

SUPPLEMENT TO “A MECHANISM DESIGN APPROACH TO  
RANKING ASYMMETRIC AUCTIONS”

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A1. OTHER AUCTION FORMATS

THE CONCLUSION IN Theorem 1 is made possible because the allocation in the SPA can be described precisely, while the possible allocations in the FPA can be narrowed down to a relatively small set. It is not necessary to know the exact allocation in the FPA.

Aside from the issue of how much rent is extracted from  $\beta_i$  types, Theorem 1 therefore really says that the SPA is a poor auction format if the objective is to generate high expected revenue. For instance, if  $\beta_w = \beta_s$  and  $u_w(\beta_w) = u_s(\beta_s) = 0$ , any auction with  $k(v) \in [v, r(v)]$  is more profitable than the SPA if condition (9) or (10) in the main paper is satisfied. In other words, it is profitable to design an auction that favors the weak bidder moderately.

It has long been understood that *optimal* auctions typically favor the weak bidder; see, for example, McAfee and McMillan (1989).<sup>1</sup> Based on this property, Klemperer (1999) argued that “it is plausible that a first-price auction may be more profitable [...] than a second-price auction.” However, this paper establishes a bound on the amount of favoritism that can safely be extended to the weak bidder. Specifically, any mechanism where the weak bidder wins more often than is efficient but less often than he would in a counterfactual symmetric auction against another weak bidder is more profitable than a SPA.

To illustrate, define a winner-pay auction to be an auction in which the winner pays a proportion  $\gamma$  of his own bid and  $(1 - \gamma)$  of the losing bid, and the loser does not pay,  $\gamma \in [0, 1]$ . The FPA corresponds to  $\gamma = 1$ , the SPA to  $\gamma = 0$ .

**PROPOSITION A1:** *Assume that (i)  $F_w \leq_{rh} F_s$ , (ii) condition (9) or (10) holds, and (iii)  $\beta_s = \beta_w$ . Then, the SPA yields strictly the lowest expected revenue of all winner-pay auctions.*

**PROOF:** Consider  $\gamma \in (0, 1]$ , that is, an auction that is not a pure SPA. In this case, the two bidders must share the same maximal bid,  $\bar{b}$ . Let  $\phi_i(b)$  denote bidder  $i$ 's inverse bidding strategy,  $i = s, w$ , where  $b \in [\beta_w, \bar{b}]$ . Assume, for the

<sup>1</sup>In Maskin and Riley's (2000) one model where the SPA is superior to the FPA, the optimal auction would in fact discriminate *against* the weak bidder. In contrast, for the proof of Theorem 1 to work, it is necessary that  $F_s$  dominates  $F_w$  in terms of the hazard rate (see footnote 12 in the main paper). An optimal auction therefore favors the weak bidder.

moment, that the bidding strategy is strictly increasing and differentiable. If bidder  $i$  has type  $v$ , his problem is

$$\max_b \int_{\beta_w}^b [v - (\gamma b + (1 - \gamma)x)] dF_j(\phi_j(x)),$$

where  $j \neq i$  denotes bidder  $i$ 's rival. The first order condition is

$$\frac{f_j(\phi_j(b))}{F_j(\phi_j(b))} \phi_j'(b) = \frac{\gamma}{v - b}.$$

In equilibrium, bidder  $i$  bids  $b$  if his type is  $v = \phi_i(b)$ . Substituting into the first order conditions produces the system of differential equations

$$\frac{f_w(\phi_w(b))}{F_w(\phi_w(b))} \phi_w'(b) = \frac{\gamma}{\phi_s(b) - b}, \quad \frac{f_s(\phi_s(b))}{F_s(\phi_s(b))} \phi_s'(b) = \frac{\gamma}{\phi_w(b) - b}.$$

The only difference from the FPA is that  $\gamma \in (0, 1]$  (the boundary conditions are the same). The proofs in Maskin and Riley (2000) can then be repeated to conclude that the auction has the same features as a FPA,  $k_\gamma(v) \in [v, r(v)]$  for all  $\gamma \in (0, 1]$ . Since bidders with type  $\beta_i$  earn zero rent for all  $\gamma \in [0, 1]$ , Theorem 1 applies directly. *Q.E.D.*

Not all auctions have the property that  $k(v) \in [v, r(v)]$ . The most prominent example is probably the all-pay auction for which  $k(v) < v$  when  $v$  is small. The reason is that a weak bidder with a low type is deterred from bidding (which is a sunk cost in an all-pay auction) when facing a rival he perceives as strong. Thus, it is not possible to rank the SPA and the all-pay auction using the method developed in this paper.

