ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

http://www.econometricsociety.org/

Econometrica, Vol. 80, No. 5 (September, 2012), 2349–2364

A MECHANISM DESIGN APPROACH TO RANKING ASYMMETRIC AUCTIONS

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Econometrica, Vol. 80, No. 5 (September, 2012), 2349–2364

A MECHANISM DESIGN APPROACH TO RANKING ASYMMETRIC AUCTIONS

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I propose a new mechanism design approach to the problem of ranking standard auctions with two heterogeneous bidders. A key feature of the approach is that it may be possible to rank two auctions even if neither dominates the other for all combinations of types. The approach simplifies the analysis and unifies results in the existing literature. Roughly speaking, the first-price auction is more profitable than the second-price auction when the strong bidder's distribution is flatter and more disperse than the weak bidder's distribution. Applications include auctions with one-sided externalities. Moreover, contrary to previous work, reserve prices are easily handled. Finally, the method can be extended to some environments with many bidders.

KEYWORDS: Asymmetric auctions, mechanism design, revenue ranking.

1. INTRODUCTION

MANY, IF NOT MOST, auctions involve bidders that are heterogeneous ex ante. For example, procurement auctions may involve domestic and foreign firms; an auction for a new license or technology may pit an incumbent against a prospective entrant; an art collector may be vying for a synergy not relevant to a bidder with unit demand, and so on. Finally, asymmetries may be created over time, as valuations in the last auction in a sequence likely depend on how many items have been won at that time.

The interest in asymmetric auctions dates back to the inception of modern auction theory, with Vickrey (1961). As Vickrey (1961) first discovered, and Myerson (1981) and Riley and Samuelson (1981) later proved more generally, the first-price auction (FPA) and second-price auction (SPA) yield the same expected revenue in the independent private values model when bidders are homogenous. However, Vickrey (1961) proved by example that revenue equivalence does not hold when bidders are heterogeneous. Thus, the question of which auction is more profitable with heterogeneous bidders is an old and fundamental question. In light of the increasing use of auctions, it is as relevant as ever. However, the problem is complicated by the fact that bidding strategies generally cannot be characterized in closed form in the FPA.

The primary objective of this note is to make a methodological contribution to the literature on asymmetric auctions. Specifically, a relatively simple mechanism design approach is proposed. With this method in hand, most existing

[1]I thank the Canada Research Chairs programme and the Social Sciences and Humanities Research Council of Canada for funding this research. I am grateful for several comments and suggestions from the editor and three anonymous referees. I also thank Bernard Lebrun, Ruqu Wang, and audiences at the University of Arizona, the University of Guelph, Queen's University, and the 10th SEAT conference. I thank A. Marcel Oestreich for research assistance.

 DOI: 10.3982/ECTA9859

results can be unified and a general environment in which the FPA dominates the SPA can be established.

In a seminal paper on auctions with two heterogeneous bidders, Maskin and Riley (2000) presented three seemingly separate classes of environments in which it is possible to rank the FPA and the SPA in terms of revenue. In two of the cases—if the strong bidder's type distribution is either a "shifted" or a "stretched" version of the weak bidder's type distribution, the FPA yields higher expected revenue than the SPA.[2]

Maskin and Riley (2000) used arguments from mechanism design to explain why the problem is nontrivial. They then abandoned the approach, concluding that "mechanism design considerations do not settle the matter of which auction generates more revenue."[3] Instead, they used the system of differential equations that describe bidding behavior to derive two technical lemmata that quantify revenue in the two auctions. Using these, they proved the superiority of the FPA in two of their models.

In contrast, I will show that mechanism design can, in fact, not only be used to address the problem, it also greatly simplifies the analysis itself. The starting point is Myerson's (1981) result that expected revenue is largely determined by the expected value of the winner's virtual valuation. The key step is to formulate the latter as the expected value of the expectation of the winner's virtual valuation *conditional* on the weak bidder's type. When bidders' supports have the same lowest end-point, the assumptions in the main theorem ensure that this conditional expectation is greater for the FPA than for the SPA for all types in the weak bidder's support. When the lower end-points differ, the proof is completed by exploiting the fact that the FPA extracts more rent from the strong bidder than does the SPA. Compared to other uses of Myerson's (1981) result, the important point is that two auctions can be ranked when neither dominates for all combinations of types.[4]

The mechanism design approach unifies and extends Maskin and Riley's results. Roughly speaking, the FPA dominates if the strong bidder's distribution

[2]Maskin and Riley's (2000) paper is the most general treatment of the revenue effects of heterogeneity in the existing literature. The remaining literature focuses on analytical or numerical examples. Vickrey (1961), Lebrun (1996), and Cheng (2006) analytically examined special cases of power distributions. Griesmer, Levitan, and Shubik (1967), Plum (1992), and Kaplan and Zamir (2010) derived bidding strategies in such environments. The numerical analysis of auctions dates back to Marshall, Meurer, Richard, and Stromquist (1994), who also focused on the power distribution. Vickrey (1961), Cheng (2010), and Gavious and Minchuk (2010) provided examples in which the SPA is superior. The FPA is superior if the two-bidder game is augmented with a type of learning or, under certain regularity conditions, if resale is allowed. See Jehiel (2011), Hafalir and Krishna (2008), and Cheng and Tan (2010).

