Handicaps in Incomplete Information All-Pay Auctions with a Diverse Set of Bidders*

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December 2011

Abstract

In many contests, a subset of contestants is granted preferential treatment which is presumably intended to be advantageous. Examples include affirmative action and biased procurement policies. In this paper, however, I show that some of the supposed beneficiaries may in fact become worse off when the favored group is diverse. The reason is that the other favored contestants become more aggressive, which may outweigh the advantage that is gained over contestants who are handicapped. The contest is modelled as an incomplete-information all-pay auction in which contestants have heterogenous and non-linear cost functions. A source of uncertainty, such as incomplete information, is crucial for the results.

JEL Classification Numbers: C72, D44, D82.

Keywords: All-Pay Auctions, Asymmetric Auctions, Contests, Handicaps, Preferential Treatment.

*I would like to thank J. Atsu Amegashie and Mike Hoy for comments on an earlier draft. I would also like to thank the Social Sciences and Humanities Research Council and the Canada Research Chairs program for funding this research. An earlier version of the paper circulated under the title "Preferential Treatment may Hurt: Another Application of the All-Pay Auction".

1 Introduction

There are many examples of contests in which a subset of contestants receive either "preferential treatment" or a "handicap". On the labor market, affirmative action may influence which job applicant wins the prize, in this case the job. The same is true in the contest to win admission into university. Internal applicants are sometimes given preference over external applicants when a firm seeks to fill a senior position. In public procurement, domestic firms may be given preferential treatment over foreign firms, and so on.

In all these examples it is a *diverse* group of contestants who are favored. Affirmative action applies to individuals with different backgrounds, internal applicants for senior positions are likely to be heterogeneous, and domestic firms may have different technologies. Another feature of the examples is that the prize is not awarded based on the identities of the contestants alone, but also on the qualifications of the contestants in question. The investment in these qualifications – obtaining an education before applying for a job, preparing for the SAT, working hard to prove one's worth to the company, or building up expertise prior to seeking a procurement contract – may entail very significant costs. Importantly, the size of this investment is endogenous; it is likely to depend on the perceived strength of the competition and on whether the contestant is given preferential treatment. Since a given contestant may not have complete information regarding the skills, costs, or preferences of his rivals, asymmetric information may also play a role in determining the magnitude of a contestant's investment.

The objective of this paper is to study the consequences of preferential treatment in contests that are characterized by *within-group diversity* and *incomplete information*. I will show that the combination of these realistic features may, somewhat perversely, produce outcomes that are arguably opposite of what intuition would suggest. Specifically, the main result is that if the group of contestants who is given preferential treatment (modelled as a handicap to their common rival) is diverse, a subset of them may participate less often, win less often, and overall be worse off when preferential treatment is introduced. Thus, the current paper serves as a note of caution; rather than "leveling the playing field", preferential treatment may, in principle, increase the severity of the problems or inequalities it perhaps intends to minimize.

The contest is modelled as a deterministic contest or, more formally, an all-pay auction. Thus, the paper is related to the literature on auctions with heterogenous participants. However, for technical reasons, most papers that compare different auctions or the consequences of changes to the auction design assume there are exactly two heterogeneous bidders.[1]

[1]Recent prominent examples include Maskin and Riley's (2000) seminal comparison of first-price and

Clearly, since the purpose of this paper is to consider a setting with two groups, at least one of which is diverse, a model with only two bidders is not adequate.[2] Recently, Parreiras and Rubinchik (2010) have examined all-pay auctions with more bidders. They point out that the addition of more bidders gives rise to several technical complications that are absent in two-bidder models. For instance, strategies may be discontinuous. However, Parreiras and Rubinchik (2010) do not examine the consequences of changes to the contest design.[3]

In this paper, I take a first step toward a more general analysis of comparative statics in all-pay auctions with more than two bidders. However, the set-up of the problem is engineered to minimize the complications from having several bidders and instead maximizing the use of insights from two-bidder auctions. In particular, the reaction to preferential treatment in an auction with two bidders is used to infer the main result.

Consider a contest with a "strong", a "weak", and a "very weak" bidder, and assume the strong bidder is handicapped. As a consequence, the two weaker bidders have less to fear from the strong bidder. However, that does not necessarily mean that they will work less hard. In fact, the "weak" bidder may push his newfound advantage by investing more aggressively. From the point of view of the "very weak" bidder, one rival has become less of a threat, but the other more of a threat. I show, under mild assumptions, that the very weak bidder would be less likely to win the prize with a small bid when the strong bidder is handicapped. The second step is to show that there are cost structures for which a monotonic equilibrium exists in which the very weak bidder wins the prize with probability zero and earns zero payoff, but that such an equilibrium does not exist without the handicap. Conditions under which the former is the unique equilibrium are also presented. The cost function of the very weak bidder must have the right amount of "curvature", not too much and not too little. Moreover, the cost functions of the weak and very weak bidders are generally not ordered (they cross).

It is interesting to contrast the results of the current paper with the literature on handicaps in complete information all-pay auctions with heterogeneous bidders. Baye et al (1993) show that it may be profitable to exclude the strongest bidders (which can be viewed as an extreme form of handicapping). Gale and Stegeman (1994) show that it is even more

second-price auctions, Hafalir and Krishna's (2008, 2009) analysis of such auctions with resale, and Hörner and Sahuguet's (2007) analysis of jump bidding.

[2]There is a small literature on handicaps in all-pay auctions with private information and precisely two bidders. Lien (1990) and Feess et al (2008) assume bidders are homogeneous and prove that the handicapped bidder wins less often than is efficient. Clark and Riis (2000) and Kirkegaard (2011a) consider the revenue effects of various forms of preferential treatment in all-pay auctions with two heterogeneous bidders.

[3]Hickman (2011) considers all-pay auctions with incomplete information and bidders who belong to one of two different groups. His model is made tractable by the assumption that there is a continuum of bidders and prizes Aided by these assumptions, Hickman (2011) analyzes different types of preferential treatment. In contrast, there is a finite number of bidders in the current paper.

profitable to impose an "additive" handicap on the strongest bidder (to win, his bid must be the largest by a pre-specified amount). Fu (2006) shows that it is also profitable to impose a "relative" handicap on the strong bidder. In all these settings, the remaining bidders are never hurt by the intervention. In fact, it follows from Siegel's (2009) more general analysis that no bidder can be hurt in a complete information all-pay auction if a rival is handicapped, regardless of the nature of that handicap. Thus, the assumption of incomplete information is critical for the results in this paper. However, it is conceivable that other sources of uncertainty may have similar effects. Following Tullock (1980), complete information contests are often modeled as non-deterministic; random factors mean a contestant may win even if he is outperformed by a rival. In such a setting, it also cannot be ruled out without further study that preferential treatment may be disadvantageous to a subset of the intended beneficiaries.

2 Model

Three risk neutral bidders compete in an all-pay auction. Each bidder has a privately known type, v, which captures how much he values winning the prize. Bidder i's type is distributed according to some strictly increasing and twice continuously differentiable distribution function, $F(v)$, with no mass points and support $[\underline{v}, \overline{v}]$, where $\overline{v} > \underline{v} > 0$, $i = 1, 2, 3$. Densities, denoted by f, are bounded above and below, away from zero. The value of not participating is zero.

Bidders can submit "bids", which depending on the context can be interpreted as either monetary bids or effort. However, in a biased auction the winner need not be the bidder with the highest bid. Instead, bids are converted into *scores*, which may be a function of the identity of the bidder, although it is always assumed that the cost of a zero score is zero. The bidder with the highest score wins (with ties being broken by the toss of a fair coin). Hence, it is useful to think of bidders as choosing scores rather than bids. As in Athey (2001, Section 4), the set of actions is then $\{out\} \cup \mathbb{R}_+$, where *out* refers to the possibility of staying out of the auction altogether. The cost for bidder i of scoring s is described by a twice continuously differentiable cost function, $c_i(s)$, with $c_i(0) = 0$, $0 < c_i'(\cdot) < \infty$, and $c_i''(\cdot) \geq 0$, $i = 1, 2, 3$. Bidder 1 is "stronger" than bidder 2 in the sense that $c_1'(s) < c_2'(s)$ for all $s \in \mathbb{R}_+$. Differences in cost functions may be due to some pre-existing discrimination or to differences in underlying bidding technologies.