## A2. TYPES WITH TWO COMPONENTS

Here, I first reexamine the ‘‘stochastic shift’’ model from Section 4.2 in the main paper. The condition that  $f_w$  is increasing is relaxed. Then, I consider a model where the strong bidder's type is obtained by multiplying two random variables. For both models, the following lemma is useful. Here, a function is said to be unimodal if it is monotonic or has an inverse-U shape (it may have regions where it is flat). If  $f_s$  is unimodal but not monotonic, let  $\widehat{v}$  denote the smallest type at the peak, such that  $f_s(\widehat{v}) > f_s(v)$  for all  $v < \widehat{v}$ .

**LEMMA A1:** *Condition (9) is satisfied if  $\beta_w = \beta_s$ ,  $f_s$  and  $f_w$  are unimodal on  $S_s$  and  $S_w$ , respectively,  $f_w(v) \geq f_s(v)$  and  $f_w(v) \geq f_s(r(v))$  for all  $v \in S_w$ , and  $\widehat{v} \leq \alpha_w$ .*

PROOF: The lemma is straightforward if  $f_s$  is monotonic. Hence, assume  $f_s$  is not monotonic;  $\hat{v}$  is in the interior of  $S_s$ . By assumption,  $\hat{v} \in S_w$ . Since  $F_s$  is more disperse than  $F_w$ ,  $f_w(v) \geq f_s(r(v))$  for all  $v \in [\beta_w, r^{-1}(\hat{v})]$ , an interval on which  $f_s$  is increasing since  $r^{-1}(\hat{v}) < \hat{v}$ . Thus, condition (9) is satisfied for all  $v \in [\beta_w, r^{-1}(\hat{v})]$ . Since  $f_w(r^{-1}(\hat{v})) \geq f_s(\hat{v})$ ,  $f_w(\hat{v}) \geq f_s(\hat{v})$ , and  $f_w$  is itself unimodal, it must hold that  $f_w(v) \geq f_s(\hat{v})$  for all  $v \in [r^{-1}(\hat{v}), \hat{v}]$ , and since  $f_s$  attains its peak at  $\hat{v}$ , condition (9) must also be satisfied on the interval  $[r^{-1}(\hat{v}), \hat{v}]$ . For  $v > \hat{v}$ ,  $f_w(v) \geq f_s(v)$  combines with the monotonicity of  $f_s$  on that region to ensure that (9) is satisfied here as well. Q.E.D.

### A2.1. Stochastic Shifts

To generalize Example 1 to permit  $G$  to be nondegenerate and obtain Proposition 1, the assumption that  $F_w$  is log-concave need only be replaced with the slightly stronger assumption that  $f_w$  is log-concave. By imposing more conditions on  $g$ , the assumption that  $f_w$  is increasing can also be relaxed. The intention is to use Lemma A1.

However, the convolution of two unimodal densities is not necessarily unimodal. Ibragimov (1956) has shown that the convolution of a log-concave density with any unimodal density is itself unimodal. Hence, a log-concave function is sometimes referred to as strongly unimodal. Since it has already been assumed that  $f_w$  is log-concave, unimodality of  $f_s$  is then guaranteed if  $g$  is unimodal.<sup>2</sup> It will also be assumed that  $\beta = 0$ , implying that  $\beta_w = \beta_s$ . Log-concavity of  $f_w$  also implies that  $F_s$  is more disperse than  $F_w$ , as mentioned in the main paper. However, to apply Lemma A1, it is also necessary that  $F_w$  is steeper than  $F_s$  on  $S_w$ . Unfortunately, a convolution may increase the density locally. Thus, additional assumptions are required.

