined by (8) and (11). Bidder $j$ bids $m_i$ if $v \geq \widetilde{v}_j$ and uses the strategy in (9) otherwise. Bidder $i$ stays out if $v < k_i(v|\widetilde{v}_j)$ and uses the strategy in (10) otherwise.*

**Proof.** Existence and uniqueness has been addressed in the main text. Given the tie-breaking rule, the same argument as in Proposition 1 can be used to show that bidder $i$ has no incentive to deviate regardless of his type, nor does bidder $j$ if his type is below $\widetilde{v}_1$. Since preferences satisfy the single-crossing condition, the best response of bidder $j$ with a higher type is to bid $m_i$, as required. ■

### 3.3 Comparative statics

It is fruitful to examine the comparative statics associated with changes in $p$. To begin, let $B$ denote the maximum bid in an auction with $p = 1$ and no budget constraint. From (7), it is evident that as $p$ varies, the maximum bid satisfies $\bar{b} = pB$ in the event there is no binding cap. Next, fix $m_i$ and assume $0 < m_i < B$. Then, the cap is binding if $p$ is high but not when $p$ is low.

If the cap is not binding, a small change in $p$ does not change the allocation since $k_i(v)$ is unaffected. Hence, each bidder participates with the same probability as before and continues to have the same probability of outbidding the rival. However, since bidder $j$ with type $v$ wins with probability $pF(k_i(v))$, a decline in $p$ means his winning probability decreases. The same is true for bidder $i$. It is also the case that both bidders are worse off.[11]

The case in which $p$ is large, or $m_i < pB$, is more interesting. Since the cap is binding, (11) applies. The right hand side of (11) is increasing in both $p$ and $\widetilde{v}_j$. Hence, the two must move in opposite directions. In other words, a decrease in $p$ implies an increase in $\widetilde{v}_j$ and thus an increase in $k_i^{-1}(v|\widetilde{v}_j)$. The implication is that bidder $i$ participates for more types and outbids bidder $j$ with larger probability the more likely it is that the object is withheld. The intuition is that when $p$ decreases, bids tend to decline as well, as bidders become more cautious. Hence, the cap $m_i$ constitutes less of a constraint, and bidder $i$ now has room to become more aggressive in his interaction with bidder $j$.

For bidder $i$, the types that stay out for large $p$ but who begin to participate when $p$ decreases evidently win more often when $p$ is lowered. It is harder to say whether bidder $i$ wins more often if his type is high, say equal to $\bar{v}$. In this case, his winning probability is $pF(\widetilde{v}_j)$, but $p$ and $F(\widetilde{v}_j)$ move in opposite directions. Nevertheless, for bidder $i$, it turns out that all participating types wins more often when $p$ decreases and $m_i < pB$. Since the opposite holds when $m_i > pB$, bidder $i$'s winning probability is hump-shaped in $p$. For bidder $j$, the winning probability is strictly increasing in $p$.

**Lemma 1** *Regardless of his type, bidder $j$'s winning probability is strictly increasing in $p$. Bidder $i$ participates with the same set of types as long as $p < m_i/B$ but the set of participating types shrinks as $p$ increases beyond $m_i/B$. For all participating types, bidder $i$'s winning probability is strictly increasing in $p$ as long as $p < m_i/B$ and strictly decreasing in $p$ thereafter.*

**Proof.** See the Appendix. ■

[11] Bidder $j$'s expected payoff can be written $p\,(vF(k_i(v)) - \alpha_j B_j(v))$, where $B_j(v)$ is the bidding strategy if $p = 1$ (and the cap is not binding). Hence, bidder $j$ is worse off when $p$ decreases. A different proof is provided in Proposition 3.

Lemma 1 implies that bidder $j$'s expected payoff is strictly increasing in $p$, while bidder $i$'s expected payoff is hump-shaped in $p$. Here, hump-shaped means that payoff is not decreasing in $p$ nor is it first strictly decreasing and then later strictly increasing; however, an interval where payoff is flat is possible (which is the case for non-participating types).

**Proposition 3** *Regardless of his type, bidder $j$'s expected payoff is strictly increasing in $p$. Regardless of his type, bidder $i$'s expected payoff is hump-shaped in $p$, and maximized when $p = m_i/B$.*

**Proof.** Let $Q_j(v|p)$ denote bidder $j$'s equilibrium winning probability given his type $v$ and the exogenous probability $p$. Myerson (1981) shows that the expected payoff of bidder $j$ with type $v$, $EU_j(v|p)$, can be written

$$EU_j(v|p) = EU_j(\underline{v}|p) + \int_{\underline{v}}^{v} Q_j(x|p)dx.^{12} \tag{12}$$

Bidder $j$'s costs are zero regardless of $p$ if his type is $\underline{v}$. Since he wins more often the higher $p$ is, $EU_j(\underline{v}|p)$ is strictly increasing in $p$. Since $Q_j(x|p)$ is increasing in $p$ as well, it follows that $EU_j(v|p)$ is strictly increasing in $p$ for all $v \in [\underline{v}, \overline{v}]$. For bidder $i$, $EU_i(\underline{v}|p) = 0$ regardless of $p$. The second part of the proposition then follows immediately from Lemma 1. ■

The main point in Section 4 is to show that the addition of a third, sufficiently weak, bidder, has the same effects as a decrease in $p$ from bidder $i$'s perspective. Thus, bidder $i$ may be made better off with more competition.