[3]To clarify, Maskin and Riley (2000) constructed one environment in which they used mechanism design to prove the SPA is more profitable. Their proofs are different when the FPA dominates.

[4]In Mares and Swinkels' (2011) comparison of auctions with handicaps, one auction dominates another because it is better for *all* pairs of types. In contrast, the standard FPA is not better than the SPA for all type pairs. This is the exact reason Maskin and Riley dismissed mechanism design.

is flatter and more disperse than the weak bidder's distribution; "shifting" and "stretching" are special cases. Maskin and Riley's results are thus corollaries. While the objective of this note is to present the mechanism design approach, an earlier version (Kirkegaard (2011)) demonstrated the scope of the main result. For instance, the FPA is superior in any environment that "lies between" the shift and stretch models in a natural way.

Several applications and extensions are possible. One application, closely related to Maskin and Riley's "shift" model, is to auctions with one-sided externalities.[5] Moreover, the mechanism design approach easily handles reserve prices. The ranking can also be extended to some many-bidder settings. These and other extensions are discussed in Section 4. Other auction formats are considered in the Supplemental Material (Kirkegaard (2012)).

2. MODEL AND PRELIMINARIES

There are two risk neutral bidders. Bidder s is strong and bidder w is weak; bidder s is more likely to value the object being sold more highly. Bidder i draws a type or valuation from a distribution function, F_i, which is continuously differentiable on its support, $S_i = [\beta_i, \alpha_i]$, $i = s, w$. The density, f_i, is strictly positive on $(\beta_i, \alpha_i]$, with $\alpha_i > \beta_i \geq 0$, $i = s, w$. Mass points are ruled out. It is assumed that $\beta_w \leq \beta_s$ and $\alpha_w < \alpha_s$. Let $C = S_s \cap S_w$ denote the common support. Thus, $C = [\beta_s, \alpha_w]$ if the supports overlap. Finally, let $F_i(v) = f_i(v) = 0$ for all $v < \beta_i$, $i = s, w$.

As in Maskin and Riley (2000), F_s dominates F_w in terms of the reverse hazard rate, $F_w \leq_{rh} F_s$. That is, $\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)}$, for all $v \in C$.[6] This assumption will be used to derive bounds on equilibrium strategies. Let $r(v) = F_s^{-1}(F_w(v))$, $v \in S_w$. Bidder s is just as likely to have a type below $r(v)$ as bidder w is to have a type below v; the two bidders have the same rank, or $F_s(r(v)) = F_w(v)$. Together, $F_w \leq_{rh} F_s$ and $\alpha_w < \alpha_s$ imply "strict" first order stochastic dominance, $r(v) > v$ for all $v \in (\beta_w, \alpha_w]$.

Let

$$J_i(v) = v - \frac{1 - F_i(v)}{f_i(v)}$$

denote bidder i's "virtual valuation." The distribution F_s is said to be "regular" if J_s is nondecreasing. It is said to dominate F_w in terms of the hazard rate, $F_w \leq_{hr} F_s$, if $\frac{f_s(v)}{1-F_s(v)} \leq \frac{f_w(v)}{1-F_w(v)}$ for all $v \in C$. Note that hazard rate dominance

[5]See Jehiel, Moldovanu, and Stacchetti (1999) for a discussion of the difficulties that arise in auctions with externalities. Their revenue results are confined to symmetric environments, however.

[6]Maskin and Riley's (2000) assumption is slightly stronger. However, reverse hazard rate dominance is sufficient to deliver the key result that bidder w is at least as aggressive as bidder s in the FPA. Kirkegaard (2009) described behavior in some environments where $F_w \leq_{rh} F_s$ is violated.

implies $J_w(v) \geq J_s(v)$ for all $v \in C$. To fix ideas, assume $F_w \leq_{hr} F_s$; stronger assumptions will be imposed later. The purpose of those assumptions is to enable the revenue ranking (while using the bounds on strategies that derive from the $F_w \leq_{rh} F_s$ assumption).

2.1. *Mechanism Design*

Myerson (1981) developed tools to construct the revenue maximizing auction. However, his analysis can be used to evaluate any auction, be it optimal or not. In particular, he showed that expected revenue in any mechanism is equal to the expected value of the winning bidder's virtual valuation, less any rent earned by the lowest types. Thus, an optimal auction favors the bidder with the higher virtual valuation, in this case bidder w. While the FPA is advantageous to bidder w, it may unfortunately be "too favorable." Thus, a method to determine whether the FPA is better, on balance, is needed. In other words, is it better to favor bidder w excessively than not to at all?[7]

Assume, for now, that the seller does not use a reserve price. This assumption is relaxed in Section 4. Then, the FPA and SPA belong to a class of mechanisms where the allocation can be described by a single function. The "tying function," $k(v)$, describes the type of bidder s that bidder w with type v would tie with, in equilibrium. Bidder w wins if bidder s's type falls below $k(v)$. Bidder s wins otherwise. Let $k(v) = \beta_s$ if bidder w loses with probability 1 in the mechanism in question.

The next step is to rewrite Myerson's revenue expression in terms of k. Specifically, the key trick is to write the expected value of the winner's virtual valuation as the expected value of a conditional expectation, or

$$(1) \qquad ER^k = \int_{\beta_w}^{\alpha_w} \left(J_w(v) F_s(k(v)) + \int_{k(v)}^{\alpha_s} J_s(x)\, dF_s(x) \right) dF_w(v) - u_w^k(\beta_w) - u_s^k(\beta_s),$$

where $u_i^k(\beta_i)$ denotes bidder i's expected utility if his type is β_i. The term in parentheses is the expected value of the winning bidder's virtual valuation, conditional on the weak bidder's type being v.