PROPOSITION A2: *The FPA yields strictly higher expected revenue than the SPA in the additive model if  $f_w$  is log-concave,  $\beta = 0$ , and  $g$  is decreasing and satisfies*

$$(A1) \quad f_w(\alpha_w) \geq \int_0^\alpha f_w(\alpha_w - z)g(z) dz.$$

PROOF: For  $v \in C = S_w$ ,

$$\frac{f_s(v)}{f_w(v)} = \frac{\int_0^\alpha f_w(v - z)g(z) dz}{f_w(v)} = \int_0^\alpha \frac{f_w(v - z)}{f_w(v)} g(z) dz,$$

<sup>2</sup>Miravete (2011) examined the properties of convolutions of two log-concave densities. He emphasized their relevance to models of asymmetric information, including multidimensional screening.

where  $f_w(v - z) = 0$  if  $v - z \leq \beta_w$ . Thus,

$$\frac{d}{dv} \left( \frac{f_s(v)}{f_w(v)} \right) \geq \int_0^\alpha \left( \frac{f'_w(v - z)}{f_w(v - z)} - \frac{f'_w(v)}{f_w(v)} \right) \frac{f_w(v - z)}{f_w(v)} g(z) dz,$$

which is positive since  $f_w$  is log-concave. Hence,  $F_s$  dominates  $F_w$  in terms of the likelihood ratio (this conclusion relies on  $\beta = 0$ ). Condition (A1) is equivalent to  $\frac{f_s(\alpha_w)}{f_w(\alpha_w)} \leq 1$ . Thus, since  $\frac{f_s(v)}{f_w(v)}$  is increasing on  $S_w$ ,  $f_w(v) \geq f_s(v)$  for all  $v \in S_w$ . Since the convolution of  $f_w$  and  $g$  is unimodal, Lemma A1 applies if  $f_s$  peaks to the left of  $\alpha_w$ . However, when  $v > \alpha_w$ ,

$$f_s(v) = \int_{\max\{\beta_w, v - \alpha\}}^{\alpha_w} g(v - z) f_w(z) dz,$$

which is decreasing in  $v$  when  $g$  is decreasing. Consequently,  $f_s$  peaks at or before  $\alpha_w$ . Q.E.D.

Condition (A1) requires that  $f_w(\alpha_w)$  exceeds a weighted average of  $f_w$  over the interval  $[\alpha_w - \alpha, \alpha_w]$  (on which  $f_w$  may be zero if  $\alpha_w - \alpha \leq \beta_w$ ). Thus, if  $\alpha$  is small, it rules out that  $f_w$  is decreasing. However,  $f_w$  may be nonmonotonic as long as it does not “dip down” too much after it has passed its peak. The condition is less restrictive if  $\alpha$  is large, such that the asymmetry between bidders is large.

### A2.2. A Multiplicative Model

Assume that the strong bidder’s type has two components,  $v$  and  $a$ . The former is drawn from  $F_w$ . The latter is drawn from  $G$ , which has support  $[\beta, \alpha]$ . The strong bidder’s type is then  $u(v, a) = va$ .

Cuculescu and Theodorescu (1998) examined multiplication of random variables. For nonnegative random variables, they showed that log-concavity of  $f_w$  must be replaced by log-concavity of  $f_w(e^v)$  to obtain results that mirror those for addition of random variables.<sup>3</sup> That is, if a random variable with this property is multiplied with another random variable with unimodal density, then the resulting variable also has unimodal density. Likewise, the multiplicative convolution of  $F_w$  and a nondegenerate random variable  $G$  is more “star disperse” than  $F_w$  itself (i.e.,  $\frac{r(v)}{v}$  is increasing).

Assume  $\beta \geq 1$ , such that  $\beta_s \geq \beta_w$  and  $F_s$  (the multiplicative convolution of  $F_w$  and  $G$ ) first order stochastically dominates  $F_w$ . Then,  $\frac{r(v)}{v}$  increasing implies  $r(v) - v$  increasing. In other words,  $F_s$  is more disperse than  $F_w$ .

<sup>3</sup>Jewitt (1987, footnote 15) made a related observation in a model of risk aversion with two sources of uncertainty. Jewitt (1987) used tools from total positivity, as in Miravete (2011).

