

Inefficiency and Nonlinear Pricing in the Optimal Multi-Unit Auction*

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Abstract

Until recently, most theoretical treatments of auctions have assumed that buyers have unit demand. In this paper we characterize the efficiency properties of the optimal auction when buyers have multi-unit demand. The auctioneer's problem is equivalent to the problem facing a monopolist capable of double segmentation. Specifically, buyers are segmented on the basis of their identity and demand characteristics. The exact efficiency properties of the optimal mechanism, be it auction or monopoly, depend upon the behavior of the elasticity of demand. Although the optimal auction is inefficient, the efficient Vickrey auction is more profitable than the uniform price auction, but no unambiguous revenue ranking of reserve prices and entry fees exists.

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1 Introduction

The majority of studies on auctions has focussed on the situation in which only a single good is for sale. Likewise, the analysis of optimal auctions has been largely constrained to environments where each buyer desires to buy only one of a number of goods offered. While important results have been established in this context, it is important to realize that in many situations the assumption of unit demand is unrealistic. Hence, in this paper we aim to extend the theory of optimal auctions to include buyers with multi-unit demand.

Throughout, we restrict attention to the independent private values model and assume that objects are identical. Buyers demand several units of the same good and marginal utility is assumed to be decreasing.¹ We consider the case where each buyer is described by a one-dimensional type, which has the advantage that we can use Myerson's (1981) approach to design the optimal auction.² In addition to deriving the optimal auction, the present analysis also yields insights on optimal multi-unit monopoly pricing.

The papers most closely related to this one are Bulow and Roberts (1989) and Maskin and Riley (1989). Bulow and Roberts assume that each buyer has unit demand and show that "the analysis of optimal auctions [...] is essentially equivalent to the analysis of standard monopoly third-degree price discrimination". So, if buyers are asymmetric, they are treated asymmetrically. Maskin and Riley, on the other hand, assume buyers are symmetric and have multi-unit demand. They note that "the optimal [auction] procedure is thus a nonlinear pricing scheme modified to take account of the supply constraint". Hence, it is hardly surprising that we find that the optimal multi-unit auction combines these features, when bidders are allowed to be asymmetric and have multi-unit demand.

Thus, the optimal auction is essentially equivalent to the optimal selling

¹Levin (1997) also considers optimal multi-unit auctions. He allows goods to be heterogeneous and explicitly assumes that goods are complements, whereas we follow Maskin and Riley (1989) (see below) in assuming a downwards sloping demand curve.

²Engelbrecht-Wiggans (1988) and Krishna and Perry (2000) establish that Myerson's (1981) celebrated Revenue Equivalence Theorem continues to hold when types are multi-dimensional. However, the optimal auction with multi-dimensional types have not been derived. Armstrong (2000) considers auctions for heterogeneous objects with additive valuations, and derives the optimal auction when the type space is finite. In a similar model, Jehiel, Meyer-ter-Vehn and Moldovanu (2003) assume the type space is infinite, but they consider only a limited class of auctions.

procedure of a monopolist capable of not only segmenting his customers based on their identity, but capable also of implementing nonlinear pricing in each of these segments. Hence, when we in the following derive properties of the optimal auction, it is simple to translate these into properties of the optimal monopoly selling procedure, and vice versa.

Considering the substantial market power that the seller commands, the efficiency properties of the optimal mechanism is obviously of interest. It will be shown that the most important factor in determining the properties of the optimal mechanism is the price sensitivity (i.e. price elasticity) of the buyers.

On this note, we observe that a central assumption in Maskin and Riley (1989), see also Maskin and Riley (1984), is that the (absolute value of the) elasticity of demand is increasing in type. That is, buyers with higher demand curves are assumed to be more price sensitive than buyers with lower demand curves. As indicated, we relax this assumption. We now outline some of the results, and highlight the importance of the assumptions concerning price sensitivity.