2.2. *Bidder Behavior in the FPA and SPA*

This paper is concerned with two suboptimal auctions, the FPA and SPA. These auctions are significant because they are commonly used in the real

[7]Bulow and Roberts (1989) suggested that auction design has parallels with third degree price discrimination. In those terms, the question is whether a large discount to the weak market is better than no discount at all. Kirkegaard (2011) argued that the conditions in Theorem 1, below, can be interpreted as conditions on the relationship between demand curves in different markets.

world. In practice, the seller may not have the knowledge or sophistication to construct the optimal auction. Let k_1, k_2 denote the tying function in the FPA and SPA, respectively.

In the SPA, I assume that each bidder follows the weakly dominant strategy of submitting a bid equal to his type (even if it has probability zero of winning). Hence, $k_2(v) = \max\{\beta_s, v\}$. Thus, bidder w's payoff is zero if his type is β_w, or $u_w^2(\beta_w) = 0$. For bidder s,

$$u_s^2(\beta_s) = \int_{\beta_w}^{\alpha_w} \max\{\beta_s - v, 0\}\, dF_w(v). \tag{2}$$

Behavior in the FPA is harder to describe. Maskin and Riley (2000) formulated the problem in terms of finding inverse bidding strategies. Then, the first order conditions produce a system of differential equations. The main problem is that this system can, in general, not be solved analytically. Thus, the best one can hope for is to derive useful qualitative properties. Next, I demonstrate the difficulties and explain the inferences that can be made. However, since the tying function is central to this paper, I use a slightly less common formulation of the problem. Both formulations were used in Lebrun (1999).

The endogenous functions are (b_w, k_1), where b_w is bidder w's bidding strategy (bidder s's strategy is $b_s(v) = b_w(k_1^{-1}(v)))$. Let $b_* \leq \beta_s$ denote the bid submitted by bidder s with type β_s. In equilibrium, $b_* \in S_w$.[8] Bidder w bids below b_* if his type is below b_*; he never wins and so $k_1(v) = \beta_s \in [v, r(v)]$ for $v \leq b_*$. Types above b_* submit "serious" bids. If (as turns out to be true) b_w and k_1 are increasing for $v > b_*$, bidder w with type v bids "as if" his type is x to maximize $(v - b_w(x))F_s(k_1(x))$. Bidder s with type $k_1(v)$ maximizes $(k_1(v) - b_w(x))F_w(x)$. In equilibrium, both problems are solved at $x = v$, by definition of b_w and k_1. For $v > b_*$, the first order conditions yield

$$\begin{aligned} k_1'(v) &= \frac{F_s(k_1(v))}{f_s(k_1(v))} \frac{f_w(v)}{F_w(v)} \frac{k_1(v) - b_w(v)}{v - b_w(v)}, \\ b_w'(v) &= \frac{f_w(v)}{F_w(v)}(k_1(v) - b_w(v)). \end{aligned} \tag{3}$$

A boundary condition is obtained by noting that the two bidders must bid the same if they have the highest possible types, or $k_1(\alpha_w) = \alpha_s$. Again, the problem is that this system cannot generally be solved analytically. However, given $F_w \leq_{rh} F_s$, it readily follows from $k_1'(v)$ that $k_1(v) \in [\max\{\beta_s, v\}, r(v)] \subset$

[8]Thus, $b_* \in S_w \cap [\beta_w, \beta_s]$. If $\beta_s = \beta_w$, then $b_* = \beta_s = \beta_w$. Maskin and Riley (2000) characterized b_*, but its exact value is unimportant for our purposes. They also pointed out that the strong bidder always bids α_w and wins if β_s is much larger than α_w. This case corresponds to $b_* = \alpha_w$.

$[v, r(v)]$ for all $v \in S_w$.[9] In other words, it is possible to bound k_1 both above and below. As in Maskin and Riley (2000), these bounds are important for the revenue ranking.

In a FPA, $u_w^1(\beta_w) = 0$ (since β_w never wins) and $u_s^1(\beta_s) = (\beta_s - b_*)F_w(b_*)$. Note that bidder s with type β_s prefers the SPA, with

$$u_s^2(\beta_s) - u_s^1(\beta_s) = \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\alpha_w} \max\{\beta_s - v, 0\}\, dF_w(v). \tag{4}$$

3. RANKING ASYMMETRIC AUCTIONS

To explain the main forces at work, consider the simpler case with $\beta_w = \beta_s$. Assume that F_s and F_w are regular and let $\kappa(v) = J_s^{-1}(J_w(v))$ (whenever it exists).[10] Assume $F_w \leq_{hr} F_s$, such that $\kappa(v) \geq v$ for all $v \in C$. Recall that in an optimal auction (subject to the constraint that the object is sold), bidder w should win if and only if bidder s has a type below $\kappa(v)$. However, note that $\kappa(v) < r(v)$ when v is large because $J_w(v) < J_s(r(v))$ when v is close to α_w.