The purpose of the paper is to examine changes in the rules of the game. Specifically, bidder 1 is handicapped, meaning that his bid is translated into a lesser score. Put differently, it becomes costlier to obtain the same score. Let $c_1^h(\cdot)$ denote bidder 1's cost function with the handicap in place, with $c_1^h(0) = 0 < c_1^{h\prime}(\cdot) < \infty$ and $c_1^{h\prime\prime}(\cdot) \geq 0$. The handicap is assumed to increase bidder 1's marginal costs but without making him weaker than bidder 2.[4]

Assumption A: The handicap increases bidder 1's marginal costs; $c_1^{h\prime}(s) > c_1'(s)$, $\forall s \in \mathbb{R}_+$.

Assumption B: The handicap either *diminishes* or *nullifies* bidder 1's advantage over bidder 2. That is, either *(B1)* $c_2'(s) > c_1^{h\prime}(s)$, $\forall s \in \mathbb{R}_+$, or *(B2)* $c_2'(s) = c_1^{h\prime}(s)$, $\forall s \in \mathbb{R}_+$.

At this point, no formal assumptions regarding c_3 are imposed. The reason is that c_3 will ultimately be constructed or reversed-engineered to establish the main point, namely that there are c_3 functions for which bidder 3 are worse off when bidder 1 is handicapped.

3 Analysis

In this paper, I restrict attention to equilibria in increasing strategies. That is, each bidder has a cut-off type below which he scores zero or stays out of the auction, and above which he enters the auction with a score that is strictly increasing in his type. It follows from Athey's (2001, Theorem 7) more general analysis that an equilibrium of this nature exists.

The analysis is initiated by considering two games, Γ_2 and Γ_2^h. In these games, bidder 3 is assumed absent. Then, the larger games in which bidder 3 is potentially active, Γ_3 and Γ_3^h, are examined. Here, the central question is whether bidder 3 would select to be active if bidders 1 and 2 continue to play the increasing strategies from Γ_2 and Γ_2^h, respectively.

3.1 Bidders 1 and 2; Γ_2 and Γ_2^h

Consider the game Γ_2^h, in which bidder 3 is not present. Under Assumption B2 the auction is symmetric and has a symmetric equilibrium. As for Assumption B1, note first that it cannot be the case that both bidders stay out of the auction with strictly positive probability. The reason is that it would pay to enter the auction with a very small score, in order to win with a non-trivial probability. Let $\varphi_i^h(s)$ denote bidder i's inverse bidding strategy among the set of types who participate, such that bidder i scores s if his type is φ_i^h, $i = 1, 2$.

Amann and Leininger (1996) and Lizzeri and Persico (2000) analyze two-bidder all-pay auctions. Amann and Leininger (1996) assume that the cost functions are identical (and linear) for the two bidders, but that types are drawn from different distribution functions.

[4] In the current model, handicapping bidder 1 is isomorphic to giving uniform preferential treatment to bidders 2 and 3. Assume that the cost for bidder i, $i = 2, 3$, of obtaining a score s declines from $c_i(s)$ to $c_i(t(s))$, where $0 \le t(s) \le s$. This amounts to a simple "change of variables" compared to the model formulated in terms of handicaps. In particular, to tie with a rival bidder who scores $t(s)$, bidder 1 has to incur the cost $c_1^h(s) = c_1(t^{-1}(s))$. The difference between the two models thus amounts to an inconsequential rescaling or renaming of scores.

However, their arguments also apply to the model in the current paper. Thus, equilibrium strategies have the properties identified in Amann and Leininger (1996, Lemmas 1-5). In particular, $F_1^h(s) \equiv F(\varphi_1^h(s))$ and $F_2^h(s) \equiv F(\varphi_2^h(s))$ have a common support of the form $[0, \overline{s}^h]$, where $\overline{s}^h$ is the common maximal score. Although a mass of types may stay out, for those types that participate $F_i^h(s)$ has no atoms, no gaps, and is strictly increasing.[5] Since strategies are monotonic, they are differentiable almost everywhere and ties occur with probability zero. Hence, upon entry, bidder 1 with type v seeks to maximize $vF(\varphi_2^h(s)) - c_1^h(s)$, where $F(\varphi_2^h(s))$ is the probability that he outscores bidder 2 (and wins) with a score of s. Thus, any interior solution to bidder 1's problem must satisfy the first order condition,

$$v\frac{dF(\varphi_2^h(s))}{ds} - c_1^{h\prime}(s) = 0.$$

The first order condition for bidder 2 is obtained in similar fashion.

In equilibrium, bidder i scores s if his type is $v = \varphi_i^h(s)$, $i = 1, 2$. Substituting these into the first order conditions gives the following pair of conditions:

$$\frac{dF(\varphi_1^h(s))}{ds} = \frac{c_2'(s)}{\varphi_2^h(s)}, \quad \frac{dF(\varphi_2^h(s))}{ds} = \frac{c_1^{h\prime}(s)}{\varphi_1^h(s)} \tag{1}$$

or

$$\varphi_1^{h\prime}(s) = \frac{c_2'(s)}{f(\varphi_1^h(s))\varphi_2^h(s)}, \quad \varphi_2^{h\prime}(s) = \frac{c_1^{h\prime}(s)}{f(\varphi_2^h(s))\varphi_1^h(s)}. \tag{2}$$

Since $\overline{s}^h$ is the common maximal bid the system of differential equations must satisfy the boundary condition $F(\varphi_1^h(\overline{s}^h)) = F(\varphi_2^h(\overline{s}^h)) = 1$. However, $\overline{s}^h$ is endogenous.

By assumption, the right hand side of the equations in (2) are continuously differentiable in φ_1^h, φ_2^h, and s. Likewise, the assumption that $\underline{v} > 0$ implies that $F(\varphi_i^h(s))$ has finite slope everywhere and that the system is Lipschitz. Thus, as in Lizzeri and Persico (2000, Section 3), for any given guess on the value of $\overline{s}^h$, the system in (1) or (2) takes a unique path as s approaches zero ("shooting backwards" from $\overline{s}^h$ to 0); see e.g. Hirsch and Smale (1974, Theorem 1, p. 297). Verifying whether the outcome is consistent with an equilibrium then helps to pinpoint the values of $\overline{s}^h$ that are equilibrium candidates.[6]

[5] In particular, no bidder scores or bids zero for a mass of types. Otherwise, the rival bidder with valuation $\underline{v} > 0$ should not score zero, but rather marginally above zero, in order to dramatically increase the probability of winning by ruling out a tie. It is for this reason that the action *out* is included in the specification of the game, as in Athey (2001, Section 4). In contrast, Amann and Leininger (1996) assume $\underline{v} = 0$, in which case the argument does not preclude a mass of types from scoring zero.

[6] Note that if φ_1^h and φ_2^h satisfy (2) then φ_i^h is increasing in s (the right hand side is strictly positive). Fixing v, the first derivative of bidder 1's payoff with respect to the score is $\left(v/\varphi_1^h(s) - 1\right) c_1^{h\prime}(s)$, which is positive when s is small (such that $\varphi_1^h(s) < v$) and negative when s is large (and $\varphi_1^h(s) > v$). Consequently, payoff is single peaked in s, and the first order conditions are sufficient if φ_1^h and φ_2^h satisfy (1).

To this end, note that since strategies are monotonic and at least one bidder participates with probability one, either $F(\varphi_1^h(0)) = 0$ or $F(\varphi_2^h(0)) = 0$ (or both). By using this condition and the requirement of a common support, $[0, \bar{s}^h]$, it will be shown that there is a unique equilibrium in increasing strategies (see Proposition 1, below). That is, there is one, and only one, value of $\bar{s}^h$ for which (1) produces an equilibrium of the game.

In the following, let φ_i denote the strategies and let $\bar{s}$ denote the maximum equilibrium score when there is no handicap. The differential equations for this case are analogous to (1). Given Assumptions A and B1, the strong bidder scores more aggressively than the weak bidder whether or not he is handicapped. Moreover, the weak bidder stays out of the auction with positive probability, whereas the strong bidder always participates. However, the strong bidder scores less aggressively when he is handicapped (although his bid may be higher). In response, the weak bidder becomes *more* aggressive, at least in the sense that he is now more likely to participate.