Figure 1 sketches two polar assumptions. In the monopoly interpretation we have drawn the inverse demand curves for three different consumers in a given market, although we imagine there is a continuum of consumers.³ In Figure 1a the consumer with the highest (inverse) demand curve is the most price sensitive, as in Maskin and Riley (1989). This can be seen by noting that the relative price reduction needed to induce this consumer to purchase more units is small relative to the other consumers. In contrast, Figure 1b depicts the possibility that price sensitivity is smaller for consumers with high demand.

Consider now the aggregate demand for one unit, and the imaginary aggregate demand for a second unit, given one unit has already been consumed. In Figure 1a the aggregate demand for one unit is dense, in the sense that the willingness to pay for the first unit does not vary much across consumers. Therefore, a small reduction in the price for the first unit purchased will imply a large increase in sales. Thus, the aggregate demand for one unit is price sensitive. On the other hand, the aggregate demand for a second unit is price insensitive, because a large price reduction is needed to significantly

³Alternatively, in the auction interpretation, the figure concentrates on one buyer (rather than one market). This buyer has private information about his demand curve, or type. There is a continuum of possible types (rather than a continuum of consumers), but we have drawn only three.

increase sales of a second unit to consumers who have already bought one unit.

We now have two imaginary aggregate demand curves, with different price sensitivities. However, we know that the more price sensitive demand is, the more is it profitable to sell. Applying this to our imaginary demands, it is clear that the monopolist maximizes profits by satisfying more of the demand for one unit than the demand for a second unit. Consequently, the outcome is that the monopolist designs the nonlinear pricing scheme in such a way that some of the consumers who buy one unit are discouraged from buying a second unit.

However, when demand is described by the demand curves in Figure 1b, the aggregate demand for a second unit is very price sensitive. Hence, the monopolist would want to sell two units to some consumers that he would never sell only one unit to. Thus, the monopolist has to sell one unit to what appears to be too many consumers, in order to take advantage of the more profitable aggregate demand for a second unit. In this case, the monopolist simply sells bundles of two units when he is not capacity constrained. If the monopolist is quantity constrained, he may not want to bundle units.

For an auctioneer, the implications are the following. When Figure 1b is relevant, the auctioneer will never decide to sell only one unit. If he can find a buyer who he would be willing to sell one unit to, it will always be profitable to sell two units to this buyer. Indeed, in these circumstances the probability that objects are sold at all in the optimal auction are higher if the auctioneer has two units to sell, rather than just one. When we say that the monopolist sells to more buyers (implying increased sales) in a given segment, the equivalent interpretation in the auction setting is that the auctioneer is willing to sell to more types of a given buyer, therefore increasing the probability of a sale.⁴

Although the optimal auction is inefficient when buyers have multi-unit demand, not all inefficient auctions perform well. We show that auctions in which the units are shared between too many buyers relative to what is efficient are revenue inferior to efficient auctions. An implication is that uniform price auctions are inferior to efficient auctions, such as the Vickrey auction. Furthermore, studying the Vickrey auction in more detail reveals

⁴In a given market, the monopolist faces a “population” distribution of consumers with different demand curves. On the other hand, the auctioneer has a probability distribution over the possible demand curves (types) that a given buyer may have.

that reserve prices and entry fees are not revenue equivalent when buyers have multi-unit demand. In fact, no unambiguous revenue ranking can be obtained.

For the monopolist, the optimal pricing policy will generally involve quantity discounts, or even quantity premia, and different consumers will purchase different quantities. However, if price sensitivity is smaller for consumers with high demand, the optimal pricing policy is extremely simple. It will be optimal to bundle units together, and the seller can then simply charge a single price for the bundle. It is also shown that if price sensitivity is not increasing or decreasing in type for all quantities, a property of the optimal pricing policy is that it induces gaps in the quantity purchased by consumers. For instance, some consumers may purchase one unit, others three, but no consumer will purchase two. Hence, the monopolist could offer several differently sized bundles. Thus, a rationale for bundling is provided, which does not rely on the cost of packaging, transportation and so on.