## 3.4 An outsider's perspective

Before turning to three-bidder auctions in earnest, consider the incentive for an outsider to start bidding in the auction when bidders $i$ and $j$ expect no other rivals to be active (and so use the strategies described above). The outsider knows bidders $i$ and $j$ believe the prize is allocated with probability $p$. However, assume the outsider instead believes the prize is allocated with probability $q$, where $q$ may or may not coincide with $p$.

Assume no bidder faces a budget constraint. Then, the outsider with type $v$ and marginal costs $\alpha_k$ would seek to maximize $vqF(\varphi_i(b))F(\varphi_j(b)) - \alpha_k b$ by bidding $b \in [0, \overline{b}]$. Using (1), the first derivative with respect to $b$ is

$$v\frac{q}{p}\left(\alpha_i \frac{F(\varphi_i(b))}{\varphi_i(b)} + \alpha_j \frac{F(\varphi_j(b))}{\varphi_j(b)}\right) - \alpha_k.$$

[12] Bidder $j$'s expected payoff is $EU_j(v) = \max_x vQ_j(x|p) - \alpha_j b_j(x)$. The Envelope Theorem yields $EU_j'(v) = Q_j(v|p)$, which in turn produces (12).

Since $\varphi_i(b)$ is increasing in $b$ and $F(v)/v$ is increasing, by assumption, this derivative is increasing in $b$. In other words, the outsider's payoff is convex in $b$, regardless of $v$, $q$, $p$, and $\alpha_k$. Thus, the optimum is at a corner; the outsider would either stay out or bid $\bar{b}$ and outbid his rivals with probability one. The latter is more likely to be optimal the higher $v$ is.

If bidder $i$ faces a binding budget constraint, the previous argument can be used to prove that no bid in $(0, m_i)$ can be optimal for the outsider.

# 4 Three-bidder all-pay auctions

Consider a standard all-pay auction, with $p = 1$. Assume now that there are three (potential) bidders, with $\alpha_1 \leq \alpha_2 < \alpha_3$. Let $B$ denote the maximum bid in an all-pay auction with just bidders 1 and 2 present, in which $p = 1$ and neither bidder face a budget constraint. Assume from now on that

$$\overline{v} - \alpha_3 B \leq 0.$$

The significance of this assumption is that bidder 3 would not find it profitable to enter the auction with a bid of $B$ if bidders 1 and 2 use the strategies from the two-bidder auction. Since bidder 3's problem is convex in $b$, his optimal action is then to stay out. In other words, there is an equilibrium in which only bidders 1 and 2 are active in the auction with positive probability, at least when neither is financially constrained.

In the remainder of the paper, I assume that bidder 2, and only bidder 2, has a binding constraint, $m_2 < B$.[13] In fact, a stronger assumption will be imposed. Let $\overline{m}$ solve

$$\overline{v} - \alpha_3 \overline{m} = 0,$$

and assume that $m_2 < \overline{m}$. This assumption implies that bidder 3, if his type is sufficiently high, would in fact want to enter the auction with a bid of $m_2$ (or slightly higher) if bidders 1 and 2 continue to follow the equilibrium strategies from the two-bidder game.

However, bidder 3's entry at high bids constitutes a challenge to bidder 1, who might then want to start bidding above $m_2$ in order to deal with the new competitor. This effect suggest that an equilibrium with a particularly simple structure may exist when $m_2 < \overline{m}$. In a *disjoint equilibrium*, bidder 1 and bidder 2 are the only bidders active at bids below $m_2$, whereas bidder 1 and bidder 3 are the only bidders active at bids above $m_2$. There is no overlap between the bids of bidder 2 and bidder 3; their equilibrium strategies are disjoint. Characterizing the equilibrium is then made easier, since only two bidders are active at any

[13] If (only) bidder 3 faces a budget constraint it remains an equilibrium for only bidders 1 and 2 to be active. The case in which bidder 1 faces a binding budget constraint is not analyzed.

bid. However, it is possible that bidder 1 bids $m_2$ for a mass of types in a disjoint equilibrium. Note the self-confirming structure of the equilibrium; if bidder 3 is not expected to be active at bids below $m_2$, then Section 3.4 reveals that his best response is indeed to refrain from bidding in this region. Likewise, the cap prevents bidder 2 from bidding above $m_2$.

Technically, to ensure the existence of an equilibrium it is necessary to carefully define the tie-breaking rule. In the following, assume that bidder 3 wins any tie he is involved in, bidder 1 wins if he is in a tie with bidder 2 only, and bidder 2 loses any tie. It will remain the case that a tie occurs with probability zero, in equilibrium. As before, it is the discontinuity in $c_2$ which may cause a mass of types to submit the same bid.[14]

In the following, it is convenient to think of $(\alpha_1, \alpha_2, \alpha_3)$ as fixed. It will then be established that a disjoint equilibrium exists for some values of $m_2$ (values close to, but below, $\overline{m}$).

## 4.1 Characterizing disjoint equilibria

I begin by outlining how a disjoint equilibrium is characterized. Details are in the proof of Proposition 4, below.