Proposition 1 (Equilibrium Properties) *There is a unique equilibrium in increasing strategies in Γ_2^h. Under Assumption B1, the weak bidder stays out with strictly positive probability and is more likely to submit low bids than the strong bidder, $F(\varphi_1^h(s)) < F(\varphi_2^h(s))$ for all $s \in [0, \bar{s}^h)$, with $F(\varphi_2^h(0)) > 0 = F(\varphi_1^h(0))$.*[7] *The same properties hold for Γ_2. Under Assumption B2, $F(\varphi_1^h(s)) = F(\varphi_2^h(s))$ for all $s \in [0, \bar{s}^h]$, with $F(\varphi_2^h(0)) = 0$.*

Proof. See the Appendix. ■

Proposition 2 (Comparative Statics) *The unique equilibrium in increasing strategies of Γ_2 compares with its counterpart in Γ_2^h as follows:*

1. *Scores are more compressed and the strong bidder scores less aggressively in Γ_2^h than in Γ_2: $\bar{s}^h < \bar{s}$, and $F(\varphi_1^h(s)) > F(\varphi_1(s))$ for all $s \in (0, \bar{s}^h]$.*

2. *The weak bidder participates more often in Γ_2^h than in Γ_2: $0 \leq F(\varphi_2^h(0)) < F(\varphi_2(0))$.*

Proof. See the Appendix. ■

Note that the strong bidder becomes less of a threat to the weak bidder when he is handicapped. For a fixed score, the weak bidder is more likely to win. Consequently, he is more likely to participate, and, if he participates, he is better off. Thus, depending on his type, v, he is either indifferent or strictly better off when he is given preferential treatment. Ex ante (before his type is known), he must therefore be strictly better off.

Corollary 1 *With just two bidders, bidder 2 is weakly better off regardless of his type if bidder 1 is handicapped, and strictly better off ex ante.*

[7] More precisely, the equilibrium is "essentially unique" because it does not matter whether bidder 2 with type $\varphi_2^h(0)$ scores zero or stays out.

3.2 Bidder 3; Γ_3 and Γ_3^h

Consider now bidder 3. If bidder 1 and bidder 2 compete as described above, does bidder 3 have an incentive to become active in the auction? If bidder 3 enters with a small bid after bidder 1 is handicapped, he is more likely to beat the strong bidder, but less likely to beat the weak bidder compared to the situation before the handicap. Of course, bidder 3 is concerned with outscoring both bidders, the probability of which is $q_3^h(s) \equiv F(\varphi_1^h(s))F(\varphi_2^h(s))$ when he submits a score of s. The incentive to enter the auction is strongest for bidder 3 if his type is $\overline{v}$, in which case he maximizes $\overline{v}q_3^h(s) - c_3(s)$. Thus, for bidder 3 to find entry profitable, there must be a score, $s > 0$, for which $\overline{v}q_3^h(s) \geq c_3(s)$.

For small scores, both $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ are steeper than $F(\varphi_1(s))$ and $F(\varphi_2(s))$, which perhaps suggests a greater return to submitting a small score for bidder 3. However, this is counteracted by the fact that bidder 2 is more likely to participate. Given (1), the derivative of $\overline{v}q_3^h(s) - c_3(s)$ at $s = 0$ is

$$\overline{v}q_3^{h\prime}(0) - c_3'(0) = \overline{v}\frac{F(\varphi_2^h(0))}{\varphi_2^h(0)}c_2'(0) - c_3'(0) \tag{3}$$

since $F(\varphi_1^h(0)) = 0$. Equation (3) is the key to the main result.

The idea is to find conditions under which $q_3^{h\prime}(0) < q_3'(0)$, where $q_3(s) \equiv F(\varphi_1(s))F(\varphi_2(s))$.[8] Then, as long as $c_3'(0)$ is neither too small nor too large, or $\overline{v}q_3^{h\prime}(0) < c_3'(0) < \overline{v}q_3'(0)$, a bid marginally above zero is unprofitable after the introduction of the handicap, but not before. Moreover, fixing $c_3'(0)$, if $c_3'(\cdot)$ is sufficiently steep there can be no profitable larger bid in Γ_3^h either. In that case, it is not an equilibrium for bidder 3 to be inactive when bidder 1 is not handicapped, but there is an equilibrium in which he stays out after bidder 1 is handicapped.[9] Equilibrium is not characterized in the former case, but it follows from Athey (2001) that one exists.[10]

There are at least two conditions for which $q_3^{h\prime}(0) < q_3'(0)$. Regardless of F, the condition is trivially satisfied if the handicap precisely nullifies bidder 1's advantage over bidder 2 (Assumption B2), in which case $q_3^{h\prime}(0) = 0 < q_3'(0)$ since $0 = F(\varphi_2^h(0)) < F(\varphi_2(0))$. To satisfy

[8]In words, $q_3^h(s)$ is flatter than $q_3(s)$ near $s = 0$. Since $q_3^h(0) = q_3(0)$, $q_3^h(s)$ must be below $q_3(s)$ for small s; after bidder 3 is given preferential treatment along with bidder 2, he faces a worse distribution of rival scores at the bottom. Since $\overline{s}^h < \overline{s}$, $q_3^h(s)$ and $q_3(s)$ must cross, but for the purposes of this paper it is sufficient to compare the two near $s = 0$.

[9]The proof of Theorem 1 describes an entire class of cost-functions (which depends on $F(\cdot)$, $c_1(\cdot)$, and $c_2(\cdot)$) for which this result holds. The opposite is also possible. Specifically, there are other c_3 functions for which bidder 3 would be inactive without preferential treatment, but active with preferential treatment.

[10]In a companion paper, Kirkegaard (2011b), equilibrium is characterized in a simplified version of the model. There, the existence of a budget constraint or bidding cap introduces a discontinuity into the cost function. In that setting, if a diverse group of bidders is handicapped, a subset of the group may participate more often, win more often, and overall be better off when handicapped.

the condition for *any* handicap that diminishes bidder 1's advantage (Assumption B1), it is sufficient to impose the additional assumption, motivated and interpreted momentarily, that the "average probability", $F(v)/v$, is strictly increasing in v. In that case, $q_3^{h\prime}(0) < q_3'(0)$ follows from (3) and the fact that $\varphi_2^h(0) < \varphi_2(0)$.

Theorem 1 *Assume either (i) that the handicap nullifies bidder 1's advantage over bidder 2 or (ii) that the handicap diminishes bidder 1's advantage and that $F(v)/v$ is strictly increasing in v. Then, there exists a strictly increasing and strictly convex cost function, c_3, for which there is no equilibrium in increasing strategies of Γ_3 where bidder 3 wins with probability zero, but for which there is an equilibrium in increasing strategies of Γ_3^h in which bidder 3 wins with probability zero.*

Proof. See the Appendix. ■

Since bidder 3 wins with positive probability for a mass of types in Γ_3, he has types for which his payoff is positive.[11] In this case, bidder 3 would go from participating with positive probability, winning with positive probability, and having positive payoff, to participating with probability zero, winning with probability zero, and having zero payoff.

Corollary 2 *Bidder 3 may be worse off when bidder 1 is handicapped.*

The "average probability", $F(v)/v$, is strictly increasing if "total probability", $F(v)$, is weakly convex. The uniform distribution with support $[\underline{v}, \overline{v}]$, $\underline{v} > 0$, is an easy example. However, since $\underline{v} > 0$, convexity is sufficient but not necessary.

More generally, $F(\underline{v})/\underline{v} = 0$ but $F(v)/v > 0$ for any $v > \underline{v} > 0$. Hence, the "average probability" must be strictly increasing close to $\underline{v}$. It is straightforward to describe environments where $F(v)/v$ is globally strictly increasing.[12] In particular, I will argue that the condition is satisfied if the "stakes" are high enough or if information is "almost" complete. To this end, consider some twice continuously differentiable distribution function, $G(v)$, with no mass points and support $[0, 1]$, say. The density, g, is bounded above and below, away from zero. No curvature assumptions are imposed.

[11] It follows from Myerson's (1981) analysis that in any mechanism where a bidder wins with positive probability for a mass of types, these types must earn strictly positive payoff, except possibly for the lowest participating type.

[12] As the literature on all-pay auctions with many bidders matures, conditions on $F(v)/v$ will perhaps find other uses. The reason is that $F(v)/v$ features directly in the system of differential equations at any score, s, where all three bidders are active, or

$$\frac{dF(\varphi_i(s))}{ds} = \frac{1}{2F(\varphi_j(s))F(\varphi_k(s))}\left[\frac{F(\varphi_j(s))}{\varphi_j(s)}c_j'(s) + \frac{F(\varphi_k(s))}{\varphi_k(s)}c_k'(s) - \frac{F(\varphi_i(s))}{\varphi_i(s)}c_i'(s)\right],$$

where i, j and k are distinct bidders.