The remainder of the paper is organized as follows. In Section 2 the analysis is initiated by taking a brief look at a monopolist who faces consumers with multi-unit demand. The aim of this exercise is to exploit the similarities between the monopolist's and the auctioneer's problem to provide some intuition for the workings of the optimal auction. Thereafter, in Section 3, the auctioneer's problem is examined. A discussion of the similarities between monopoly and auction follows in Section 4. In Section 5 the connection between elasticity of demand and efficiency is examined rigorously. Section 6 reveals that a certain class of inefficient auctions are not as profitable as efficient auctions, and demonstrates that reserve prices and entry fees are not revenue equivalent in auctions with multi-unit demand. Section 7 offers lessons for a monopolist. Section 8 concludes.

2 The monopolist

In this section we will present a largely graphical analysis of some of the issues which arise when a monopolist faces consumers with multi-unit demand. We should note that goods are allocated on the basis of marginal revenue, rather than willingness to pay, which gives rise to inefficiencies. It will be fruitful to distinguish between two forms of inefficiency.

The term *allocative inefficiency* is used to describe the fact that sold goods are, in general, not allocated efficiently. That is, sold units can usually

be redistributed among consumers to yield a higher surplus. Furthermore, *output inefficiency* occurs when the seller does not meet demand when he has the capacity to do so, or does not exhaust capacity when unable to meet demand.⁵

While we assume in this paper that the seller can induce both types of inefficiency through his pricing policy, in some environments it may not be possible to do so. For instance, output inefficiency requires that the monopolist can credibly withhold supply. The Coase Conjecture illustrates the consequences of relaxing this assumption. Kirkegaard (2004) derives an upper bound on the value of being able to ration supply, causing output inefficiency. Likewise, if there is an active resale market, it may be difficult for the monopolist to effectively price discriminate across markets or units, thereby limiting the scope of allocative inefficiency.

It is assumed that the monopolist has constant marginal costs, which have been normalized to zero. The monopolist operates in n markets. These markets, or segments, may be identical but need not be. At present, however, we will focus on one of these markets. In this market there is a continuum of consumers, indexed by v , $v \in [\underline{v}, \bar{v}]$, where v describes the consumer's demand curve. In the next section we will allow for general demand functions, but in this section it suffices to consider an example. Specifically, consumer v is willing to pay v for one unit, and an additional kv for a second unit. By assuming $k \leq 1$, we limit attention to demand curves that are not upward sloping. While the individual's demand curve is described by v , the monopolist faces a market demand curve that depends on how the consumers are "distributed". Hence, let $F(v)$ denote the mass of consumers who has demand curves that are lower than consumer v 's demand curve, and assume that $F'(v) = f(v) > 0$ and that $F(\underline{v}) = 0$. Let $F(\bar{v}) = 1$, i.e. the mass of consumers is one.

Consequently, if the unit price is v the mass of consumers who will buy at least one good is $1 - F(v)$, while a smaller portion, namely $1 - F(\frac{1}{k}v)$, will buy two goods. To simplify the graphical presentation, assume that $F(v)$ is the uniform distribution on the unit interval, i.e. $F(v) = v$, $v \in [0, 1]$. Figure 2a then shows the inverse demand curves, $P_1(Q) = 1 - Q$ and $P_2(Q) = k(1 - Q)$, where $P_j(Q)$ is the price at which Q consumers are willing to buy at least j

⁵Armstrong (2000) defines strong and weak efficiency. Strong efficiency describes outcomes that are both allocative and output efficient, while weak efficiency is synonymous with allocative efficiency.

units, $j = 1, 2$. Figure 2a also depicts the marginal revenue (MR) curves.