Let $\widehat{v}_3$ denote bidder 3's lowest active type, i.e. the type that bids exactly $m_2$. Let $\widehat{v}_1$ denote bidder 1's highest type that bids $m_2$. Thus, if $v > \widehat{v}_i$, bidder $i$ bids above $m_2$, $i = 1, 3$. As before, let $\widetilde{v}_1$ denote bidder 1's lowest type that bids $m_2$. In other words, bidder 1 bids below $m_2$ if his type is below $\widetilde{v}_1$. He bids precisely $m_2$ if his type is in the interval $[\widetilde{v}_1, \widehat{v}_1]$.

For bids above $m_2$, the interaction between bidder 1 and bidder 3 is as described in Section 3.1. In particular, the two bidders must share the same maximal bid, $\bar{b}$. A tying function, $k_3(v)$, can then be obtained from the first order conditions and the boundary condition $k_3(\overline{v}) = \overline{v}$. Given the tying function, $\widehat{v}_1$ and $\widehat{v}_3 = k_3(\widehat{v}_1)$ are derived from the fact that bidder 3 with type $\widehat{v}_3$ must be indifferent between not participating and bidding $m_2$. Finally, bidding strategies for bidder $i$ with type above $\widehat{v}_i$, $i = 1, 3$, can be derived by "integrating up" the first order conditions.

For bids below $m_2$, bidders 1 and 2 must realize that their bids can be successful only if bidder 3's type is below $\widehat{v}_3$, which occurs with probability $F(\widehat{v}_3)$. Thus, the analysis in Section 3.2 applies when $p$ is set equal to $F(\widehat{v}_3)$. In particular, for given $\widehat{v}_3$, a candidate for $\widetilde{v}_1$ can be derived by following the procedure described there. It is necessary to check that $\widetilde{v}_1 \leq \widehat{v}_1$, as a disjoint equilibrium does not exist otherwise.

[14]Bidder 2 and bidder 3 must bid $m_2$ with probability zero. Otherwise, bidder 1 would have an incentive to bid slightly above $m_2$ (to experience a jump in his winning probability) when he is supposed to bid slightly below $m_2$. However, bidder 1 can bid $m_2$ with positive probability, because it is impossible for bidder 2 to bid above $m_2$. Given the tie-breaking rule, there is therefore no incentive for bidder 2 to jump from a bid below $m_2$ to a bid of $m_2$.

Since bidding strategies follow directly from the first order conditions once $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_3)$ is known, the triplet $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_3)$ completely characterizes the disjoint equilibrium. The following Proposition proves the existence and uniqueness of a disjoint equilibrium when $m_2$ is not too small (the constraint that $\widetilde{v}_1 \leq \widehat{v}_1$ may be violated if $m_2$ is small). Bidders 2 and 3, the weaker bidders, are relatively unsuccessful in such an equilibrium.

**Proposition 4** *Assume that $\overline{m} < B$. Then, there exists some $\underline{m} \in (0, \overline{m})$, such that a unique disjoint equilibrium exists if $m_2 \in (\underline{m}, \overline{m})$.*

**Proof.** See the Appendix. ■

**Corollary 1** *In a disjoint equilibrium, bidder 1 is the only bidder to participate with probability one. Ex ante, bidder 1 wins more often than bidders 2 and 3 combined.*

**Proof.** See the Appendix. ■

## 4.2 Comparative statics

In a two-bidder auction, Section 3.3 reveals that bidder 2 may benefit if the prize is known to be withheld with some probability. In a disjoint equilibrium of a three-bidder auction, the role of bidder 3 is precisely to make low bids less likely to be successful. Hence, bidder 2 may be made better off when bidder 3 is invited to join the auction. In other words, bidder 2 may benefit from what appears to be increased competition.

**Corollary 2** *Assume $m_2 \in (\underline{m}, \overline{m})$. Then, bidder 2 wins at least as often and is weakly better off, regardless of his type, in the three-bidder disjoint equilibrium than in the two-bidder all-pay auction.*

**Proof.** In either setting, the cap is binding for bidder 2. Hence, the analysis in Sections 3.2 and 3.3 applies. The case in which bidder 3 is absent corresponds to $p = 1$. When bidder 3 is present, $p = F(\widehat{v}_3) < 1$. The corollary then follows from Lemma 1 and Proposition 3. In fact, all participating types are strictly better off when bidder 3 is present. ■

A related, and stronger, point comes from considering changes in bidder 3's costs, $\alpha_3$, when $\alpha_1, \alpha_2$, and $m_2$ are fixed. Assume $m_2 < B$. Let $\overline{\alpha}_3$ solve $\overline{v} - \overline{\alpha}_3 m_2 = 0$. If $\alpha_3 = \overline{\alpha}_3$ a trivial disjoint equilibrium exists in which $\widehat{v}_1 = \widehat{v}_3 = \overline{v}$, which corresponds to bidder 3 participating with probability zero. However, as in the proof of Proposition 4, it can be shown that a unique disjoint equilibrium, with $\widehat{v}_3 < \overline{v}$, exists if $\alpha_3$ is not too far below $\overline{\alpha}_3$. Let $\underline{\alpha}_3$ denote the smallest value of $\alpha_3$ for which a disjoint equilibrium exists. In this case, bidder 2 benefits from decreases in $\alpha_3$, $\alpha_3 \in (\underline{\alpha}_3, \overline{\alpha}_3]$, because bidder 3 participates with higher probability the lower his costs are ($\widehat{v}_3$, and therefore $p = F(\widehat{v}_3)$, decreases when $\alpha_3$ decreases). That is, bidder 2 is better off the more competitive bidder 3 is, as long as a disjoint equilibrium exists.