Assume F has been obtained by shifting and rescaling G, such that $F(v) = G\left(\frac{v-\underline{v}}{\overline{v}-\underline{v}}\right)$, $v \in [\underline{v}, \overline{v}]$, with $\underline{v} > 0$. Then,

$$\frac{d}{dv}\left(\frac{F(v)}{v}\right) = \frac{g\left(\frac{v-\underline{v}}{\overline{v}-\underline{v}}\right)}{v^2}\left[\frac{v}{\overline{v}-\underline{v}} - \frac{G\left(\frac{v-\underline{v}}{\overline{v}-\underline{v}}\right)}{g\left(\frac{v-\underline{v}}{\overline{v}-\underline{v}}\right)}\right] \geq \frac{g\left(\frac{v-\underline{v}}{\overline{v}-\underline{v}}\right)}{v^2}\left[\frac{\underline{v}}{\overline{v}-\underline{v}} - \max_{x\in[0,1]}\left(\frac{G(x)}{g(x)}\right)\right]. \quad (4)$$

The last term inside the brackets is bounded above, by the assumption that g is bounded away from zero. However, the first term can be made arbitrarily large by shifting the support sufficiently far to the right. Specifically, fixing $\overline{v} - \underline{v}$, (4) is positive if $\underline{v}$ is sufficiently large. Hence, $F(v)/v$ is monotonic when $\underline{v}$ is large, even if $G(v)/v$ is not. Thus, the monotonicity condition holds in auctions with large stakes. Likewise, fixing $\overline{v}$ or $\underline{v}$, (4) is positive if $\overline{v} - \underline{v}$ is sufficiently small. Thus, the monotonicity condition holds when information is "almost complete". Of course, complete information corresponds to the case where F is degenerate, or $\overline{v} - \underline{v} = 0$. Following Siegel (2009), in the latter case no bidder can be hurt if a rival is handicapped (regardless of nature of the handicap).[13] However, if $\overline{v} - \underline{v}$ is strictly positive, but small, there are cost structures for which bidder 3 is hurt.

The weakness of Theorem 1 is that uniqueness is not addressed. In other words, it is unclear whether there are other equilibria of Γ_3^h which are better for bidder 3. Uniqueness is considered next.

4 Uniqueness

The existing papers that address uniqueness – Amann and Leininger (1996), Lizzeri and Persico (2000), and Siegel (2011) – explicitly and deliberately assume that there are only two bidders. As in Proposition 1, a unique equilibrium is characterized by shooting backwards from the (endogenously determined) maximum score, $\overline{s}$. In their examination of many bidder all-pay auctions, Parreiras and Rubinchik (2010, Section 4) admit the possibility of multiple equilibria. With more bidders, strategies need not be continuous and the number and identity of active bidders at any score is endogenously determined and may change over the range of equilibrium bids.[14] As Siegel (2011) states in his conclusion, "Another research direction is to extend the model and results to more than two players and additional signal distributions. This seems to be a non-trivial task, because much of the equilibrium analysis is driven

[13] More formally, Siegel (2009, Theorem 1) proves that no bidder can be hurt when a rival's reach decreases. A bidder's reach is defined as the highest score he would be willing to submit in order to win with certainty. In contests with complete information, any bidder's reach is common knowledge.

[14] Thus, at some scores the system of differential equations may look like the system in (1), while at other scores it may take the form described in footnote 12.

by these assumptions." Proving uniqueness of equilibrium in generality is also beyond the scope of the current paper. Instead, I will examine situations in which it is an equilibrium for bidder 3 to stay out of the auction after the handicap is introduced, and then derive conditions under which this equilibrium can be proven to be the unique equilibrium.

The conclusion in Theorem 1 was obtained by utilizing results for two-player auctions, thus sidestepping the need to examine the system of differential equations in the three-bidder auction. The idea in this section is similar. Here, I utilize results from mechanism design to sidestep the system of differential equations that arises if bidder 3 is active at some score, in some equilibrium. Mechanism design establishes a positive link between a bidder's probability of winning and his payoff, and this relationship can be used as the basis for a proof by contradiction. This is most easily illustrated for the case where the handicap nullifies bidder 1's advantage over bidder 2.

Proposition 3 *Assume $c_2'(s) = c_1^{h\prime}(s)$ for all $s \in \mathbb{R}_+$ and that c_3 is such that there is an equilibrium where bidder 3 stays out of Γ_3^h. Then, this equilibrium is the only equilibrium in increasing strategies where bidders 1 and 2 use symmetric strategies.*

Proof. See the Appendix. ■

The case where the handicap only diminishes but do not nullify bidder 1's advantage is harder. A partial uniqueness result is obtained for the special case where F is the uniform distribution.[15] In this case, mechanism design reveals that bidder 2 is supposed to stay out with greater probability in any equilibrium where bidder 3 is active. However, this property can be contradicted, at least for some c_3 functions.

Proposition 4 *Assume F is the uniform distribution and that the handicap diminishes but does not nullify bidder 1's advantage over bidder 2. Then, there exists a strictly increasing and strictly convex cost function, c_3, for which there is no equilibrium in increasing strategies of Γ_3 where bidder 3 wins with probability zero, but for which bidder 3 wins with probability zero in the unique equilibrium in increasing strategies of Γ_3^h.*

Proof. See the Appendix. The Appendix also contains an example. Note that the set of c_3 functions satisfying Proposition 4 is a proper subset of the set satisfying Theorem 1. ■

[15] Milgrom (2004) proposes an asymmetric auction model where bidders' types, v, are drawn from uniform distributions but where bidder i's valuation is determined by some function $g_i(v)$. In the current paper, the heterogeneity is described instead by different cost functions.

5 Comparing bidders

It is possible to make the relationship between the three bidders more precise. For c_3 to satisfy Theorem 1, it is necessary that a score of $\overline{s}^h$ is prohibitively expensive for bidder 3. On the other hand, it is not prohibitively expensive for bidders 1 and 2, since it is an equilibrium score. Hence, bidders 1 and 2 have an advantage over bidder 3 at high scores.

Next, consider low scores. Since bidder 3 is active when there is no handicap in place, there must be some profitable score if bidders 1 and 2 use their strategies from Γ_2. For instance, if bidder 3 profits by submitting a bid marginally above zero (as in the construct in the proof of Theorem 1), then

$$0 < \overline{v}q_3'(0) - c_3'(0) = \overline{v}\frac{F(\varphi_2(0))}{\varphi_2(0)}c_2'(0) - c_3'(0) < c_2'(0) - c_3'(0),$$

where the first equality follows from (3) and the last inequality from the assumption that $F(v)/v$ is strictly increasing. In conclusion, $c_3'(0) < c_2'(0)$. Thus, bidder 3 has a cost advantage over bidder 2 for low scores, but bidder 2 has the advantage when scores are high. Indeed, in the example in the Appendix, $c_2'(0) > c_1^{h\prime}(0) > c_3'(0)$. However, the cost functions used in the proof of Proposition 4 also have the property that bidders 1 and 2 with type $\underline{v}$ are willing to pay more to win with certainty than is bidder 3 with type $\overline{v}$, or $c_1^{-1}(\underline{v}) > c_2^{-1}(\underline{v}) > c_3^{-1}(\overline{v})$. In Siegel's (2009) terminology, bidders 1 and 2 have larger reaches than bidder 3, regardless of which types are compared.

6 Conclusion

This paper considered a contest with a number of realistic features: There are more than two bidders, bidders are heterogenous, and information is incomplete. In this environment, preferential treatment may have unintended consequences. Specifically, when a diverse group of bidders is given preferential treatment, a subset of the intended beneficiaries may become worse off. The reason is that the dynamics within the "favored" group changes. In particular, the stronger of the favored bidders may become more aggressive. From the point of view of the weaker of the favored bidders, this effect may outweigh the advantage that is gained over the handicapped bidder.

References

Amann, E. and Leininger, W., 1996, Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case, *Games and Economic Behavior*, 14: 1-18.

Athey, S., 2001, Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information, *Econometrica*, 69: 861-889.

Baye, M.R., Kovonock, D., and de Vries, C.G., 1993, Rigging the Lobbying Process: An Application of the All-Pay Auction, *American Economic Review*, 83 (1): 289-294.

Clark, D.J. and Riis, C., 2000, Allocation efficiency in a competitive bribery game, *Journal of Economic Behavior & Organization*, 42: 109-124.

Feess, E., Muehlheusser, G., and Walzl, M., 2008, Unfair contests, *Journal of Economics*, 93 (3): 267-291.

Fu, Q., 2006, A Theory of Affirmative Action in College Admissions, *Economic Inquiry*, 44 (3): 420-428.

Gale, I. and Stegeman, M., 1994, Exclusion in All-Pay Auction, mimeo.

Hafalir, I. and Krishna, V., 2008, Asymmetric Auctions with Resale, *American Economic Review*, 98 (1): 87-112.