Next, assume that the seller employs nonlinear pricing. The monopolist's problem is uncomplicated when he is not capacity constrained in that he sets prices at the level that ensures $MR_1 = MR_2 = 0$, implying that half the consumers buy two goods, and the remaining half buy none. This outcome is output inefficient.

To get to the central part of the monopolist's reasoning, assume that he is capacity constrained (as the auctioneer typically is). For instance, assume that at most Q^* units can be produced in the market we consider. Figure 2 illustrates the problem and its solution. By adding the two MR curves for each price, the AMR curve has been obtained and is displayed in Figure 2b. This curve is a useful tool because it links aggregate supply and marginal revenues. The monopolist will charge a price of p_1 for the first unit, and will offer a quantity discount such that the second unit will cost p_2 . The corresponding quantities, Q_1 and Q_2 , adds up to Q^* . The important feature of the solution is that at these quantities the marginal revenue for the first and second good is the same.

However, basing the price structure on equalizing marginal costs produces allocative inefficiency. It is apparent from Figure 2 that some of the Q_2 consumers who buy two units have a lower utility of their second item compared with the consumers that would be willing to pay between p_2 and p_1 for one unit. Hence, to increase consumers' surplus, even without increasing output, some of the consumers who bought two goods should give one of them to consumers who did not get any.

It is interesting to observe that Figure 2 can be reinterpreted. Consider a monopolist who sells goods in two different markets, both characterized by all consumers having unit demand. $P_1(Q)$ and $P_2(Q)$ are then the inverse demand curves in these two markets. The second market does not value the good as highly as the first market does.⁶ Again, let the monopolist be capacity constrained, but this time the problem is how to divide Q^* units between the two markets. Obviously, the previous analysis carries through and we are again faced with inefficiencies.

As mentioned, it will be shown that the optimal multi-unit demand auction (with n bidders) corresponds to a monopoly in which the seller can engage in nonlinear pricing in n distinct markets. Hence, both interpretations of Figure 1 are relevant. First, within each market, the monopolist will

⁶Note that \bar{v} and $F(\cdot)$ differ in the two markets, i.e. they are asymmetric.

design the nonlinear pricing schedule to equate the marginal revenue of the last sold units. Second, across markets, prices will be determined in such a way that the marginal revenue of the last unit is the same in all markets. At least, this is the case when the elasticity of demand is independent of type (as in the current example) or increasing in type. Marginal revenue curves are not as well behaved when the elasticity of demand is decreasing in type.

It is useful, for future reference, to note that revenue in the market analyzed in Figure 1 can be calculated as $R(Q_1, Q_2) = \sum_{j=1}^2 \int_0^{Q_j} MR_j(Q) dQ$, or alternatively as

$$R(Q_1, Q_2) = \int_{\underline{v}}^{\bar{v}} \left(\sum_{j=1}^2 MR_j(v) q_j(v) \right) f(v) dv, \quad (1)$$

where $MR_j(v)$ is the marginal revenue attributable to consumer v on unit j , and $q_j(v)$ is one if consumer v purchased at least j units, and zero otherwise. Here, $Q_j = \int_{\underline{v}}^{\bar{v}} q_j(v) f(v) dv = 1 - F(v_j^*)$, where consumer v_j^* is the consumer of the lowest type to buy at least j units, $v_j^* = \inf[v | q_j(v) = 1]$.

In summary, the monopolist not only supplies an inefficient amount of goods, he also distributes the goods he does decide to sell inefficiently. In the next section the auctioneer's optimization problem will be formally described, and we will later highlight similarities between the auctioneer's and the monopolist's problem. Thus, it will be shown that the optimal auction creates the same types of inefficiencies as the monopoly just analyzed.

3 The auctioneer

The auctioneer has H homogeneous objects (units). Analogously to assuming zero marginal costs for the monopolist, it is assumed that the auctioneer's valuation (offer cost) of each unit is zero. As evident from the following assumption, the demand structure is allowed to be much more general than that used in the previous section.