Turning to reverse hazard rate dominance, note first that, for any  $v \in C$ ,

$$\frac{F_s(v)}{F_w(v)} = \int_{\beta}^{\alpha} \frac{F_w\left(\frac{v}{z}\right)}{F_w(v)} g(z) dz.$$

Since  $f_w(e^v)$  is log-concave,  $F_w(e^v)$  is log-concave as well. Equivalently,  $vf_w(v)/F_w(v)$  is decreasing. Thus, since  $z \geq 1$ ,

$$\frac{d}{dv} \left( \frac{F_w\left(\frac{v}{z}\right)}{F_w(v)} \right) = \frac{1}{v} \frac{F_w\left(\frac{v}{z}\right)}{F_w(v)} \left( \frac{\left(\frac{v}{z}\right) f_w\left(\frac{v}{z}\right)}{F_w\left(\frac{v}{z}\right)} - \frac{vf_w(v)}{F_w(v)} \right) \geq 0,$$

and  $F_w \leq_{rh} F_s$  follows. A counterpart to Proposition 1 in the main paper is now immediate.

**PROPOSITION A3:** *The FPA yields strictly higher expected revenue than the SPA in the multiplicative model if  $f_w(e^v)$  is increasing and log-concave.*

The proof is identical to the proof of Proposition 1 in the main paper.

To relax the assumption that  $f_w$  is monotonic, it is necessary to impose more restrictions on  $g$  instead. As in the additive model, Lemma A1 is used. The proof of the following proposition is omitted since it is analogous to the proof of Proposition A2.

**PROPOSITION A4:** *The FPA yields strictly higher expected revenue than the SPA in the multiplicative model if  $f_w(e^v)$  is log-concave,  $\beta = 1$ , and  $g$  is decreasing and satisfies*

$$(A2) \quad f_w(\alpha_w) \geq \int_1^{\alpha} f_w\left(\frac{\alpha_w}{z}\right) \frac{g(z)}{z} dz.$$

### A3. AUCTIONS WITH MANY STRONG BIDDERS

This part of the Supplemental Material provides the details behind the claim in Section 4.4 of the main paper that the FPA is superior to the SPA if the asymmetry is large enough and  $m \geq 2$ . Let  $\bar{b}_s = b_s(\alpha_s)$  and  $\bar{b}_w = b_w(\alpha_w)$  denote the maximum bids of the strong and weak bidders, respectively. When  $m \geq 2$ , it is possible that  $\bar{b}_s > \bar{b}_w$ .

#### A3.1. Small Overlap

Assume that  $\alpha_w$  is “close” to  $\beta_s$  such that there is little overlap between the supports. As a starting point, if  $\beta_s = \alpha_w$ , then (i)  $J_w(\alpha_w) = \alpha_w = \beta_s > J_s(\beta_s)$ ,

and (ii)  $\bar{b}_s > \bar{b}_w$  in a FPA (a strong bidder with type  $\beta_s$  bids  $\beta_s = \alpha_w \geq \bar{b}_w$ , and his strategy is strictly increasing). If  $\alpha_w$  is “slightly above”  $\beta_s$ , it must remain the case that  $\bar{b}_s > \bar{b}_w$ , or  $k_1(\alpha_w) < \alpha_s$ , with  $J_w(\alpha_w) > J_s(x)$  for all  $x \in [\beta_s, k_1(\alpha_w)]$ . Moreover, by continuity,

$$(A3) \quad J_w(v) > J_s(x) \quad \text{for all } v \in [\beta_s, \alpha_w] \text{ and } x \in [\beta_s, k_1(\alpha_w)],$$

when  $\beta_s$  and  $\alpha_w$  are sufficiently close.<sup>4</sup> In the following, when the overlap is said to be “small,” it should be taken to mean that (A3) is satisfied.

In this case, the FPA yields higher expected revenue than the SPA because the weak bidders are winning more often against strong bidders with inferior virtual valuation. Recall that the two are revenue equivalent if there is no overlap.