Hafalir, I. and Krishna, V., 2009, Revenue and Efficiency Effects of Resale in First-Price Auctions, *Journal of Mathematical Economics*, 45 (9-10): 589-602.

Hickman, B.R., 2011, Human Capital Investment, Race Gaps, And Affirmative Action: A Game-Theoretic Analysis of College Admissions, mimeo.

Hirsch, M. and Smale, S., 1974, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press.

Hörner, J. and Sahuguet, N., 2007, Costly Signalling in Auctions, *Review of Economic Studies*, 74: 173-206.

Kirkegaard, R., 2011a, Favoritism in Asymmetric Contests: Head Starts and Handicaps, mimeo.

Kirkegaard, R., 2011b, A Poor Bidder's Perspective on All-Pay Auctions: More Competitors, Please, mimeo.

Lien, D., 1990, Corruption and Allocation Efficiency, *Journal of Development Economics*, 33: 153-164.

Lizzeri, A. and Persico, N., 2000, Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price, *Games and Economic Behavior*, 30: 83-114.

Maskin, E. and Riley, J., 2000, Asymmetric Auctions, *Review of Economic Studies*, 67, 413-438.

Milgrom, P., 2004, *Putting Auction Theory to Work*, Cambridge University Press.

Myerson, R.B., 1981, Optimal Auction Design, *Mathematics of Operations Research*, 6: 58-73.

Parreiras, S. and Rubinchik, A., 2010, Contests with Three or More Heterogeneous Agents, *Games and Economic Behavior*, 68: 703-715.

Siegel, R., 2009, All-Pay Contests, *Econometrica*. 77(1): 71-92

Siegel, R., 2011, Asymmetric Contests with Interdependent Valuations and Incomplete Information, mimeo.

Tullock, G., 1980, Efficient Rent Seeking, in *Toward a Theory of the Rent-Seeking Society*, edited by Buchanan, J.M., Tollison, R.D., and Tullock, G., Texas A&M University Press.

Appendix: Proofs

Proof of Proposition 1. Consider the game Γ_2^h. The system described by (1) is monotonic in $\overline{s}^h$. If $\overline{s}^h$ is reduced to $\widetilde{s}^h < \overline{s}^h$, it must be the case that $\varphi_i^h(\widetilde{s}^h) = \overline{v}$ after the change, but that $\varphi_i(\widetilde{s}^h) < \overline{v}$ before the change. Given (1), $F(\varphi_j^h(s))$ must therefore be strictly flatter than before at $s = \widetilde{s}^h$. Continuing this argument as s is reduced implies that $F(\varphi_1^h(\cdot))$ and $F(\varphi_2^h(\cdot))$ shift up when $\overline{s}^h$ is reduced. Thus, there is precisely one value for which $\overline{s}^h$ and the resulting unique paths of $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ satisfy the equilibrium requirement that $F(\varphi_1^h(0)) = 0$ and $F(\varphi_2^h(0)) \geq 0$ or vice versa.

Next, note that if $F(\varphi_1^h(s)) = F(\varphi_2^h(s))$ then $\varphi_1^h(s) = \varphi_2^h(s)$. Under Assumption B1, (1) then implies that $F(\varphi_1^h(s))$ is strictly steeper than $F(\varphi_2^h(s))$. Thus, $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ coincide at most once. In fact, this occurs at $\overline{s}^h$, since $F(\varphi_1^h(\overline{s}^h)) = F(\varphi_2^h(\overline{s}^h)) = 1$. Since $F(\varphi_1^h(s))$ is strictly steeper than $F(\varphi_2^h(s))$ at $\overline{s}^h$, the implication is that $F(\varphi_1^h(s)) < F(\varphi_2^h(s))$ for all $s \in [0, \overline{s}^h)$, as claimed. These arguments also apply to Γ_2. Under Assumption B2, it is easy to establish a contradiction if a candidate solution violates $\varphi_1^h(s) = \varphi_2^h(s)$ anywhere. Thus, $F(\varphi_1^h(s)) = F(\varphi_2^h(s)) = 0$ in this case. ■

Proof of Proposition 2. Assume that $\overline{s}^h = \overline{s}$. In this case $\varphi_i^h(\overline{s}) = \varphi_i(\overline{s}) = \overline{v}$, $i = 1, 2$, and it follows from Assumption A that $F(\varphi_2^h(\cdot))$ is strictly steeper than $F(\varphi_2(\cdot))$ at $\overline{s}$. Hence, $\varphi_2^h < \varphi_2$ immediately to the left of $\overline{s}$, which implies that $F(\varphi_1^h(\cdot))$ is strictly steeper than $F(\varphi_1(\cdot))$, and which in turn implies that $\varphi_1^h < \varphi_1$. These arguments repeat themselves as s is reduced even further, and it follows that $F(\varphi_1^h(\cdot))$ becomes zero for some $s > 0$. As discussed earlier, this contradicts that $\overline{s}^h$ and φ_1^h, φ_2^h form an equilibrium. It is easily seen that $\overline{s}^h > \overline{s}$ would lead to a similar contradiction. Thus, $\overline{s}^h < \overline{s}$, and $F(\varphi_1^h(\overline{s}^h)) = 1 > F(\varphi_1(\overline{s}^h))$.

Moving to the left, consider the first $s \in (0, \overline{s}^h)$ to the left of $\overline{s}^h$, if it exists, for which $F(\varphi_1^h(\cdot))$ and $F(\varphi_1(\cdot))$ coincide, or $\varphi_1^h = \varphi_1$. Since $F(\varphi_1^h(\cdot)) > F(\varphi_1(\cdot))$ to the right of this point, by definition, $F(\varphi_1^h(\cdot))$ must be at least as steep as $F(\varphi_1(\cdot))$, which implies that $\varphi_2^h \leq \varphi_2$ or $F(\varphi_2^h) \leq F(\varphi_2)$. As $\varphi_1^h = \varphi_1$, it follows from Assumption A and (1) that $F(\varphi_2^h(\cdot))$ is strictly steeper than $F(\varphi_2(\cdot))$ at this point, which means that $\varphi_2^h < \varphi_2$ just to the left of this score. As before, this leads to the conclusion that $F(\varphi_1^h(\cdot))$ is steeper than $F(\varphi_1(\cdot))$ to the left of this point. Once again, this can be ruled out, because $F(\varphi_1^h(\cdot))$ becomes zero for some $s > 0$. Thus, by contradiction, $F(\varphi_1^h(s)) > F(\varphi_1(s))$ for all $s \in (0, \overline{s}^h]$.

Assumption A and $\varphi_1^h(0) = \varphi_1(0) = \underline{v}$ imply that $F(\varphi_2^h(\cdot))$ is strictly steeper than $F(\varphi_2(\cdot))$ at $s = 0$. Assume now that $F(\varphi_2^h(0)) \geq F(\varphi_2(0))$ or $\varphi_2^h(0) \geq \varphi_2(0)$. Then, $\varphi_2^h > \varphi_2$ for small, strictly positive s, implying that $F(\varphi_1^h(\cdot))$ is strictly flatter than $F(\varphi_1(\cdot))$. Consequently, $\varphi_1^h < \varphi_1$ for small s. However, this contradicts the property that bidder 1 is less aggressive when he is handicapped. ■

Proof of Theorem 1. To begin, note that $q_3^h(s) \equiv F(\varphi_1^h(s))F(\varphi_2^h(s))$ is exogenous to the games Γ_3 and Γ_3^h, since it depends only on the strategies φ_1^h and φ_2^h from the game Γ_2^h. For similar reasons, $q_3(s) \equiv F(\varphi_1(s))F(\varphi_2(s))$ is also exogenous to the games Γ_3 and Γ_3^h. In particular, neither depend on $c_3(\cdot)$, since bidder 3 is assumed absent from Γ_2 and Γ_2^h.

So, under condition *(ii),* assume c_3 takes the form $c_3(s) = \alpha\overline{v}q_3^h(s)$ on $[0, \overline{s}^h]$, with $1 < \alpha < q_3'(0)/q_3^{h\prime}(0)$. This is a strictly increasing and strictly convex function because $q_3^h(s)$ has these properties when $F(v)/v$ is strictly increasing. Since $\alpha > 1$, $\overline{v}q_3^h(s) - c_3(s) \leq 0$ reveals that there is no incentive for bidder 3 to become active in Γ_3^h when bidders 1 and 2 follow the increasing strategies from Γ_2^h. Given bidder 3 stays out with probability one, there is no incentive for bidders 1 and 2 to deviate from the increasing strategies in Γ_2^h. Thus, it is an equilibrium for bidder 3 to always stay out, and for bidders 1 and 2 to continue using the increasing strategies from Γ_2^h; all bidders are best responding given their belief, and beliefs are consistent with strategies. Clearly, bidder 3 wins the auction with probability zero.