**Corollary 3** *Assume $m_2 < B$ and $\alpha_3 \in (\underline{\alpha}_3, \overline{\alpha}_3]$. Then, regardless of his type, bidder 2's winning probability and expected payoff is weakly decreasing in $\alpha_3$, $\alpha_3 \in (\underline{\alpha}_3, \overline{\alpha}_3]$, in the three-bidder disjoint equilibrium.*

**Proof.** When $\alpha_3$ increases, it is evident from (4) that $k_3(v)$ increases for all $v \in [\underline{v}, \overline{v})$. Since $\widehat{v}_3$ must be indifferent between staying out and bidding $m_2$,

$$\widehat{v}_3 F(\widehat{v}_1) - \alpha_3 m_2 = 0.$$

Since the last term increases when $\alpha_3$ increases, $\widehat{v}_3 F(\widehat{v}_1)$ must increase as well. I will now argue that $\widehat{v}_3 = k_3(\widehat{v}_1)$ must increase. If it decreases, then $\widehat{v}_1$ must necessarily decrease as well, since $k_3(v)$ increases for given $v$ when $\alpha_3$ increases. However, that leads to the contradiction that $\widehat{v}_3 F(\widehat{v}_1)$ decreases, since both terms decrease. Since $\widehat{v}_3$ and thus $p = F(\widehat{v}_3)$ increase in $\alpha_3$, the corollary follows from Lemma 1 and Proposition 3. ■

Changes in $\alpha_3$ could come about as a consequence of preferential treatment. Assume, for instance, that bidder 3 wins whenever $b_3 \geq \gamma \max\{b_1, b_2\}$, with $\gamma \in (0, 1)$. Then, he need only incur cost $\alpha_3' \max\{b_1, b_2\}$ to win, where $\alpha_3' = \alpha_3 \gamma < \alpha_3$. Corollary 3 then signifies that bidder 2 may *benefit* from preferential treatment of bidder 3.[15] Put differently, bidder 2 could be hurt when bidder 3 is handicapped ($\gamma > 1$). Since the latter could arguably be interpreted as preferential treatment of bidders 1 and 2, this points to a situation where bidder 2 is worse off when he is part of a diverse group of bidders that is being favoured. Kirkegaard (2011) examines a similar point in a more general model with continuous but non-linear costs. In particular, he shows that a bidder may benefit when he is a member of a diverse group of bidders that is being handicapped.

## 4.3 Extensions: More caps, more bidders

It has been assumed that only one bidder has a (binding) cap. However, it is also sometimes possible to characterize an equilibrium when both bidders 2 and 3 have binding caps. Specifically, assume bidder 3's cap is so high that it is just shy of the maximum bid in the game where he is unconstrained, with $m_3 > m_2$. Starting from the disjoint equilibrium described

[15] It can be verified that bidder 1 is worse off. Likewise, it is easy to check that if bidder 1 *or* bidder 2 is given preferential treatment (as captured by a decrease in $\alpha_1$ or a decrease in $\alpha_2$ and an increase in $m_2$, respectively) then the remaining bidders are adversely affected.

above, bidder 1 now need only bid $m_3$ to secure a win once bidder 3 is capped. Thus, for bids between $m_2$ and $m_3$, where only bidders 1 and 3 are active, equilibrium strategies would be described as in Section 3.2. Although this is likely to change $(\widehat{v}_1, \widehat{v}_3)$, equilibrium strategies for bids between 0 and $m_2$ can nevertheless be derived in a manner similar to that described in Section 4.1.

In the resulting equilibrium, bidder 2 bids in the interval $[0, m_2]$ (when active), while bidder 3 bids in the interval $[m_2, m_3]$. Note that bidder 1 bids $m_2$ and $m_3$ with positive probability. Assume now that there is a fourth bidder, with higher marginal costs and a higher cap as well, $\alpha_4 > \alpha_3$ and $m_4 > m_3$. Depending on the parameter values, a disjoint equilibrium may now exist in which bidder 4 bids in the interval $[m_3, m_4]$ (when active). More and more bidders, with higher marginal costs and higher caps, can be added to the game in this manner. The resulting equilibrium uses the logic on Section 4.1 to "piece together" the kind of two-player auction described in Section 3.2.

# 5 Conclusion

The analysis of auctions with more than two heterogenous bidders is generally quite difficult. In the case of all-pay auctions, a particular complication is that bidding strategies may be discontinuous. In this paper, I present an environment with multiple heterogenous bidders, in which equilibrium can in fact be characterized.

One of the advantages of being able to characterize equilibrium is that comparative statics are made easier. In this regard, the main result in the current paper is that a weak bidder may be better off with what appears to be more intense competition. More concretely, the bidder may become better off when an additional, weaker, bidder joins the competition.