**PROPOSITION A5:** *Assume that  $F_w \leq_{rh} F_s$  and the overlap is small. Then, the FPA generates strictly higher expected revenue than the SPA when  $m \geq 2$ ,  $n \geq 1$ .*

**PROOF:** Both auctions ensure that  $u_i^k(\beta_i) = 0$ ,  $i = s, w$ . A weak bidder with type below  $\beta_s$  loses both auctions (competition between the strong bidders ensures that any serious bid must be at least  $\beta_s$ ). By (i), a weak bidder with type  $v \in (\beta_s, \alpha_w]$  wins more often in the FPA than in the SPA. By (ii) or (A3), the winner’s virtual valuation is no lower in the FPA, and may be higher. In other words,  $D_m$  is positive. *Q.E.D.*

**EXAMPLE 1—Continued:** Consider a many-bidder extension of Example 1, with  $m \geq 2$ . If  $F_s$  is shifted far to the right such that there is no overlap between supports, then the two auctions are revenue equivalent. The same is true if  $a = 0$ , in which case bidders are homogenous. Proposition A5 then states that the FPA is superior for large “interior” values of  $a$ . A comparison cannot be made for small values. Recall that Proposition A5 does not require  $F_s$  to be a “shifted” version of  $F_w$ .

### A3.2. Large Stretches

Assume the asymmetry between bidders is so large that  $\bar{b}_s > \bar{b}_w$ . Define  $\bar{\alpha}_s \equiv k_1(\alpha_w)$  as the highest strong type that competes with the weak bidders. A strong bidder outbids the weak bidders with probability 1 if his type exceeds  $\bar{\alpha}_s$ ,  $\bar{\alpha}_s < \alpha_s$ .

Consider the consequences of “stretching” the strong bidder’s distribution, transforming  $F_s$  with support  $[\beta_s, \alpha_s]$  to  $F_s^\lambda$  with support  $[\beta_s, \alpha_s^\lambda]$ ,  $\alpha_s^\lambda > \alpha_s$ , such that  $F_s^\lambda = \lambda F_s$  on the subinterval  $v \in [\beta_s, \alpha_s]$ , with  $\lambda \in (0, 1)$ . More concisely,

<sup>4</sup> $F_s$  need not be more disperse than  $F_w$ . For instance, the former could have a smaller support than the latter. It is a general property that  $J_i(\alpha_i) = \alpha_i$  and  $J_i(\beta_i) < \beta_i$ ,  $i = s, w$ .

$F_s$  is a truncation of  $F_s^\lambda$ . Importantly,  $F_s$  and  $F_s^\lambda$  have the same reverse hazard rate on  $[\beta_s, \alpha_s]$  and therefore on  $[\beta_s, \bar{\alpha}_s]$ . Thus, if  $F_s$  dominates  $F_w$  in terms of the reverse hazard rate, so does  $F_s^\lambda$ . Likewise, the system of first order conditions from the bidders' maximization problems is unchanged at bids below  $\bar{b}_w$ . This can be seen by examining the systems in Maskin and Riley (2000) or Lebrun (2006). The implication is that weak bidders regardless of type and strong bidders with type below  $\bar{\alpha}_s$  use the exact same strategy in either case. Consequently,  $k_1$  is the same in both environments.

For types in  $[\beta_s, \bar{\alpha}_s]$ , the strong bidders' virtual valuation is

$$J_s^\lambda(v) = v - \frac{1 - F_s^\lambda(v)}{f_s^\lambda(v)} = v - \frac{\frac{1}{\lambda} - F_s(v)}{f_s(v)}$$

as a function of  $\lambda$ . The important property is that  $J_s^\lambda$  decreases without bound as  $F_s$  is stretched more and more (i.e., as  $\lambda$  decreases and goes to zero). Thus,

$$(A4) \quad J_w(v) \geq J_s^\lambda(x) \quad \text{for all } v \in [\beta_w, \alpha_w] \text{ and } x \in [\beta_s, \bar{\alpha}_s]$$

when  $F_s$  is stretched sufficiently much. In the following, when  $F_s$  is said to be stretched "a lot," it should be taken to mean that (A4) is satisfied.

**PROPOSITION A6:** *Assume that  $F_w \leq_{rh} F_s$  and  $F_s$  is stretched a lot. Then, the FPA generates strictly higher expected revenue than the SPA when  $m \geq 2$ ,  $n \geq 1$ .*

The proof is identical to the proof of Proposition A5.

**EXAMPLE 2—Continued:** Proposition A6 applies directly if  $F_w$  is a truncation of  $F_s$ , in which case  $F_s(v) = \lambda F_w(v)$  on  $v \in [\beta_w, \alpha_w]$ . As with Example 1, the two auctions are revenue equivalent if the bidders are homogenous, or  $\lambda = 1$ . A comparison cannot be made if  $\lambda$  is close to 1, or the asymmetry is small. By Proposition A6, however, the FPA is superior when  $\lambda$  is close to zero. Note that Proposition A6 does not require  $F_s$  and  $F_w$  to be related in any way other than through reverse hazard rate dominance (it does not imply one is a truncation of the other), nor does it require log-concavity.