Consider now the game without a handicap, Γ_3. If bidder 3 wins with probability zero, no strictly positive score or bid can be rationalized. Thus, bidder 3 must either stay out, or bid zero. The arguments leading to Proposition 1 still apply, and bidder 1 and bidder 2's increasing strategies from Γ_2 are the unique pair of candidates for their increasing equilibrium strategies. However, since $\alpha < q_3'(0)/q_3^{h\prime}(0)$, $\overline{v}q_3(s) - c_3(s) = \overline{v}q_3(s) - \alpha\overline{v}q_3^h(s)$ is positive for small s. Thus, bidder 3 should deviate; there is no equilibrium in increasing strategies of Γ_3 where bidder 3 wins with probability 0. The proof is similar for condition *(i)*; simply pick any convex function which satisfies $c_3'(s) > \overline{v}q_3^{h\prime}(s)$ and $c_3'(0) < \overline{v}q_3'(0)$. ■

Proof of Proposition 3. Let $EU_i(v)$ denote bidder i's payoff if his type is v, in equilibrium. It follows from Myerson (1981) that $EU_i(v)$ can be written as

$$EU_i(v) = EU_i(\underline{v}) + \int_{\underline{v}}^{v} p_i(x)dx, \tag{5}$$

where $p_i(x)$ is bidder i's equilibrium winning probability if his type is x (Myerson's argument is unaffected by non-linear costs). Now, in any symmetric equilibrium in increasing strategies, bidders 1 and 2 win with probability zero if they have type $\underline{v}$. Thus, type $\underline{v}$ earns zero payoff. By symmetry, bidder 1 with type v outbids bidder 2 with type below v. Thus, when bidder 3 stays out, $p_1(v) = F(v)$. However, if bidder 3 is active then bidder 1 must also outbid him in order to win and thus $\widetilde{p}_1(v) \leq F(v)$, where the tilde from now on refers to an equilibrium candidate where bidder 3 is active. Hence, bidders 1 and 2 must be better off when bidder 3 stays out. Assume now that there is an equilibrium where bidder 3 enters. To rationalize entry by a set of types with positive mass it must be the case that the bid submitted by type

$\overline{v}$, $\overline{s}_3$, satisfies $EU_3(\overline{v}) = \overline{v}\widetilde{q}_3^h(\overline{s}_3) - c_3(\overline{s}_3) > 0$. However, by assumption, $\overline{v}q_3^h(\overline{s}_3) - c_3(\overline{s}_3) \leq 0$, where q_3^h is the distribution of the winning bid in the equilibrium where bidder 3 stays out. Hence, $\overline{s}_3 < \overline{s}^h$ and $\widetilde{q}_3^h(\overline{s}_3) > q_3^h(\overline{s}_3)$ or, equivalently, $F(\widetilde{\varphi}_1(\overline{s}_3)) > F(\varphi_1^h(\overline{s}_3))$. Bidder 1 with type $\widetilde{\varphi}_1(\overline{s}_3)$ wins with probability $\widetilde{p}_1 = p_1$ since he outbids bidder 3 with probability 1. However, his bid is lower in the new equilibrium (or bidder 3 would not enter). Thus, bidder 1 with type $\widetilde{\varphi}_1(\overline{s}_3)$ is strictly better off, which is a contradiction. ■

Proof of Proposition 4. The cost functions will be assumed to satisfy

$$\underline{v} - c_2\left(c_3^{-1}(\overline{v})\right) > 0. \tag{6}$$

Bidder 2 with type $\underline{v}$ is willing to pay more to win for sure than is bidder 3 with type $\overline{v}$.

The proof consists of a number of steps. First, some properties of equilibria in which bidder 3 is active are derived. Second, these are used to establish a contradiction. The superscript h on e.g. φ_i^h, q_i^h, EU_i^h, and $\overline{s}^h$ refers to the equilibrium in which bidder 3 is inactive. When h is omitted, it signifies that other equilibria are considered in the game Γ_3^h.

Step 1 (Equilibrium Properties I; Ranking payoffs and maximum scores): Let $s_i(v)$ denote bidder i's strategy and, as in Parreiras and Rubinchik (2010), let $\varphi_i(s) = \max\{\underline{v}, \sup\{v|s_i(v) \leq s\}\}$, $i = 1, 2, 3$. Since $c_1^h(\cdot) \leq c_2(\cdot)$, a revealed preference argument as in Parreiras and Rubinchik (2010, Lemma 3), easily establishes that $\overline{s}_1 \equiv s_1(\overline{v}) \geq s_2(\overline{v}) \equiv \overline{s}_2$. Since at least two bidders must share the same maximum score, $\overline{s}_1 = \max\{\overline{s}_2, \overline{s}_3\}$.

If $s_2(v) \leq s_1(v)$ for some $v \in [\underline{v}, \overline{v}]$ then

$$\begin{aligned}
EU_1(v) &= vF\left(\varphi_3(s_1(v))\right) F(\varphi_2(s_1(v))) - c_1^h(s_1(v)) \\
&\geq vF\left(\varphi_3(s_2(v))\right) F(v) - c_1^h(s_2(v)) \text{ since } s_1\text{, not } s_2\text{, is the equilibrium strategy} \\
&\geq vF\left(\varphi_3(s_2(v))\right) F(v) - c_2(s_2(v)) \\
&\geq vF\left(\varphi_3(s_2(v))\right) F(\varphi_1(s_2(v))) - c_2(s_2(v)) = EU_2(v).
\end{aligned}$$

Hence, $EU_2(v) > EU_1(v)$ necessitates $s_2(v) > s_1(v)$. This implies $p_2(v) \geq p_1(v)$ and therefore, from (5), $EU_2'(v) \geq EU_1'(v)$. Thus, $EU_2 > EU_1$ to the right of v, which again necessitates $s_2 > s_1$, and so on. Hence, $EU_2(x) > EU_1(x)$ and $s_2(x) > s_1(x)$ for all $x \in [v, \overline{v}]$. However, the latter contradicts $s_1(\overline{v}) \geq s_2(\overline{v})$. Thus, $EU_1(v) \geq EU_2(v)$ for all $v \in [\underline{v}, \overline{v}]$. An implication is that $p_2(\underline{v}) = 0$ and $EU_2(\underline{v}) = 0$ in any equilibrium. If $p_2(\underline{v}) > 0$ then bidder 2 with type $\underline{v}$ either (i) scores zero but bidders 1 and 3 do not participate with probability one, or (ii) scores strictly above zero. The first possibility violates $EU_1(v) \geq EU_2(v)$ for all v close to $\underline{v}$, while the second possibility can never form part of an equilibrium since it implies the support of the winning score is removed from zero.

Next, $EU_2(\underline{v}) = 0$ implies that $\overline{s}_2 > \overline{s}_3$. Otherwise, if $\overline{s}_3 \geq \overline{s}_2$, and thus $\overline{s}_1 = \overline{s}_3$, bidder 2 would win with probability one by bidding $\overline{s}_3$, and thereby obtain strictly positive payoff, by (6), even if his type is $\underline{v}$, which contradicts $EU_2(\underline{v}) = 0$. Hence, $\overline{s}_1 = \overline{s}_2 > \overline{s}_3$. Thus, at scores above $\overline{s}_3$, the interaction between bidders 1 and 2 must be described as in the system (2). Let $\overline{s} = \overline{s}_1 = \overline{s}_2$. As has already been noticed in the proof of Proposition 1, the system is monotonic in $\overline{s}$. If $\overline{s} > \overline{s}^h$, then $F(\varphi_1)$ and $F(\varphi_2)$ moves down, implying that $q_3(s) = F(\varphi_1(s))F(\varphi_2(s))$ is reduced (falls below $q_3^h(s)$) for any $s \geq \overline{s}_3$). In that case, however, it is irrational for bidder 3 with type $\overline{v}$ to bid $\overline{s}_3$ in the new "equilibrium" if he would not have found it profitable to do so before. Thus, if there is an equilibrium where bidder 3 stays out, then in any equilibrium where he enters it must be the case that $\overline{s} < \overline{s}^h$. Since bidders 1 and 2 win with probability one when bidding $\overline{s}$, the conclusion is that these bidders are better off in any equilibrium where bidder 3 participates, at least if they have type $\overline{v}$, or $EU_i(\overline{v}) > EU_i^h(\overline{v})$, $i = 1, 2, 3$, where $EU_3^h(\overline{v}) = 0$.