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# Appendix: Proofs

**Proof of Lemma 1.** Consider bidder $i$ first. When $pB < m_i$, a small increase in $p$ does not affect $k_i^{-1}(v)$ or $F(k_i^{-1}(v))$. Since bidder $i$ wins with probability $pF(k_i^{-1}(v))$ when his type, $v$, exceeds $k_i(\underline{v})$, all participating types wins with strictly higher probability when $p$ increases and $pB < m_i$. Hence, bidder $i$ wins more often ex ante.

Next, assume $pB > m_i$, in which case $\widetilde{v}_j < \overline{v}$. As explained in the text, an increase in $p$ leads to lower $\widetilde{v}_j$ and thus a higher value of $k_i(\underline{v}|\widetilde{v}_j)$. Thus, bidder $i$ stays out for more types. Consider now the types that participate. As a preliminary step, note, from (8), that

$$\frac{\partial k_i(v|\widetilde{v}_j)}{\partial \widetilde{v}_j} = -\frac{\alpha_j}{\alpha_i}\frac{k_i(v|\widetilde{v}_j)}{f(k_i(v|\widetilde{v}_j))}\frac{f(\widetilde{v}_j)}{\widetilde{v}_j} < 0 \tag{13}$$

for any $v \in [\underline{v}, \overline{v}]$. Evaluating (8) at $v = k^{-1}(z|\widetilde{v}_j)$ yields

$$\alpha_i \int_z^{\overline{v}} \frac{f(x)}{x} dx = \alpha_j \int_{k^{-1}(z|\widetilde{v}_j)}^{\widetilde{v}_j} \frac{f(x)}{x} dx$$

for $z \in [k_i(\underline{v}|\widetilde{v}_j), \overline{v}]$. Hence,

$$\frac{\partial k_i^{-1}(v|\widetilde{v}_j)}{\partial \widetilde{v}_j} = \frac{f(\widetilde{v}_j)}{\widetilde{v}_j}\frac{k_i^{-1}(v|\widetilde{v}_j)}{f(k_i^{-1}(v|\widetilde{v}_j))} > 0. \tag{14}$$

Now, (11) must hold for all $p$ such that $pB > m_i$. Thus, (11) can be used to infer how $\widetilde{v}_j$ changes when $p$ changes. Note that the derivative of the right hand side of (11) with respect to $\widetilde{v}_j$ involves both (13) and (14). Using these, (11) yields

$$\frac{d\widetilde{v}_j}{dp} = -\left(p\frac{f(\widetilde{v}_j)}{\widetilde{v}_j}\right)^{-1}\left(\underline{v}\frac{\alpha_j}{\alpha_i}k_i(\underline{v}|\widetilde{v}_j) + \int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} \frac{k_i^{-1}(x|\widetilde{v}_j)}{f(k_i^{-1}(x|\widetilde{v}_j))} f(x)dx\right)^{-1}\left(\int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} k_i^{-1}(x|\widetilde{v}_j)f(x)dx\right).$$

Bidder $i$'s winning probability, assuming he participates, is $pF(k_i^{-1}(v|\widetilde{v}_j))$, the derivative of which is

$$\begin{aligned}\frac{\partial pF(k_i^{-1}(v|\widetilde{v}_j))}{\partial p} &= F(k_i^{-1}(v|\widetilde{v}_j)) + pf(k_i^{-1}(v|\widetilde{v}_j))\frac{\partial k_i^{-1}(v|\widetilde{v}_j)}{\partial \widetilde{v}_j}\frac{d\widetilde{v}_j}{dp} \\ &= k_i^{-1}(v|\widetilde{v}_j)\left(\frac{F(k_i^{-1}(v|\widetilde{v}_j))}{k_i^{-1}(v|\widetilde{v}_j)} + p\frac{f(\widetilde{v}_j)}{\widetilde{v}_j}\frac{d\widetilde{v}_j}{dp}\right),\end{aligned}$$

the sign of which is determined by the term in the parenthesis. The aim is to show this is negative. The last term in the parenthesis is negative and independent of $v$. The first term, however, is increasing in $v$, by the assumption that $F(v)/v$ is increasing. Hence, the sum is negative for all $v \in [k_i(\underline{v}|\widetilde{v}_j), \overline{v}]$ if it is negative at $v = \overline{v}$. Since $k_i^{-1}(\overline{v}|\widetilde{v}_j) = \widetilde{v}_j$,

$$\begin{aligned}
\frac{\partial pF(k_i^{-1}(\overline{v}|\widetilde{v}_j))}{\partial p} &= F(\widetilde{v}_j) + pf(\widetilde{v}_j)\frac{d\widetilde{v}_j}{dp} \\
&\propto \frac{1}{\widetilde{v}_j}\left(\underline{v}\frac{\alpha_j}{\alpha_i}k_i(\underline{v}|\widetilde{v}_j) + \int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} \frac{k_i^{-1}(x|\widetilde{v}_j)}{f(k_i^{-1}(x|\widetilde{v}_j))}f(x)dx\right) \\
&\quad -\frac{1}{F(\widetilde{v}_j)}\left(\int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} k_i^{-1}(x|\widetilde{v}_j)f(x)dx\right).
\end{aligned}$$