ASSUMPTION A: There are n buyers. Buyer i is described by a type, v_i , drawn at random from the interval $[\underline{v}_i, \bar{v}_i]$, using the cumulative distribution function $F_i(v_i)$ which is strictly increasing and twice continuously differentiable, with $F_i(\underline{v}_i) = 0$ and $F_i(\bar{v}_i) = 1$, $i = 1, 2, \dots, n$. Regardless of type, buyer i demands h_i units, where $g_{ij}(v_i) \geq 0$ denotes the willingness to pay

for the j th unit, with $g_{ij}(v_i) = 0$ for $j > h_i$ ⁷ Demand is not upwards sloping, $g_{ij}(v_i) \geq g_{ij+1}(v_i)$. $g_{ij}(v_i)$ is twice continuously differentiable with $g'_{ij}(v_i) > 0$, for $j = 1, \dots, h_i$, implying higher types have higher demand, and demand curves do not cross.

When the monopolist was considered earlier, $F_i(v)$ was used to trace out the demand curve in market i , but here $F_i(v)$ is used to pick the type of buyer i taking part in the auction. Only the buyer himself knows his type.

The seller aims to construct an optimal auction, or selling mechanism. We focus on direct mechanisms, i.e. mechanisms in which a buyer reports his type, as the revelation principle states that any allocation that can be implemented using an indirect mechanism can also be brought about using a direct mechanism. Hence, we follow the approach taken by Myerson (1981), Maskin and Riley (1989), and many others. See also Mas-Colell et al (1995).

Thus, buyer i reports a type, x_i , to the seller, $x_i \in [\underline{v}_i, \bar{v}_i]$. In equilibrium, reports are truthful, implying that $x_{-i} = v_{-i}$, where x_{-i} is the vector of reports submitted by buyer i 's competitors. Let $R_i(x_i, v_{-i})$ denote buyer i 's expected payment and let $q_{ij}(x_i, v_{-i})$ denote the allocation of unit j to buyer i , given reports. $q_{ij}(x_i, v_{-i})$ is one if buyer i receives at least j units. Any allocation must satisfy the feasibility constraints,

$$0 \leq q_{ij}(x_i, x_{-i}) \leq 1, \quad (2)$$

$$q_{ij}(x_i, x_{-i}) \geq q_{ij+1}(x_i, x_{-i}) \quad (3)$$

and

$$\sum_{i=1}^n \sum_{j=1}^{h_i} q_{ij}(x_i, x_{-i}) \leq H. \quad (4)$$

Naturally, the buyer does not know other buyers' types, or their reports. Consequently, let $q_{ij}(x_i)$ be the expected value of $q_{ij}(x_i, v_{-i})$, with expectations taken over v_{-i} , and define $R_i(x_i)$ in a similar way. A report of x_i produces expected payoff of

$$E\pi_i(x_i, v_i) = \sum_{j=1}^{h_i} g_{ij}(v_i)q_{ij}(x_i) - R_i(x_i). \quad (5)$$

⁷The properties of the optimal auction are less clear cut if the number of units demanded depends on type.

It is an equilibrium condition that v_i be the optimal report. This is also often referred to as the incentive compatibility (IC) condition. To simplify notation somewhat, the subscript i is being suppressed in the following. Incentive compatibility implies, for a type v buyer contemplating reporting x , that,

$$E\pi(v, v) \geq \sum_{j=1}^h g_j(v)q_j(x) - R(x) = E\pi(x, x) - \sum_{j=1}^h (g_j(x) - g_j(v))q_j(x). \quad (6)$$

By reversing the role of x and v in (6) the IC condition for a type x buyer is obtained. Combining these two versions of (6) yields

$$\sum_{i=1}^h \frac{g_j(v) - g_j(x)}{v - x} q_j(v) \geq \frac{E\pi(v, v) - E\pi(x, x)}{v - x} \geq \sum_{i=1}^h \frac{g_j(v) - g_j(x)}{v - x} q_j(x). \quad (7)$$