#### A4. BOUNDING THE ALLOCATION

Recall from (3) in the main paper that the system of differential equations can be written

$$(A5) \quad k_1'(v) = \frac{F_s(k_1(v))}{f_s(k_1(v))} \frac{f_w(v)}{F_w(v)} \frac{k_1(v) - b_w(v)}{v - b_w(v)},$$

$$b_w'(v) = \frac{f_w(v)}{F_w(v)} (k_1(v) - b_w(v))$$

for  $v \in (b_*, \alpha_w]$ , where  $b_* \in [\beta_w, \beta_s]$  is the smallest serious bid (see Maskin and Riley (2000) for a characterization of  $b_*$ ). The system must satisfy the condition  $k_1(\alpha_w) = \alpha_s$ . It is also the case that  $k_1(b_*) = \beta_s$  and  $b_w(b_*) = b_*$ . Fix  $k_1$  and  $v \in (b_*, \alpha_w]$ , with  $k_1 \geq v$ , and note that

$$(A6) \quad \frac{d}{db_w} \left( \frac{k_1 - b_w}{v - b_w} \right) = \frac{k_1 - v}{(v - b_w)^2} \geq 0.$$

The implication is that if  $b_w$  can be bounded, then the last term in  $k'_1(v)$  can be bounded as well.

To illustrate the usefulness of bounds on  $b_w$ , start with the crudest bound, namely,  $b_w(v) > b_*$  for  $v \in (b_*, \alpha_w]$ . Consider the function  $\bar{k}$ , with domain  $(b_*, \alpha_w]$ . Assume  $\bar{k}(\alpha_w) = \alpha_s$ , with

$$(A7) \quad \bar{k}'(v) = \frac{F_s(\bar{k}(v))}{f_s(\bar{k}(v))} \frac{f_w(v)}{F_w(v)} \frac{\bar{k}(v) - b_*}{v - b_*}$$

for  $v \in (b_*, \alpha_w]$ . Assuming  $F_w \leq_{rh} F_s$ , the same proof as in footnote 9 of the main paper proves that  $\bar{k} \in [v, r(v)]$ . Compare now  $k_1$  and  $\bar{k}$ . Should they coincide at some  $v \in (b_*, \alpha_w]$ , then  $k'_1(v) \geq \bar{k}'(v)$ , by (A6). Hence,  $\bar{k}$  can cross  $k_1$  at most once, and then from above. If this occurs in the interior, then  $k_1(\alpha_w) = \bar{k}(\alpha_w)$  is violated because  $k'_1(\alpha_w) > \bar{k}'(\alpha_w)$  due to the assumption that  $k_1(\alpha_w) - \alpha_w = \alpha_s - \alpha_w > 0$ . Consequently,  $\bar{k}$  is an upper bound on  $k_1$ , on  $(b_*, \alpha_w]$ . The assumption that  $\alpha_s > \alpha_w$  can be replaced by the joint assumption that  $\alpha_s = \alpha_w$  and  $F_s$  log-concave.<sup>5</sup> Recall that  $\bar{k}(v) \leq r(v)$ , so the bound is tighter than  $r(v)$ . Clearly,  $\bar{k}$  depends only on the primitives,  $F_s$  and  $F_w$ , and the fixed number  $b_*$  (which Maskin and Riley (2000) derived).

Like  $r(v)$ ,  $\bar{k}(v)$  can be implicitly characterized. Since (A7) is a separable differential equation, it is helpful to rewrite it as

$$\frac{f_s(\bar{k})}{F_s(\bar{k})} \frac{1}{\bar{k} - b_*} d\bar{k} = \frac{f_w(v)}{F_w(v)} \frac{1}{v - b_*} dv.$$

With the boundary condition  $\bar{k}(\alpha_w) = \alpha_s$ ,  $\bar{k}$  is then implicitly characterized by

$$(A8) \quad \int_{\bar{k}(v)}^{\alpha_s} \frac{f_s(x)}{F_s(x)} \frac{1}{x - b_*} dx = \int_v^{\alpha_w} \frac{f_w(x)}{F_w(x)} \frac{1}{x - b_*} dx,$$