Step 2 (Equilibrium Properties II; Winning and entry probabilities): This step utilizes that F is the uniform distribution. The ex ante probability that bidder i wins is

$$EW_i = \int_{\underline{v}}^{\overline{v}} p_i(x) f(x) dx = \frac{1}{\overline{v} - \underline{v}} \int_{\underline{v}}^{\overline{v}} p_i(x) dx.$$

Since the object is sold with probability one in any equilibrium, $\sum EW_i = 1$. In the equilibrium where bidder 3 stays out, $EW_3^h = 0$. If bidder 3 enters the auction with positive probability then $EW_3 > 0$ and $EW_1 + EW_2$ declines. However, since $EU_2(\underline{v}) = 0$, note that

$$EW_2 = \frac{EU_2(\overline{v})}{\overline{v} - \underline{v}},$$

by (5). Since $EU_2(\overline{v}) > EU_2^h(\overline{v})$, $EW_2 > EW_2^h$. Thus, it is necessary that $EW_1 < EW_1^h$. At the same time, however, $EU_1(\overline{v}) > EU_1^h(\overline{v})$, or by (5),

$$EU_1(\underline{v}) + \int_{\underline{v}}^{\overline{v}} p_1(x) dx > EU_1^h(\underline{v}) + \int_{\underline{v}}^{\overline{v}} p_1^h(x) dx,$$

which can be rewritten

$$\frac{EU_1(\underline{v})}{\overline{v} - \underline{v}} + EW_1 > \frac{EU_1^h(\underline{v})}{\overline{v} - \underline{v}} + EW_1^h.$$

In conclusion, it is necessary that $EU_1(\underline{v}) > EU_1^h(\underline{v}) > 0$. However, since $s_1(\underline{v}) = s_1^h(\underline{v}) = 0$, this in turn necessitates that $p_1(\underline{v}) > p_1^h(\underline{v}) > 0$. In words, bidder 1 with the lowest type must necessarily win more often in the equilibrium where bidder 3 is active. Thus, neither bidder 2 nor bidder 3 participate with probability one, and so $p_2(\underline{v}) = p_3(\underline{v}) = 0 = EU_2(\underline{v}) = EU_3(\underline{v})$.

Since bidder 3 is active with some positive probability, however, $p_1(\underline{v}) > p_1^h(\underline{v})$ necessitates that $\varphi_2(0) > \varphi_2^h(0)$, i.e. that bidder 2 stays out more often in the new equilibrium. It is these properties that will be contradicted in step 3.

Step 3 (Bidder 2's entry incentive): Since $\overline{s}_1 = \overline{s}_2 > \overline{s}_3$, any equilibrium is in large part characterized by the pair $(\overline{s}_1, \overline{s}_3)$. Holding $\overline{s}_3$ fixed, bidder 3's winning probability $p_3(\overline{v}) = q_3(\overline{s}_3)$ is determined by $\overline{s}_1$, because $\varphi_1(\overline{s}_3)$ and $\varphi_2(\overline{s}_3)$ are determined by shooting backwards from $\overline{s}_1$. Thus, it is meaningful to write e.g. $\varphi_1(\overline{s}_3|\overline{s}_1)$. Note that $q_3(\overline{s}_3|\overline{s}_1) = F(\varphi_1(\overline{s}_3|\overline{s}_1))F(\varphi_2(\overline{s}_3|\overline{s}_1))$ is decreasing in $\overline{s}_1$, because the system (2) is monotonic in $\overline{s}_1$.

Let $v_2^c = \varphi_2^h(0)$. Supposedly, bidder 2 with type v_2^c should stay out in any equilibrium where bidder 3 is active. However, if bidder 2 with type v_2^c bids $\overline{s}_3$, his payoff is

$$U_2^c(\overline{s}_1, \overline{s}_3) = v_2^c F(\varphi_1(\overline{s}_3|\overline{s}_1)) - c_2(\overline{s}_3).$$

Note that $U_2^c(\overline{s}_1, \overline{s}_3)$ is decreasing in $\overline{s}_1$. At $\overline{s}_3$, φ_1 is described by (2), and so in any equilibrium where bidder 3 is active (or $\overline{s}_3 > 0$)

$$\frac{\partial U_2^c(\overline{s}_1, \overline{s}_3)}{\partial \overline{s}_3} = v_2^c \frac{c_2'(\overline{s}_3)}{\varphi_2(\overline{s}_3|\overline{s}_1)} - c_2'(\overline{s}_3) < 0$$

since, in equilibrium, $v_2^c < \varphi_2(\overline{s}_3)$. Let $r_2(v_2^c)$ denote the reach of bidder 2 with type v_2^c, defined by $v_2^c - c_2(r_2(v_2^c)) = 0$, and recall $r_2(v_2^c) > c_3^{-1}(\overline{v})$, by (6).

For any $\overline{s}_3 \in [0, r_2(v_2^c)]$, define $\overline{s}_1^*(\overline{s}_3)$ to satisfy $U_2^c(\overline{s}_1^*(\overline{s}_3), \overline{s}_3) = 0$. Note that $\overline{s}_1^*(0) = \overline{s}_1^h$ where $(\overline{s}_1, \overline{s}_3) = (\overline{s}_1^h, 0)$ describes the equilibrium where bidder 3 stays out. Likewise, $\overline{s}_1^*(r_2(v_2^c)) = r_2(v_2^c)$, in which case $\overline{s}_1 = \overline{s}_3$. Since $U_2^c(\overline{s}_1, \overline{s}_3)$ is decreasing in both arguments, $\overline{s}_1^*(\overline{s}_3)$ is a decreasing function with $\overline{s}_1^{*\prime}(0) = 0$ and $\overline{s}_1^{*\prime}(\overline{s}_3) < 0$ for $\overline{s}_3 > 0$. Fixing $\overline{s}_3$, bidder 2 with type v_2^c has an incentive to enter the auction (with a bid of $\overline{s}_3$) if $\overline{s}_1 < \overline{s}_1^*(\overline{s}_3)$.

Next, return to the winning probabilities in Step 2. Since $EU_2(\underline{v}) = EU_3(\underline{v}) = 0$ and $EU_1(\underline{v}) = \underline{v}p_1(\underline{v})$, the ex ante winning probabilities are

$$\begin{aligned}
EW_3(\overline{s}_1, \overline{s}_3) &= \frac{EU_3(\overline{v})}{\overline{v} - \underline{v}} = \frac{1}{\overline{v} - \underline{v}} \left(\overline{v} F(\varphi_1(\overline{s}_3))F(\varphi_2(\overline{s}_3)) - c_3(\overline{s}_3)\right) \\
EW_2(\overline{s}_1, \overline{s}_3) &= \frac{EU_2(\overline{v})}{\overline{v} - \underline{v}} = \frac{1}{\overline{v} - \underline{v}} \left(\overline{v} - c_2(\overline{s}_1)\right) \\
EW_1(\overline{s}_1, \overline{s}_3) &= \frac{EU_1(\overline{v}) - EU_1(\underline{v})}{\overline{v} - \underline{v}} = \frac{1}{\overline{v} - \underline{v}} \left(\overline{v} - c_1^h(\overline{s}_1) - \underline{v}p_1(\underline{v})\right).
\end{aligned}$$

Since the winning probabilities sum to one,

$$c_3(\overline{s}_3) = \underline{v}(1 - p_1(\underline{v})) + \overline{v}\left(1 + F(\varphi_1(\overline{s}_3))F(\varphi_2(\overline{s}_3))\right) - c_1^h(\overline{s}_1) - c_2(\overline{s}_1).$$

Let $P = p_1^h(\underline{v})$. Recall any equilibrium candidate with entry by bidder 3 is admissible only if $p_1(\underline{v}) > P$. Let

$$A(\overline{s}_1, \overline{s}_3) = \underline{v}(1 - P) + \overline{v}\left(1 + F(\varphi_1(\overline{s}_3|\overline{s}_1))F(\varphi_2(\overline{s}_3|\overline{s}_1))\right) - c_1^h(\overline{s}_1) - c_2(\overline{s}_1)$$

and define $A^*(\overline{s}_3) = A(\overline{s}_1^*(\overline{s}_3), \overline{s}_3)$. Since $A(\overline{s}_1, \overline{s}_3)$ is decreasing in $\overline{s}_1$, $A(\overline{s}_1, \overline{s}_3) \leq A^*(\overline{s}_3)$ for all $\overline{s}_1 \geq \overline{s}_1^*(\overline{s}_3)$. Now, consider a cost-function c_3 with the property that $c_3(\overline{s}_3) \geq A^*(\overline{s}_3)$ for all $\overline{s}_3 \in [0, r_2(v_2^c)]$. Then, $c_3(\overline{s}_3) \geq A^*(\overline{s}_3) \geq A(\overline{s}_1, \overline{s}_3)$ for all $\overline{s}_1 \geq \overline{s}_1^*(\overline{s}_3)$. Hence, either: (i) if $\overline{s}_1 \geq \overline{s}_1^*(\overline{s}_3)$, then $c_3(\overline{s}_3) \geq A(\overline{s}_1, \overline{s}_3)$ and so $p_1(\underline{v}) \leq P$, which contradicts the requirement that $p_1(\underline{v}) > P$, or (ii) if $\overline{s}_1 < \overline{s}_1^*(\overline{s}_3)$ then $U_2^c(\overline{s}_1, \overline{s}_3) > 0$, which contradicts the requirement that bidder 2 with type v_2^c stay out. Thus, any $(\overline{s}_1, \overline{s}_3)$ pair leads to a contradiction.