From (7) it is apparent that any allocation where $q_j(\cdot)$ is increasing can be implemented, i.e. any mechanism which gives rise to increasing $q_j(\cdot)$ functions is incentive compatible. Therefore, there exists some incentive compatible mechanism that always allocates the objects efficiently.⁸ As can be seen by letting x approach v , $dE\pi(v, v)/dv = \sum_{j=1}^h g'_j(v)q_j(v)$. An alternative expression for $E\pi(v, v)$ is thus obtainable, i.e.

$$E\pi(v, v) = \int_{\underline{v}}^v \sum_{j=1}^h g'_j(z)q_j(z)dz + E\pi(\underline{v}, \underline{v}). \quad (8)$$

Combining (5) and (8) produces an expression for what a type v buyer expects to pay in the auction, namely

$$R(v) = \sum_{j=1}^h g_j(v)q_j(v) - \int_{\underline{v}}^v \sum_{j=1}^h g'_j(z)q_j(z)dz - E\pi(\underline{v}, \underline{v}). \quad (9)$$

Importantly, we can now derive the revenue the seller can expect to extract from each buyer, as this is equal to what the buyer expects to pay ex ante, i.e. before he learns his type:

⁸In an efficient auction a type v player is at least as likely to win j objects than a type x player, when $v > x$. Since $q_j(v) \geq q_j(x)$ the inequality in (7) is satisfied.

$$ER = \int_{\underline{v}}^{\bar{v}} \left[\sum_{j=1}^h g_j(v) q_j(v) - \int_{\underline{v}}^v \sum_{j=1}^h g'_j(z) q_j(z) dz - E\pi(\underline{v}, \underline{v}) \right] f(v) dv.$$

Integration by parts simplifies the expression,

$$ER = \int_{\underline{v}}^{\bar{v}} \left[\sum_{j=1}^h \left(g_j(v) - g'_j(v) \frac{1 - F(v)}{f(v)} \right) q_j(v) \right] f(v) dv - E\pi(\underline{v}, \underline{v}). \quad (10)$$

Since the functions $g_j(\cdot)$ and $F(\cdot)$ are exogenous to the seller, he must maximize expected revenue by determining $E\pi(\underline{v}, \underline{v})$ and $q_j(v)$, $\forall v, j = 1, 2, \dots, h$. The fact that (10) depends only on $E\pi(\underline{v}, \underline{v})$ and on the allocation, $q_j(v)$, gives rise to the so called Revenue Equivalence Theorem.⁹

Proposition 1 (Revenue Equivalence) *Any two auctions that allocate the goods in the same way, and between which each buyer would be indifferent if he had the lowest possible type, generate the same expected revenue.*

In addition to the IC condition, we also require that the auction satisfies the participation constraint, i.e. all buyers choose voluntarily to take part in the auction. Hence, expected payoff from participating must exceed that from ignoring the auction, $E\pi(v, v) \geq 0$. Recall from (10) that any rent the seller permits the type \underline{v} buyer to extract decreases expected revenue, wherefore it is optimal to minimize this amount. Hence, the optimal auction is designed in such a way that $E\pi(\underline{v}, \underline{v}) = 0$.¹⁰ The seller then has one instrument left to maximize profit, namely the rules that govern the probabilities of winning.

Defining $J_{ij}(\cdot)$ as

$$J_{ij}(v) = g_{ij}(v) - g'_{ij}(v) \frac{1 - F_i(v)}{f_i(v)}, \quad (11)$$

⁹See Myerson (1981) and Riley and Samuelson (1981) for the first Revenue Equivalence Theorems, in which unit demand is assumed. The most general Revenue Equivalence Theorem, encompassing Proposition 1, is to be found in Engelbrecht-Wiggans (1988). See also Krishna and Perry (2000).