<sup>5</sup>In this case,  $k_1$  and  $\bar{k}$  coincide and are tangent at  $v = \alpha_w$ . Assume now that  $k_1 > \bar{k}$  for some  $v$ . Then, by log-concavity,  $\frac{F_s(k_1)}{f_s(k_1)} \geq \frac{F_s(\bar{k})}{f_s(\bar{k})}$ . Thus, by (A6),  $k_1$  is strictly steeper than  $\bar{k}$ . Hence,  $k_1$  and  $\bar{k}$  diverge, and  $k_1(\alpha_w) = \bar{k}(\alpha_w)$  is impossible. Thus,  $k_1(v) \leq \bar{k}(v)$  for all  $v \in (b_*, \alpha_w]$ .



for  $v \in (b_*, \alpha_w]$ . Incidentally, the tying function in an asymmetric all-pay auction can be characterized implicitly in much the same way; see Amann and Leininger (1996).

The upper bound can be tightened further, and a lower bound established as well, by exploiting tighter bounds on  $b_w(v)$ . To begin, let  $b_s$  denote the strong bidder's strategy in the asymmetric auction. Let  $b_{ii}$  denote bidders' strategy in a symmetric auction in which both bidders are of kind  $i$ ,  $i = s, w$ . Then, Maskin and Riley's (2000) Proposition 3.5 implies that  $b_{ww}(v) \leq b_w(v)$ ,  $v \in S_w$ , and  $b_{ss}(v) \geq b_s(v)$ ,  $v \in S_s$ . That is, bidders are more aggressive when they face a strong rather than a weak bidder. Let  $b^*$  denote the maximum bid in the asymmetric auction. Since the purpose is to bound  $b_w$ ,

$$b_w(v) \geq \max\{b_*, b_{ww}(v)\} = \max\left\{b_*, v - \int_{\beta_w}^v \frac{F_w(x)}{F_w(v)} dv\right\}$$

is immediately useful. If  $\beta_w = \beta_s$ , then  $b_{ww}(v) \geq b_* = \beta_w$  for all  $v \in S_w$ . An upper bound on  $k_1$  can now be derived by replacing  $b_*$  in (A7) with  $\max\{b_*, b_{ww}(v)\}$ .

It is harder to derive a lower bound on  $k_1$ . To do so, an upper bound on  $b_w$  is needed. Recalling that  $k_1(v) \leq r(v)$  leads to the conclusion that  $b_w(v) = b_s(k_1(v)) \leq b_s(r(v))$ . Hence,

$$\begin{aligned} b_w(v) &\leq b_s(r(v)) \leq \min\{b^*, b_{ss}(r(v))\} \\ &= \min\left\{b^*, r(v) - \int_{\beta_s}^{r(v)} \frac{F_s(x)}{F_s(r(v))} dv\right\}. \end{aligned}$$

There are two weaknesses here. First,  $b^*$  is endogenous and, unlike  $b_*$ , cannot generally be characterized. Second, it is entirely possible that  $\min\{b^*, b_{ss}(r(v))\}$  exceeds  $v$ . An alternative is to begin with

$$\begin{aligned} (v - b_w(v))F_s(r(v)) &\geq (v - b_w(v))F_s(k_1(v)) \\ &= \int_{\beta_w}^v F_s(k_1(x)) dx \\ &\geq \int_{\beta_w}^v F_s(x) dx, \end{aligned}$$

where the inequalities come from  $r(v) \geq k_1(v) \geq v$ . The equality is due to Myerson (1981). Hence,

$$b_w(v) \leq v - \int_{\beta_w}^v \frac{F_s(x)}{F_w(v)} dx,$$

since  $F_s(r(v)) = F_w(v)$ . Clearly, this bound on  $b_w$  does not exceed  $v$ .

The next example illustrates the main points.