To evaluate the above, note that the two terms, by letting $z = k_i^{-1}(x|\widetilde{v}_j)$, can be written

$$\int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} \frac{k_i^{-1}(x|\widetilde{v}_j)}{f(k_i^{-1}(x|\widetilde{v}_j))}f(x)dx = \int_{\underline{v}}^{\widetilde{v}_j} \frac{z}{f(z)}f(k_i(z|\widetilde{v}_j)k_i'(z|\widetilde{v}_j)dz = \int_{\underline{v}}^{\widetilde{v}_j} \frac{\alpha_j}{\alpha_i}k_i(z|\widetilde{v}_j)dz$$

and

$$\int_{k_i(\underline{v}|\widetilde{v}_j)}^{\overline{v}} k_i^{-1}(x|\widetilde{v}_j)f(x)dx = \int_{\underline{v}}^{\widetilde{v}_j} zf(k_i(z|\widetilde{v}_j))k_i'(z|\widetilde{v}_j)dz = \int_{\underline{v}}^{\widetilde{v}_j} \frac{\alpha_j}{\alpha_i}k_i(z|\widetilde{v}_j)f(z)dz \qquad (15)$$

since, from e.g. (8),

$$k_i'(z|\widetilde{v}_j) = \frac{\alpha_j}{\alpha_i}\frac{k_i(z|\widetilde{v}_j)}{f(k_i(z|\widetilde{v}_j))}\frac{f(z)}{z}.$$

Note that (15) confirms that bidder $i$ with type $\overline{v}$ bids the same as bidder $j$ with type $\widetilde{v}_j$. With these in hand,

$$\begin{aligned}
\frac{\partial pF(k_i^{-1}(\overline{v}|\widetilde{v}_j))}{\partial p} &\propto \frac{1}{\widetilde{v}_j}\left(\underline{v}\frac{\alpha_j}{\alpha_i}k_i(\underline{v}|\widetilde{v}_j) + \int_{\underline{v}}^{\widetilde{v}_j} \frac{\alpha_j}{\alpha_i}k_i(z|\widetilde{v}_j)dz\right) - \frac{1}{F(\widetilde{v}_j)}\left(\int_{\underline{v}}^{\widetilde{v}_j} \frac{\alpha_j}{\alpha_i}k_i(z|\widetilde{v}_j)f(z)dz\right) \\
&\propto \frac{1}{\widetilde{v}_j}\left(\underline{v}k_i(\underline{v}|\widetilde{v}_j) + \int_{\underline{v}}^{\widetilde{v}_j} k_i(z|\widetilde{v}_j)dz\right) - \frac{1}{F(\widetilde{v}_j)}\left(F(\underline{v})k_i(\underline{v}|\widetilde{v}_j) + \int_{\underline{v}}^{\widetilde{v}_j} k_i(z|\widetilde{v}_j)f(z)dz\right),
\end{aligned}$$

since $F(\underline{v}) = 0$. Define $H(v) = \frac{v}{\overline{v}}$, $v \in [0, \overline{v}]$, the uniform distribution over $[0, \overline{v}]$, and let $h(v)$ denote its density. Then,

$$\frac{\partial pF(k_i^{-1}(\overline{v}|\widetilde{v}_j))}{\partial p} \propto \left(\frac{H(\underline{v})}{H(\widetilde{v}_j)}k_i(\underline{v}|\widetilde{v}_j) + \int_{\underline{v}}^{\widetilde{v}_j} k_i(z|\widetilde{v}_j)\frac{h(z)}{H(\widetilde{v}_j)}dz\right) - \left(\frac{F(\underline{v})}{F(\widetilde{v}_j)}k_i(\underline{v}|\widetilde{v}_j) + \int_{\underline{v}}^{\widetilde{v}_j} k_i(z|\widetilde{v}_j)\frac{f(z)}{F(\widetilde{v}_j)}dz\right).$$

The first term is the expectation of the non-decreasing function $k_i(\max\{\underline{v}, z\}|\widetilde{v}_j)$ when $z$ is drawn from the (truncated) distribution $\frac{H(z)}{H(\widetilde{v}_j)} = \frac{z}{\widetilde{v}_j}$. The second term is the expectation when $z$ is drawn from $\frac{F(z)}{F(\widetilde{v}_j)}$. Since $z \leq \widetilde{v}_j$, $\frac{F(z)}{z} \leq \frac{F(\widetilde{v}_j)}{\widetilde{v}_j}$, by assumption. Hence,

$$\frac{F(z)}{F(\widetilde{v}_j)} \leq \frac{z}{\widetilde{v}_j} = \frac{H(z)}{H(\widetilde{v}_j)}.$$

In words, the distribution $\frac{F(z)}{F(\widetilde{v}_j)}$ first order stochastically dominates the distribution $\frac{H(z)}{H(\widetilde{v}_j)}$. Thus, the expectation of $k_i(\max\{\underline{v}, z\}|\widetilde{v}_j)$ is higher when $z$ is drawn from the former rather than the latter. It follows that

$$\frac{\partial pF(k_i^{-1}(\overline{v}|\widetilde{v}_j))}{\partial p} < 0,$$

as I wanted to prove.