¹⁰Since $g'_i(\cdot)$ and $q_i(\cdot)$ are both non-negative, the former by assumption, $E\pi(v, v) \geq E\pi(\underline{v}, \underline{v})$, cf. (8). The participation constraint is therefore fulfilled for all types.

the objective is thus to maximize

$$E \left[\sum_{i=1}^n \sum_{j=1}^{h_i} J_{ij}(v_i) q_{ij}(v_i, v_{-i}) \right], \quad (12)$$

over the winning probabilities, and subject to (2), (3), (4) and (7).¹¹

The advantage of formulating expected revenue as in (12) is that it lends itself to pointwise maximization. That is, maximizing the term inside the expectations operator for every realization of types, subject to (2) and (4), and then taking the expectation over the possible types yields an obvious candidate for maximal revenue.

Given the standard regularity condition that $J_{ij}(v_i)$ is increasing in v_i , pointwise maximization implies that the higher v_i is the higher is $q_{ij}(v_i, v_{-i})$. Consequently, the IC condition, (7), is satisfied.

Finally, to be optimal the allocation resulting from pointwise maximization must also satisfy the last remaining feasibility constraint, (3). Maskin and Riley's (1989) assumption on preferences, namely that a buyer is more price sensitive the higher his type is, is sufficient to ensure that this condition is satisfied. The reason is explained in Section 5. Then, pointwise maximization yields maximal revenue.

On the other hand, when buyers are less price sensitive the higher their type is, (3) may turn out to be a binding constraint. In this case, maximal revenue can be obtained by pointwise maximization subject to (2), (4) and (3). For details, see Section 5.

To illustrate, assume that all buyers have unit demand, i.e. $h_i = 1$, $i = 1, \dots, n$. This case has been studied thoroughly in the literature. Assuming that $J_{i1}(v_i)$ is increasing for all i , expected revenue is maximized if, for every realization of types, the H buyers with the highest values of $J_{i1}(v_i)$ are rewarded one unit each, as long as $J_{i1}(v_i) \geq 0$. If $J_{i1}(v_i)$ can take negative values, output inefficiency can occur. Likewise, allocative inefficiency is possible as long as buyers are asymmetric, since in that case the buyers with the highest J_{i1} values are not necessarily the ones with the highest g_{i1} values.

In the next section we compare monopoly and auction, and show that the optimal auction can sensibly be described in terms that are familiar from monopoly analysis.

¹¹To obtain (12), start by rewriting (10) for each buyer i while recalling that $q_{ij}(v_i)$ is the expectation of $q_{ij}(v_i, v_{-i})$ over v_{-i} . Finally, sum over the buyers and remember that $E\pi_i(v_i, v_i) = 0$ in the optimal auction.

4 The equivalence of monopoly and auction

As mentioned in the introduction, Bulow and Roberts (1989) established the essential equivalence of monopoly and auction when buyers have unit demand. Bulow and Klemperer (1996) use this to show that an (efficient) English auction with n bidders yields higher expected revenue than the optimal auction with $n - 1$ bidders. See also Klemperer (1999).

Allowing for multi-unit demand, we argue that the optimal auction is equivalent to a monopoly which optimally employs segmentation with non-linear pricing. In other words, third degree price discrimination is used, as shown by Bulow and Roberts, to divide the market into n segments, and nonlinear pricing is thereafter applied to each of these segments.

Thus, assume that the monopolist is capable of such segmentation. Then, if the monopolist has a random capacity, expected monopoly profits are

$$\sum_{i=1}^n \int_{\underline{v}_i}^{\bar{v}_i} \sum_{j=1}^{h_i} MR_{ij}(v_i) E [q_{ij}^M(v_i, k)] f_i(v_i) dv_i, \quad (13)$$

where $q_j^M(v, k)$ is determined by the price schedule chosen by the monopolist when the capacity is k . It is one if a consumer whose demand curve is described by v wins at least j units, and zero otherwise. The expectation is taken over the set of possible capacities.¹²