Now, *fixing bidder w's type* at v, the difference between a FPA and a SPA is that bidder w wins in the former but loses in the latter if bidder s's type is in $[k_2(v), k_1(v)]$. When $\beta_w = \beta_s$, $k_2(v) = v$ and so $\kappa(v) \geq k_2(v)$. If $k_1 \leq \kappa$, the allocation in the FPA is more profitable; bidder w is eating his way into an area where his virtual valuation exceeds J_s. However, if $k_1 > \kappa$, the trend reverses—bidder w is now winning too often. This occurs when v is close to α_w, since $k_1(\alpha_w) = \alpha_s > \kappa(\alpha_w)$. Hence, depending on the strong bidder's *actual* type, switching from the SPA to the FPA may or may not increase the winner's virtual valuation. Therefore, Maskin and Riley (2000) concluded that mechanism design is of no use in determining which auction is more profitable.

However, the concern is with expected revenue. All that is required is to determine which effect dominates *in expectation*. The advantage of formulating expected revenue as in (1) is precisely that it brings this trade-off to the fore.[11]

Using (1) and $u_w^1(\beta_w) = u_w^2(\beta_w) = 0$, the difference between the FPA and the SPA in terms of expected revenue is

$$ER^1 - ER^2 = \int_{\beta_w}^{\alpha_w} D(v|k_1, k_2)\, dF_w(v) + u_s^2(\beta_s) - u_s^1(\beta_s), \tag{5}$$

[9]If $k_1(v) = v$ for any $v > b_*$, then $k_1'(v) \leq 1$, since $F_w \leq_{rh} F_s$. Thus, $k_1(v)$ can cross the 45° line at most once, and then from above. If it does so, however, then $k_1(\alpha_w) = \alpha_s > \alpha_w$ is violated. Hence, $k_1(v) \geq v$ for all $v \in S_w$. If $k_1(v) = r(v) > v$, then $k_1'(v) > \frac{f_w(v)}{f_s(r(v))} = r'(v)$. By a similar argument as before, $k_1(v) \leq r(v)$ for all $v \in S_w$.

[10]Regularity implies that $\kappa(v)$ is unique whenever it exists. J_s exceeds $J_w(v)$ if and only if the strong bidder's type is larger than $\kappa(v)$.

[11]Thus, (1) is the counterpart to Lemma 4.1 and Lemma 4.2 in Maskin and Riley (2000), in which they derived expressions for revenue in the FPA and SPA, respectively. The proofs of these lemmata are somewhat technical and offer little economic insight.

where

$$D(v|k_1,k_2)=\int_{k_2(v)}^{k_1(v)}(J_w(v)-J_s(x))\,dF_s(x). \tag{6}$$

$D(v|k_1,k_2)$ measures the consequences of the change in allocation for a fixed value of v—the seller obtains $J_w(v)$ by sacrificing $J_s(x)$ when he moves from a SPA to a FPA. In other words, D quantifies the trade-off discussed earlier.

The allocation is the same in both auctions if bidder w's type is below b_*; he loses, or $k_1=k_2=\beta_s$. Thus, $D(v|k_1,k_2)=0$ for $v\in[\beta_w,b_*]$. Using (4) to expand (5),

$$\begin{aligned} ER^1-ER^2=&\int_{\beta_w}^{b_*}(b_*-v)\,dF_w(v)\\ &+\int_{b_*}^{\alpha_w}(\max\{\beta_s-v,0\}+D(v|k_1,k_2))\,dF_w(v). \end{aligned} \tag{7}$$

The first term is trivially positive. The FPA is thus superior if the integrand in the second term is positive for all $v\in[b_*,\alpha_w]$.[12] As will be shown, this turns out to be the case if

$$\int_{v}^{k_1(v)}(f_w(v)-f_s(x))\,dx\geq 0\quad\text{for all } v\in S_w. \tag{8}$$

However, (8) contains an endogenous variable. At this stage, little more is known than $k_1(v)\in[v,r(v)]$. Thus, one possibility is to assume that (8) holds for all functions with this property. Since this condition may be cumbersome to check, a more straightforward sufficient condition is attractive. In particular, (8) is implied by

$$f_w(v)\geq f_s(x)\quad\text{for all } x\in[v,r(v)] \text{ and all } v\in S_w. \tag{9}$$

Alternatively, (8) can be replaced by a condition with similar flavor, namely,

$$F_s \text{ is regular}\quad\text{and}\quad\int_{v}^{r(v)}(f_w(v)-f_s(x))\,dx\geq 0\quad\text{for all } v\in S_w. \tag{10}$$

[12]If $\alpha_w=\alpha_s$, then $k_1'(\alpha_w)=f_w(\alpha_w)/f_s(\alpha_w)$, by (3), and, under appropriate differentiability assumptions, $D(\alpha_w|k_1,k_2)=D'(\alpha_w|k_1,k_2)=0>D''(\alpha_w|k_1,k_2)$ whenever $f_w(\alpha_w)\neq f_s(\alpha_w)$. Hence, D is negative when v is close to α_w. Thus, the approach does not work if $\alpha_w=\alpha_s$. Likewise, if $C\neq\emptyset$, then $F_w\leq_{hr}F_s$ is necessary. Otherwise, $J_s(v)>J_w(v)$ for some $v\in C$. Then, $D(v|k_1,k_2)<0$ if k_1 is close to v, which cannot be ruled out without tighter bounds on k_1.