Finally, consider bidder $j$. When $pB < m_i$, bidder $j$'s probability of outbidding bidder $i$ is strictly positive and independent of $p$. Thus, an increase in $p$ increases $pF(k_i(v))$. When $pB > m_i$, both terms increase when $p$ increases. ■

**Proof of Proposition 4.** The proof is in three steps.

Step 1 (Constructing an equilibrium candidate): For bids above $m$, where only bidders 1 and 3 are active, the approach in Section 3.1 reveals that $k_3(v)$ is implicitly given by

$$\alpha_3 \int_{k_3(v)}^{\overline{v}} \frac{f(x)}{x} dx = \alpha_1 \int_{v}^{\overline{v}} \frac{f(x)}{x} dx. \tag{16}$$

As before, once $k_3(v)$ has been derived, bidding strategies are characterized by the first order conditions. Since $b_1(\widehat{v}_1) = m_2$, by assumption,

$$b_1(v) = m_2 + \int_{\widehat{v}_1}^{v} \frac{k_3(x)}{\alpha_3} f(x) dx,\ v \in [\widehat{v}_1, \overline{v}]\,. \tag{17}$$

Bidder 3's bid can be derived in a similar manner, or directly from $b_3(v) = b_1(k_3^{-1}(v))$.

To find $\widehat{v}_1$ it is useful to reconsider bidder 3's problem. In a disjoint equilibrium, bidder 3 is supposed to stay out if his type is below $\widehat{v}_3$, and then jump to a bid of $m_2$ if his type is $\widehat{v}_3$. Consequently, bidder 3 must be indifferent between the two actions if his type is $\widehat{v}_3$, or

$$\widehat{v}_3 F(\widehat{v}_1) - \alpha_3 m_2 = 0, \tag{18}$$

since a bid of $m_2$ wins the auction if bidder 1's type is below $\widehat{v}_1$ (given the tie-breaking rule). For the first order conditions to be satisfied it must hold that $\widehat{v}_3 = k_3(\widehat{v}_1)$, or

$$k_3(\widehat{v}_1) F(\widehat{v}_1) - \alpha_3 m_2 = 0, \tag{19}$$

which has a unique solution. Thus, $\widehat{v}_1$ and $\widehat{v}_3$ have been identified, as have strategies for bidder $i$ with type $v \in [\widehat{v}_i, \overline{v}]$, $i = 1, 3$.

Consider now bids below $m_2$. For bidder 1, such a bid wins only if it beats bidder 2 and bidder 3 stays out. The situation is similar for bidder 2. Hence, everything is as in Section 3.2, with $j = 1$, $i = 2$, $p = F(\widehat{v}_3)$, where the latter is unique. Thus, the approach in Section 3.2 identifies at most one $\widetilde{v}_1$ candidate.

Bidding strategies for those types that bid below $m_2$ is then given by (9) and (10). However, it is not guaranteed that a $\widetilde{v}_1$ candidate exists at all. The reason is that if $p = F(\widehat{v}_3)$ is small, then there may be no $\widetilde{v}_1 \in [\underline{v}, \overline{v}]$ that satisfies (11). In this case, $p$ is so small that bidder 2's financial constraint is not binding. Moreover, the equilibrium construct assumes that $\widetilde{v}_1$ is bounded above by $\widehat{v}_1$, or $\widetilde{v}_1 \leq \widehat{v}_1$. Thus, the existence of a disjoint equilibrium is not guaranteed.

Step 2 (Existence): Recall that any $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_2)$ triplet completely characterizes an equilibrium candidate, since bidding strategies immediately follow from (5), (6) (for bids above $m_2$) and (9), (10) (for bids below $m_2$). In Section 3.2, the strategy of bidder $i = 2$ was written $b_2(v|\widetilde{v}_1, p)$. Since $p = F(\widehat{v}_3) = F(k_3(\widehat{v}_1))$, I will, with some abuse of notation, now write it as $b_2(v|\widetilde{v}_1, \widehat{v}_1)$. Then, $\widehat{v}_3$ is derived from $\widehat{v}_3 = k_3(\widehat{v}_1)$, while the pair $(\widetilde{v}_1, \widehat{v}_1)$ is derived from the pair of equations

$$\frac{k_3(\widehat{v}_1)F(\widehat{v}_1)}{\alpha_3} = m_2 \quad (20)$$

$$b_2(\overline{v}|\widetilde{v}_1, \widehat{v}_1) = m_2. \quad (21)$$

To form a feasible equilibrium candidate, it is required that $\widetilde{v}_1 \leq \widehat{v}_1 < \overline{v}$. Note that if $m_2 = \overline{m}$ then (20) is satisfied at $\widehat{v}_1 = \overline{v}$ and if $m_2 < \overline{m}$ it is satisfied at some $\widehat{v}_1 < \overline{v}$. Second, $B = b_2(\overline{v}|\overline{v}, \overline{v}) > \overline{m}$, by assumption. Since $b_2(\overline{v}|\widetilde{v}_1, \widehat{v}_1)$ is strictly increasing in $\widetilde{v}_1$ and is zero at $\widetilde{v}_1 = \underline{v}$, there is some $\widetilde{v}_1 \in (\underline{v}, \overline{v})$ for which $b_2(\overline{v}|\widetilde{v}_1, \overline{v}) = \overline{m}$, satisfying (21). Thus, both conditions are satisfied at $m_2 = \overline{m}$, by some $\widetilde{v}_1 < \widehat{v}_1 = \overline{v}$. It is also the case that $k_3(\widehat{v}_1)F(\widehat{v}_1)$ and $b_2(\overline{v}|\widetilde{v}_1, \widehat{v}_1)$ are continuous and increasing in $\widehat{v}_1$ and $\widetilde{v}_1$. Thus, by continuity and monotonicity, all conditions remain satisfied if $m_2$ is reduced slightly.