EXAMPLE: Assume bidder  $i$ ,  $i = s, w$ , draws a type from  $F_i$ , where

$$F_i(v) = \left(\frac{v}{\alpha_i}\right)^{\gamma_i}, \quad v \in [0, \alpha_i],$$

with  $\alpha_s = \frac{4}{3} > 1 = \alpha_w$  and  $\gamma_s = 2 > 1 = \gamma_w$ . These parameters fit with Cheng's (2006) assumptions. In this case, it is possible to solve the model analytically. Specifically, Cheng (2006) showed that bidding strategies are linear. Thus,  $k_1(v) = \frac{4}{3}v$ , which is also linear. The ex ante probability that bidder  $w$  wins can then be calculated to be  $W(k_1) = \frac{1}{3}$ . In contrast, if  $r(v) = \frac{4}{3}\sqrt{v}$  is used as an upper bound, then it is possible to conclude only that bidder  $w$  wins (ex ante) with probability no greater than  $W(r) = \frac{1}{2}$ . This is a general conclusion when using  $r(v)$  as the bound on  $k_1(v)$ , and shows just how loose a bound it is. Now, using the bound derived from  $b_w(v) \geq b_* = 0$ , (A8) yields  $\bar{k}(v) = 4\frac{v}{v+2}$  and  $W(\bar{k}) \approx 0.403$ . If  $b_w(v) \geq b_{ww}(v) = \frac{1}{2}v$  is used to derive an upper bound, the result is the solution to

$$\frac{dk_{ww}}{dv} = \frac{k_{ww}}{2} \frac{1}{v} \frac{k_{ww} - \frac{1}{2}v}{v - \frac{1}{2}v}, \quad k_{ww}(1) = \frac{4}{3},$$

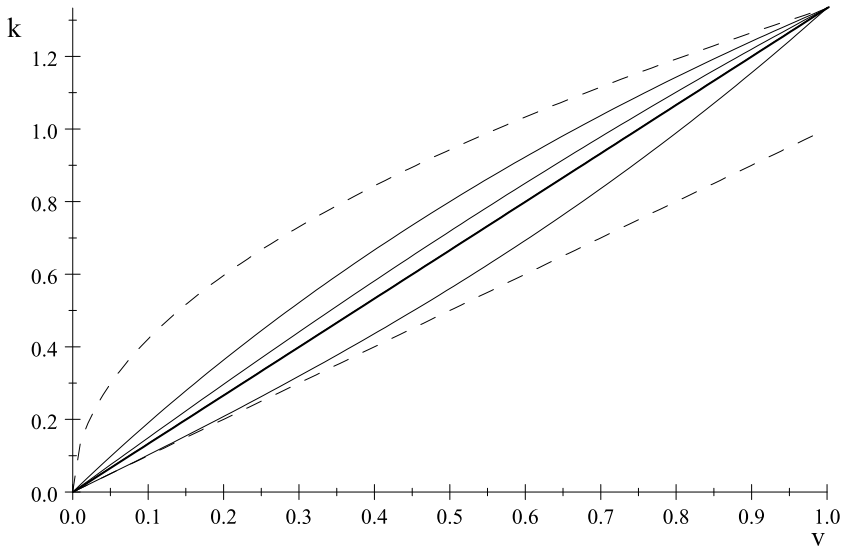


FIGURE A1.—Bounds on  $k_1$ .

which is  $k_{ww}(v) = 12v(v^{3/2} + 8)^{-1}$ , with  $W(k_{ww}) \approx 0.360$ . Turning to a lower bound, note that

$$b_w(v) \leq v - \int_0^v \frac{F_s(x)}{F_w(v)} dx = v - \frac{3}{16}v^2.$$

Using this in place of  $b_*$  in (A7) yields the less palatable lower bound

$$\underline{k}(v) = \left( \frac{3}{4\sqrt{v}} \frac{e^{-8/(3v)}}{e^{-8/3}} + \frac{1}{\sqrt{v}} e^{-8/(3v)} \int_v^1 \frac{8}{3x^{5/2}} e^{8/(3x)} dx \right)^{-1},$$

with  $W(\underline{k}) \approx 0.281$ . In comparison, using only  $k_1(v) \geq v$  as the lower bound would yield  $W(v) = \frac{3}{16} = 0.1875$ . Figure A1 illustrates the bounds just derived. The dashed curves are the bounds on  $k_1$  that exist in the current literature, namely,  $r(v)$  and  $v$ , respectively. The fat line is the true  $k_1$ . The remaining curves are the two upper bounds and the lower bound just derived.

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