Either condition implies $F_w \leq_{hr} F_s$. Clearly, (9) implies the second part of (10).[13]

THEOREM 1: *Assume that $F_w \leq_{rh} F_s$ and that either condition (9) or (10) holds. Then, the FPA generates strictly higher expected revenue than the SPA.*

PROOF: Because $F_w \leq_{rh} F_s$, $r(v) \geq k_1(v) \geq k_2(v) = \max\{\beta_s, v\} \geq \beta_s$. Let $I(v|k_1, k_2)$ denote the integrand in the second term in (7). Simple integration yields

$$\begin{aligned} I(v|k_1, k_2) = {} & \max\{\beta_s - v, 0\} + J_w(v)(F_s(k_1) - F_s(k_2)) \\ & + k_1(1 - F_s(k_1)) - k_2(1 - F_s(k_2)), \end{aligned}$$

which can be rearranged to produce

$$\begin{aligned} & I(v|k_1, k_2) \\ & \quad = \max\{\beta_s - v, 0\} - (k_2 - v)(1 - F_s(k_2)) \\ & \qquad + \frac{1 - F_s(k_1)}{f_w(v)}\left[f_w(v)(k_1 - v) - \frac{1 - F_w(v)}{1 - F_s(k_1)}(F_s(k_1) - F_s(k_2))\right]. \end{aligned}$$

Since $k_2 - v = \max\{\beta_s - v, 0\}$, the first two terms reduce to $\max\{\beta_s - v, 0\}F_s(k_2) = 0$. Finally, $1 - F_s(k_1) \geq 1 - F_w(v)$ because $k_1(v) \leq r(v)$, and it then follows that

$$\begin{aligned} I(v|k_1, k_2) & \geq \frac{1 - F_s(k_1)}{f_w(v)}\left[f_w(v)(k_1 - v) - (F_s(k_1) - F_s(k_2))\right] \\ & \propto \int_v^{k_1} (f_w(v) - f_s(x))\, dx, \end{aligned}$$

where the last step uses $f_s(x) = 0$ for $x \in [v, \beta_s)$. Then, if (9) holds, $I(v|k_1, k_2) \geq 0$ for all $v \in [b_*, \alpha_w]$. Indeed, since $k_1(v) < r(v)$ almost always, the inequality is strict almost always. Hence, the FPA is strictly more profitable than the SPA.

If (10) holds, then, fixing v, $D(v|k_1, k_2)$ is either increasing (if $J_w(v) \geq J_s(r(v))$), decreasing (if $J_w(v) \leq J_s(k_2(v))$), or inverse-U shaped (if $J_s(r(v)) \geq J_w(v) \geq J_s(k_2(v))$) in k_1 on $[k_2(v), r(v)]$. Thus, $I(v|k_1, k_2)$ is minimized (with respect to k_1) at one of the end-points. The function is nonnegative at $k_1 = k_2$.

[13]Both conditions imply D is positive if $v \in C$. It is more difficult to determine the sign of D when $v \in (b_*, \beta_s)$. However, the extra rent appropriated from bidder s solves the problem; $\beta_s - v + D(v|k_1, k_2) \geq 0$.

Hence, it is nonnegative for all $k_1 \in [k_2(v), r(v)]$ if it is nonnegative at $k_1 = r(v)$. Using the same steps as before,

$$I(v|k_1, k_2) = \frac{1 - F_s(r(v))}{f_w(v)} \int_v^{r(v)} (f_w(v) - f_s(x))\, dx$$

at $k_1 = r(v)$, which is nonnegative, by (10). *Q.E.D.*

Condition (9) has an interesting interpretation. Since $f_w(v) \geq f_s(v)$, the vertical distance between F_w and F_s is increasing in v on S_w. Likewise, $f_w(v) \geq f_s(r(v))$ implies that $r'(v) \geq 1$. Hence, the horizontal difference between F_s and F_w, $r(v) - v$, is also increasing on S_w. Formally, F_w is smaller than F_s in the *dispersive order*.[14] Moreover, since $F_w \leq_{rh} F_s$, the ratio $\frac{F_w(v)}{F_s(v)}$ is decreasing on C. In summary, the auction environment is one in which the absolute difference between bidders grows but the relative difference diminishes as the stakes get higher.

Condition (10) says that F_s is flatter than F_w "on average," on $[v, r(v)]$. What regularity buys is that it is sufficient to check that (8) is satisfied when evaluated at $k_1 = r(v)$.

4. DISCUSSION AND EXTENSIONS

I begin by comparing Theorem 1 to Maskin and Riley's examples. Next, applications and extensions are explored. The Supplemental Material (Kirkegaard (2012)) examines other auction formats. In particular, the SPA is worse than any convex combination of a SPA and a FPA.

4.1. *Maskin and Riley's Models*

Maskin and Riley (2000) showed that the FPA dominates if F_s is either a "shifted" or a "stretched" version of F_w. In fact, their assumptions imply (9). Thus, Maskin and Riley's (2000) examples are corollaries of Theorem 1, as verified next.