Finally, to prove that $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_2)$ characterizes a disjoint equilibrium, profitable deviations must be ruled out. First, local or small deviations can be ruled out for bidder 2 and bidder 3, as well as for bidder 1 with types outside the interval $[\widetilde{v}_1, \widehat{v}_1]$. The reason is that strategies are derived from first order conditions (and these conditions are sufficient, as explained in an earlier footnote). Large deviations must then be ruled out. Bidder 2 is unable to jump to bids in excess of $m_2$, and the tie-breaking rule ensures he has no incentive to jump to $m_2$, given the first order condition specifies what the optimal bid in the range $[0, m_2]$ is. Bidder 3 has no incentive to pick bids in the range $(0, m_2)$, due to the convexity of his payoff function in that range. The tie-breaking rule guarantees that bidder 3 wins if he is in a tie at bid $m_2$, so he has no incentive to jump from such a bid when his equilibrium strategy dictates that he bids $m_2$. Since the single-crossing condition is satisfied and type $\widehat{v}_3$ earns zero payoff, there is no incentive to change the entry decision for types below or above $\widehat{v}_3$. Consider now bidder 1. For him, the probability of a tie at a bid of $m_2$ is zero, so there is no jump in payoff from bidding marginally higher than $m_2$. The first order conditions then ensure that bidder 1 with type outside $[\widetilde{v}_1, \widehat{v}_1]$ is playing a best response. The single-crossing condition then implies that types between $\widetilde{v}_1$ and $\widehat{v}_1$ must be maximizing by bidding exactly $m_2$, as required. In summary, there is no incentive to deviate.

Step 3 (Uniqueness): Assuming existence, $\widehat{v}_1$ is unique because the left hand side of (20) is strictly increasing in $\widehat{v}_1$. It follows that $\widehat{v}_3 = k_3(\widehat{v}_1)$ is unique as well. Finally, for this (and any other) fixed value of $\widehat{v}_1$, the left hand side of (21) is strictly increasing in $\widetilde{v}_1$, which means that $\widetilde{v}_1$ is unique as well. ■

**Proof of Corollary 1.** At least one bidder must submit strictly positive bids with probability one, or there would be an incentive to submit a small bid for someone who is supposed to bid zero or stay out. By construction, bidder 3 stays out bids with probability $F(\widehat{v}_3) > 0$ and bidder 2 with probability $F(k_2(\underline{v}|\widetilde{v}_1)) > 0$. Hence, bidder one is the only bidder to submit strictly positive bids with probability one.

For the second part, it is easily verified that $(i)$ $k_2(v|\widetilde{v}_1) > v$, $v \in [\underline{v}, \widetilde{v}_1)$, $(ii)$ $k_2(v|\widetilde{v}_1)$ is decreasing in $\widetilde{v}_1$, and $(iii)$ $k_3(v) > k_2(v|\overline{v}) > v$, $v \in [\underline{v}, \overline{v})$. Thus, bidder 1 wins with probability $F(k_3(v)) > F(v)$ when his type is $v \in [\widehat{v}_1, \overline{v})$, and probability $F(k_3(\widehat{v}_1)) > F(\widehat{v}_1) > F(v)$ when his type is $v \in [\widetilde{v}_1, \widehat{v}_1)$. When $v \in [\underline{v}, \widetilde{v}_1)$, his winning probability is $F(\widehat{v}_3)F(k_2(v|\widetilde{v}_1))$, which is no smaller than $F(k_3(\widehat{v}_1))F(k_2(v|\widehat{v}_1))$. Viewing the last expression as a function of $\widehat{v}_1$, observation $(ii)$ and the assumption that $F(v)/v$ is strictly increasing in $v$ can be used to show that it is single-peaked in $\widehat{v}_1$, for a fixed $v$. Thus, the expression is minimized when $\widehat{v}_1 = \overline{v}$ or $\widehat{v}_1 = v$. At $\widehat{v}_1 = \overline{v}$, $F(k_3(\widehat{v}_1))F(k_2(v|\widehat{v}_1)) = F(k_2(v|\overline{v})) > F(v)$, while at $\widehat{v}_1 = v$, $F(k_3(\widehat{v}_1))F(k_2(v|\widehat{v}_1)) = F(k_3(v)) > F(v)$. In summary, it has been shown that bidder 1's winning probability strictly exceeds $F(v)$ for all $v \in [\underline{v}, \overline{v})$. Hence, his ex ante winning probability is strictly larger than

$$\int_{\underline{v}}^{\overline{v}} F(v)f(v)dv = \frac{1}{2},$$

which proves the last part of the Corollary. ■