However, it remains to establish that $c_3(s) \geq A^*(s)$ is a sensible restriction to impose on a cost function. Note first that $A^*(0) = A(\overline{s}_1^*(0), 0) = A(\overline{s}_1^h, 0) = 0$ (the easiest way to see this is to recall that $P = p_1^h(\underline{v})$ which is precisely $p_1(\underline{v})$ when $(\overline{s}_1, \overline{s}_3) = (\overline{s}_1^h, 0)$, in which case $c_3(\overline{s}_3)$ coincides with $A(\overline{s}_1, \overline{s}_3)$ and equals zero). Moreover,

$$\begin{aligned} A^{*\prime}(\overline{s}_3) &= \overline{v}\frac{\partial F(\varphi_1(\overline{s}_3|\overline{s}_1^*))F(\varphi_2(\overline{s}_3|\overline{s}_1^*))}{\partial \overline{s}_3} + \overline{v}\frac{\partial F(\varphi_1(\overline{s}_3|\overline{s}_1^*))F(\varphi_2(\overline{s}_3|\overline{s}_1^*))}{\partial \overline{s}_1}\overline{s}_1^{*\prime}(\overline{s}_3) \\ &\quad - \left(c_1^{h\prime}(\overline{s}_1^*) + c_2'(\overline{s}_1^*)\right)\overline{s}_1^{*\prime}(\overline{s}_3). \end{aligned}$$

The first term is positive. The second is the product of two negative terms. In the third term, $\overline{s}_1^{*\prime}(\overline{s}_3)$ is negative. Thus, $A^{*\prime}(\overline{s}_3) \geq 0$, with

$$A^{*\prime}(0) = \overline{v}\frac{\partial F(\varphi_1(\overline{s}_3|\overline{s}_1^*))F(\varphi_2(\overline{s}_3|\overline{s}_1^*))}{\partial \overline{s}_3}_{|(\overline{s}_1,\overline{s}_3)=(\overline{s}_1^h,0)} = \overline{v}q_3^{h\prime}(0),$$

since $\overline{s}_1^{*\prime}(0) = 0$. More generally, since the last two terms are positive, using (2) yields

$$\begin{aligned} A^{*\prime}(s) &\geq \overline{v}\frac{\partial F(\varphi_1(s|\overline{s}_1^*))F(\varphi_2(s|\overline{s}_1^*))}{\partial s} \\ &= \overline{v}\left(\frac{F(\varphi_1(s|\overline{s}_1^*))}{\varphi_1(s|\overline{s}_1^*)}c_1^{h\prime}(s) + \frac{F(\varphi_2(s|\overline{s}_1^*))}{\varphi_2(s|\overline{s}_1^*)}c_2'(s)\right). \end{aligned}$$

Since $\overline{s}_1^* \leq \overline{s}_1^h$ and the system is monotonic in $\overline{s}_1$, $\varphi_i(s|\overline{s}_1^*) \geq \varphi_i(s|\overline{s}_1^h) = \varphi_i^h(s)$, $i = 1, 2$. Hence

$$A^{*\prime}(s) \geq \overline{v}\left(\frac{F(\varphi_1^h(s))}{\varphi_1^h(s)}c_1^{h\prime}(s) + \frac{F(\varphi_2^h(s))}{\varphi_2^h(s)}c_2'(s)\right) = \overline{v}q_3^{h\prime}(s).$$

Finally, since $A^{*\prime}(s) \geq \overline{v}q_3^{h\prime}(s)$, it follows that $c_3(s) \geq A^*(s) \geq \overline{v}q_3^h(s)$

Step 4 (Conclusion): Let q_3 denote the distribution of the winning bid in the game Γ_2. As in Section 3, $q_3'(0) > q_3^{h\prime}(0)$. Assume that $c_3'(0) \in (\overline{v}q_3^{h\prime}(0), \overline{v}q_3'(0))$. Then, by the argument in Theorem 1, there is no equilibrium in increasing strategies in Γ_3 in which bidder 3 stays out. Moreover, assume that c_3 satisfies (6) and further that $c_3(s) \geq A^*(s)$ for all $s \in [0, r_2(v_2^c)]$, where $A^*(0) = 0$ and $A^{*\prime}(0) = \overline{v}q_3^{h\prime}(0)$. Since $c_3(s) \geq \overline{v}q_3^h(s)$, there is an equilibrium of Γ_3^h where bidder 3 stays out. By the argument in Step 3, there are no other equilibria in increasing strategies. Note that the assumptions imposed here are not mutually exclusive, but that there are other cost functions for which there is an equilibrium of Γ_3^h where bidder 3 stays out but where the uniqueness proof does not apply. ■

Example of Proposition 4: Assume $F(v) = v - 1$, $v \in [1, 2]$. Assume $c_2(s) = 2s$, and, after the handicap, $c_1^h(s) = \frac{3}{2}s$. The assumption of linear costs is for simplicity. The advantage is that it turns the system of differential equations in the two-player auction into an autonomous system, which has been studied by Amann and Leininger (1996). The properties of an equilibrium where bidder 3 is inactive are then easily determined. In such an equilibrium, bidder 2 stays out if and only if his type falls below $v_2^c = 2^{\frac{1}{4}} \approx 1.189$. Bidder 2's reach is $r_2(v_2^c) = 2^{-\frac{3}{4}} \approx 0.595$ if his type is v_2^c. The maximum score is $\overline{s}^h = \frac{8}{7} - \frac{2}{7}\sqrt[4]{2} \approx 0.803$ in an equilibrium where bidder 3 is inactive. The thin curve in Figure A.1 depicts $q_3^h(s)$, while the fat curve illustrates the function $A^*(s)/\overline{v}$. In comparison, the dashed curve describes the distribution of the winning bid, $q_3(s)$, in the game Γ_2, when it is assumed that $c_1(s) = \frac{1}{8}s$. The cost function $c_3(s) = s(1+s)^4$, for example, satisfies $\frac{c_3(s)}{\overline{v}} \geq \frac{A^*(s)}{\overline{v}}$ or $c_3(s) \geq A^*(s)$ and at the same time satisfies $\frac{c_3'(0)}{\overline{v}} < q_3'(0)$. Hence, if the cost functions are

$$c_1(s) = \frac{1}{8}s,\ c_1^h(s) = \frac{3}{2}s,\ c_2(s) = 2s \text{ and } c_3(s) = s(1+s)^4$$

then there is no equilibrium of Γ_3 where bidder 3 stays out and the only equilibrium of Γ_3^h in increasing strategies is the one where bidder 3 does in fact stay out. Note that $c_2'(0) > c_1^{h\prime}(0) > c_3'(0)$.

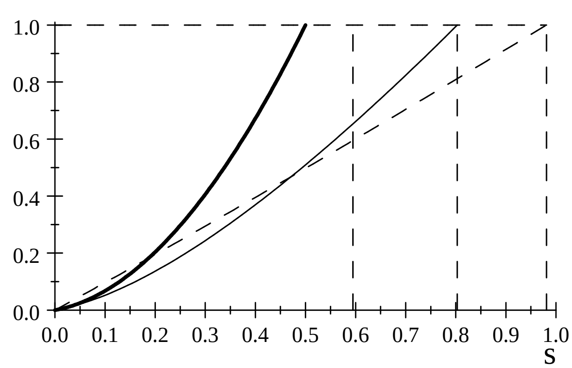

Figure A.1: Example of Proposition 4.