EXAMPLE 1—Horizontal Shifts: Assume that F_s is convex and F_w is obtained by shifting F_s to the left. That is, $F_w(v) = F_s(v + a)$, for $v \in [\beta_w, \alpha_w]$, where $a = \beta_s - \beta_w = \alpha_s - \alpha_w > 0$ and $\beta_w \geq 0$. Since $r(v) = v + a$, $f_w(v) = f_s(r(v)) \geq f_s(x)$

[14] See Shaked and Shantikumar (2007) for a detailed treatment of stochastic orders. In auction theory, the dispersive order has recently been used by, for example, Ganuza and Penalva (2010) and Mares and Swinkels (2011). It, and other orders of spread, were interpreted in Kirkegaard (2011).

for all $x \le r(v)$, by convexity. Thus, (9) is satisfied. Assume F_s is log-concave, implying

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v+a)}{F_s(v+a)} \le \frac{f_s(v)}{F_s(v)} \quad \text{for all } v \in C,$$

or $F_w \le_{rh} F_s$. Hence, the assumptions of the theorem are satisfied.

REMARK 1: Maskin and Riley (2000) assumed that $J_w(v) \le 0$ for v close to β_w. This assumption is not necessary here because (7) makes better use of the fact that more rent is extracted from bidder s with type β_s in the FPA. On the other hand, Maskin and Riley (2000) allowed for a mass point at β_i, or $F_i(\beta_i) \ge 0$.

EXAMPLE 2—Truncations and Stretches: Assume F_s is log-concave and that F_w is a truncation of F_s, that is, $F_w(v) = \frac{F_s(v)}{F_s(\alpha_w)}$, $v \in [\beta_w, \alpha_w]$, with $\beta_w = \beta_s < \alpha_w < \alpha_s$. Since $\frac{F_w(v)}{F_s(v)}$ is constant on C, $F_w \le_{rh} F_s$. By log-concavity,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v)}{F_s(v)} \ge \frac{f_s(x)}{F_s(x)}$$

for any $x \in [v, r(v)]$. For any x in this range, $F_s(x) \le F_w(v)$. The above inequality then necessitates that $f_s(x) \le f_w(v)$ for all $x \in [v, r(v)]$, implying (9).

REMARK 2: Maskin and Riley's (2000) setup is slightly different. They said that F_s is obtained by "stretching" F_w, so that $F_s(v) = \lambda F_w(v)$ on $[\beta_w, \alpha_w]$, for some $\lambda \in (0, 1)$. This leaves the problem of what F_s looks like on $(\alpha_w, \alpha_s]$, and they were then forced to make restrictive assumptions on this as well (compare (4.13) in their paper to (9) in the current paper). Unfortunately, their conditions rule out the much-studied class of convex power distributions, where $F_s(v) = (v/\alpha_s)^{\gamma_s}$ for some $\gamma_s > 1$, $v \in [0, \alpha_s]$. In contrast, the power distribution satisfies all the assumptions in Example 2.

Kirkegaard (2011) showed that if F_s is convex, any environment that "lies between" the shift and stretch models in a natural way satisfies the conditions in Theorem 1. Similar results are derived for the cases where f_s is decreasing or nonmonotonic.

4.2. *Stochastic Shifts*

Example 1 can be interpreted as follows: Bidders independently draw a valuation, v, from the distribution F_w. For bidder w, this valuation is his type. For bidder s, it is only a component of his type. He may expect synergies between existing objects and the object for sale, or he may suffer a negative externality should bidder w win. Let the additional component be worth a, $a \ge 0$. His type is then $u_s(v, a) = v + a$. Contrary to Example 1, assume a is private information. I show that the revenue ranking survives under assumptions that are similar in spirit to those in Example 1.

Unless stated otherwise, assume a is drawn from a nondegenerate distribution, G, with no mass-points, density g, and support $[\beta, \alpha]$, $0 \leq \beta < \alpha < \infty$. The strong bidder's distribution, F_s, is given by the convolution of F_w and G.[15]

A random variable, F, is said to be *dispersive* if the convolution of F and *any* other distribution is more disperse than F. In other words, F is dispersive if adding more noise makes the resulting distribution more disperse. A fundamental result, due to Droste and Wefelmeyer (1985), building on Lewis and Thompson (1981), says that F is dispersive if and only if its density is log-concave. Hence, assume f_w is log-concave; F_s is then more disperse than F_w.[16] As in Example 1, assume F_w is convex.

PROPOSITION 1: *The FPA yields strictly higher expected revenue than the SPA in the "stochastic shift" model if f_w is increasing and log-concave.*

PROOF: If $C \neq \emptyset$, then, for any $v \in C = [\beta_w + \beta, \alpha_w]$,

$$\frac{F_s(v)}{F_w(v)} = \int_{\beta}^{\min\{\alpha, v-\beta_w\}} \frac{F_w(v-z)}{F_w(v)} g(z)\, dz$$

is increasing in v because F_w is log-concave. Thus, $F_w \leq_{rh} F_s$. Since F_s is more disperse than F_w, $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$. Since f_w is increasing, $f_w(v) \geq f_w(x) \geq f_s(r(x))$ for all $x \in [\beta_w, v]$ and any $v \in S_w$. Hence, $f_w(v) \geq f_s(z)$ for all $z \leq r(v)$. Condition (9) is thus satisfied. *Q.E.D.*

Bidder s is "strong" because he fears a negative externality, for example. Consider now an alternative model in which bidder w is "weak" because he incurs a positive externality if bidder s wins. When the distribution of the externality, G, is degenerate, the two models are essentially equivalent and the FPA dominates, at least as long as $\beta_w \geq 0$ (as in Example 1). The next example allows G to be nondegenerate.

EXAMPLE 3—Positive Externalities: Bidder s draws his type from F_s, which is regular and has $\beta_s = 0$. There is an implicit or explicit minimum bid of zero.

[15]This convolution has the property that $f_s(\alpha_s) = 0$, which violates the assumptions that are typically imposed to analyze the FPA. Nevertheless, this complication affects only the known proofs of uniqueness, such as that in Lebrun (2006). Kirkegaard (2011) contains a more detailed discussion.

[16]F_w is log-concave (as in Example 1) because f_w is log-concave; see, for example, Bagnoli and Bergstrom (2005)). The assumption that f_w is increasing can be replaced by conditions on g. See the Supplemental Material. The Supplemental Material also considers multiplicative uncertainty, $u_s(v, a) = va$.

Note that bidder s will always bid. Bidder w's type, $v = x - a$, is then the difference between his consumption value of the object (x) and his payoff if bidder s wins (a). Here, x and a are drawn from F_s and G, respectively. Assume $a = \beta_s = 0$ with probability $p \in (0, 1)$ and $a = \alpha_s$ with probability $1 - p$. Bidder w does not bid if his type falls below 0 (or $a = \alpha_s$). Hence, $k_1(v) = k_2(v) = \beta_s$ and so $D(v|k_1, k_2) = 0$ for $v \leq 0$. If $v \in C = S_s$, then $F_w(v) = (1 - p) + pF_s(v)$, and thus $F_w \leq_{rh} F_s$ and $J_w(v) = J_s(v)$ for $v \in C$. By regularity, $D(v|k_1, k_2) \leq 0$ for all $v \in C$. Hence the SPA is superior.

REMARK 3: Example 3 can be viewed as a simplified version of Maskin and Riley's third example, in which F_w has a mass-point at β_s and $\alpha_w = \alpha_s$. They also used mechanism design to prove that the SPA dominates. Thus, the addition of the mechanism design approach in Theorem 1 methodologically unifies their three examples.

Technically, an important difference between the negative- and positive-externality models is that $\alpha_w < \alpha_s$ in the former but not necessarily the latter. Proposition 1 and Example 3 suggest that the nature of the externality is important. Similarly, Proposition 1 says that the ranking in Example 1 is robust to private information about a negative externality. However, Examples 1 and 3 suggest that the ranking is more sensitive when positive externalities are involved. Of course, in the latter case, a minimum bid of zero may function as a binding reserve price. I turn to reserve prices next.

4.3. *Reserve Prices*

As in Maskin and Riley (2000), reserve prices have so far been all but ignored. Reserve prices are often used in practice, however. Fortunately, reserve prices are easily handled.[17] Let τ denote the reserve price.

Assume that the reserve price has "bite," but is not prohibitive, or $\tau \in [\beta_s, \alpha_w) \neq \emptyset$.[18] A reserve price "shuts out" some types, but this effect is the same in both auctions. For those types that are not excluded, it remains the case that the FPA dominates contingent on the weak bidder's type. Thus, the revenue ranking is intact. If $\tau < \beta_s$, the proof is completed by modifying (4) and proceeding as in Theorem 1.

[17]Proposition 2 applies only to one-dimensional types. If a bidder experiences an externality if his rival wins, as in Section 4.2, then his type is multidimensional when a binding reserve price is in effect. The reason is that he values the two outcomes "the rival wins" and "no one wins" differently. In contrast, the latter outcome is ruled out in Example 3, since bidder s always bids.

[18]The two auctions are revenue equivalent if $\tau \geq \alpha_w$ because, in that case, the winner is the same in the two auctions and $u_s^1(\beta_s) = u_s^2(\beta_s)$. In general, bidding strategies depend on the reserve price. In particular, $k_1(v)$ depends on τ even if $v > \tau$, although that dependence is suppressed here. However, the property that $k_1(v) \in [v, r(v)]$ for all $v \in [\tau, \alpha_w]$ is unchanged. See Lebrun (1999).

PROPOSITION 2: *Assume that* (i) $F_w \leq_{rh} F_s$, (ii) *condition* (9) *or* (10) *holds, and* (iii) $\tau < \alpha_w$. *Then, the FPA with reserve price* τ *generates strictly higher expected revenue than the SPA with the same reserve price*.

PROOF: Assume first that $\tau \in [\beta_s, \alpha_w) \neq \emptyset$. Because $\tau \geq \beta_s \geq \beta_w$, $u_i^k(\beta_i) = 0$ for $i = s, w$ and $k = 1, 2$. A bidder stays out of the auction if and only if his type is below τ. Modifying (1) yields

$$ER^k(\tau) = \int_{\beta_w}^{\tau} \left(J_w(v) \times 0 + \int_{\tau}^{\alpha_s} J_s(x)\, dF_s(x) \right) dF_w(v)$$
$$+ \int_{\tau}^{\alpha_w} \left(J_w(v) F_s(k(v)) + \int_{k(v)}^{\alpha_s} J_s(x)\, dF_s(x) \right) dF_w(v).$$

Hence,

$$ER^1(\tau) - ER^2(\tau) = \int_{\tau}^{\alpha_w} D(v|k_1, k_2)\, dF_w(v),$$

which, as shown in the proof of Theorem 1, is positive since $\tau \geq \beta_s$. Consider next the case where $\tau < \min\{\beta_s, \alpha_w\}$. The reserve price has no effect in either auction if $\tau \leq \beta_w$. Thus, assume $\beta_w < \tau < \min\{\beta_s, \alpha_w\}$, and, as before, let b_* denote the bid submitted by bidder s with type β_s. It is possible that b_* coincides with τ; at the very least, $\min\{\beta_s, \alpha_w\} \geq b_* \geq \tau$. Hence, (4) becomes

$$u_s^2(\beta_s) - u_s^1(\beta_s)$$
$$= F_w(\tau)(\beta_s - \tau) + \int_{\tau}^{\alpha_w} \max\{\beta_s - v, 0\}\, dF_w(v) - F_w(b_*)(\beta_s - b_*)$$
$$= \int_{b_*}^{\alpha_w} \max\{\beta_s - v, 0\}\, dF_w(v)$$
$$+ \int_{\tau}^{b_*} (b_* - v)\, dF_w(v) + F_w(\tau)(b_* - \tau).$$

An expression similar to (7) can now be obtained and the proof concluded by following the same steps as in Theorem 1. *Q.E.D.*

A reserve price may reverse the ranking if (9) and (10) are violated. As explained in footnote 12, $D(v|k_1, k_2)$ is negative if v is close to α_w and $\alpha_w = \alpha_s$. Hence, if τ is large, only types for which $D(v|k_1, k_2) \leq 0$ are active. The SPA is thus superior. However, Lebrun (1996) considered a model (see Section 4.5) with $\alpha_w = \alpha_s$, in which the FPA dominates when $\tau = 0$. See also the discussion following Remark 3.

4.4. *Larger Auctions*

Consider auctions with $m \geq 1$ strong bidders and $n \geq 1$ weak bidders. With symmetric and monotonic strategies within each group, the auction winner must have the highest type within his group. Hence, (1) becomes

$$ER^k = \int_{\beta_w}^{\alpha_w} \left(J_w(v) F_s(k(v))^m + \int_{k(v)}^{\alpha_s} J_s(s)\, dF_s(x)^m \right) dF_w(v)^n \\ - n u_w^k(\beta_w) - m u_s^k(\beta_s).$$

The counterpart to (6) is

$$D_m(v|k_1, k_2) = \int_{k_2(v)}^{k_1(v)} (J_w(v) - J_s(x))\, dF_s(x)^m.$$

Once again, it can be shown that $k_1(v) \in [v, r(v)]$ and $u_w^k(\beta_w) = 0$ in both auctions. If $m = 1$, the boundary condition remains $k_1(\alpha_w) = \alpha_s$, and (4) is as before, but with $F_w(v)^n$ in place of $F_w(v)$. Hence, (7) is largely unchanged, with $F_w(v)^n$ replacing $F_w(v)$. Thus, the proof of Theorem 1 applies to auctions with $n > 1$ weak bidders.

PROPOSITION 3: *Assume that $F_w \leq_{rh} F_s$ and that condition* (9) *or* (10) *holds. Then, the FPA generates strictly higher expected revenue than the SPA when $n \geq m = 1$.*

Assume now that $m \geq 2$ and $\beta_s < \alpha_w$. Moreover, assume the asymmetry is so small that $k_1(\alpha_w) = \alpha_s$. Since $J_w(\alpha_w) = \alpha_w$, evaluating D_m at $v = \alpha_w$ yields

$$D_m(\alpha_w|k_1, k_2) = \alpha_w(1 - F_s(\alpha_w)^m) - \int_{\alpha_w}^{\alpha_s} J_s(x)\, dF_s(x)^m < 0.$$

Thus, when $m \geq 2$, the approach does not work in general, for the same reason it does not work when $\alpha_w = \alpha_s$ and $m = 1$. Here, the problem is caused by $k_1(\alpha_w) = \alpha_s$. In the Supplemental Material, I show that the ranking can sometimes be resurrected if the asymmetry is "large enough," in which case $k_1(\alpha_w) < \alpha_s$ is possible. For example, the FPA dominates if the overlap of supports is small in the "shift" model, or if F_s is stretched a lot in the "stretch" model.

4.5. *Concluding Remarks*

The proof of Theorem 1 involves two steps. In the first step, mechanism design is used to obtain a simple expression for the difference in expected revenues between the FPA and the SPA. The second step is to sign the difference.

In this regard, while conditions (9) and (10) are restrictive, it is easy to see why they are imposed and note directions in which one can hope to relax them.

First, the theorem relies on $k_1(v) \in [v, r(v)]$, which is all that can be inferred from the current literature. Clearly, a tighter upper bound would permit the conditions to be relaxed. As I show in the Supplemental Material, progress in this direction can be made from k_1' in the system (3) by exploiting known lower bounds on the serious bids. Likewise, Mares and Swinkels (2011) developed tools that may help bound the slope of k_1.

Second, the theorem stringently requires that the integrand, $I(v|k_1, k_2)$, in (7) is nonnegative for *every* $v \in S_w$. If $\beta_w = \beta_s$ and $F_w \leq_{hr} F_s$, then $I(v|k_1, k_2) \geq 0$ when v is close to β_w, but not necessarily when v is large (see, e.g., footnote 12). In such cases, Lebrun (1996) observed that a tighter *lower* bound on k_1 may be useful. Then, $I(v|k_1, k_2)$ can be bounded below when v is small and I is positive. This may be enough to outweigh the negative values when v is large.

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Manuscript received February, 2011; final revision received December, 2011.