On the other hand, expected revenue in an auction is

$$\sum_{i=1}^n \int_{\underline{v}_i}^{\bar{v}_i} \sum_{j=1}^{h_i} J_{ij}(v_i) E [q_{ij}^A(v_i, v_{-i})] f_i(v_i) dv_i, \quad (14)$$

where $q_{ij}^A(v_i, v_{-i})$, is determined by the rules of the auction, as stipulated by the auctioneer, and by the competitors' types. It is one if a buyer with type v_i wins at least j units, and zero otherwise. The expectation is taken over the set of possible types.¹³

Clearly, (13) and (14) are very similar expressions. Furthermore, the constraints in the two problems are similar as well. For every realization

¹²(13) is obtained in the following way. For each of the n markets write revenue as in (1) for a given capacity, then sum over the markets and finally take the expectation over capacities.

¹³To obtain (14) sum (10) over the n buyers, and recall that $q_{ij}(v_i)$ is defined as $E [q_{ij}(v_i, v_{-i})]$.

of k , the monopolist must respect the constraints that $0 \leq q_{ij}(v_i, k) \leq 1$, $q_{ij}(v_i, k) \geq q_{i(j+1)}(v_i, k)$ and $\sum \sum q_{ij}(v_i, k) \leq k$. These correspond roughly to (2), (3) and (4).¹⁴ Off course, $q_{ij}^M(v_i, k)$ depends on k whereas $q_{ij}^A(v_i, v_{-i})$ depends on v_{-i} . Bulow and Roberts (1989) observe, however, that in both cases the arguments other than v_i (k and v_{-i} , respectively) are random variables.

Given these similarities, the $J_{ij}(\cdot)$ function plays the same role in the auctioneer's maximization problem as the $MR_{ij}(\cdot)$ function plays in the monopolist's problem. That is, given some similar capacity (and incentive) constraints, the objective is to maximize the sum of J_{ij} 's or MR_{ij} 's of the winners.

Consider then the monopolist's problem in more detail. Since the monopolist is assumed to be able to discriminate between markets we can focus on how to determine the price schedule within a given market. Then, in market i , consider the incremental demand for a j 'th unit, i.e. the imaginary aggregate demand for j units given $j - 1$ units have been purchased. The function $g_{ij}(v_i)$ can be thought of as the unit specific (inverse) demand curve for unit j . If the price is $g_{ij}(v_i)$, $Q_i(v_i) = 1 - F_i(v_i)$ consumers buy a j th unit.

For future reference, observe that this demand curve has slope

$$\frac{dg_{ij}(v_i)}{dQ_i} = \frac{g'_{ij}(v_i)}{dQ_i/dv_i} = -\frac{g'_{ij}(v_i)}{f_i(v_i)}, \quad (15)$$

and that the (numerical value of the price) elasticity is

$$\widehat{\varepsilon}_{ij}(v_i) = -\frac{dQ_i}{dg_{ij}(v_i)} \frac{g_{ij}(v_i)}{Q_i} = \frac{f_i(v_i)}{1 - F_i(v_i)} \frac{g_{ij}(v_i)}{g'_{ij}(v_i)}. \quad (16)$$

Now, profit attributable to the sale of the incremental j th unit is $g_{ij}(v)Q_i(v_i)$. Marginal revenue can thus be derived,

$$\begin{aligned} MR_{ij}(v_i) &= \frac{d}{dQ_i}(g_{ij}(v_i)Q_i(v_i)) = g_{ij}(v_i) + g'_{ij}(v_i) \frac{Q_i}{dQ_i/dv_i} \\ &= g_{ij}(v_i) - g'_{ij}(v_i) \frac{1 - F_i(v_i)}{f_i(v_i)} = J_{ij}(v_i). \end{aligned} \quad (17)$$

¹⁴See Bulow and Roberts (1989) for a more thorough discussion.

Thus, not only does $J_{ij}(\cdot)$ play the same role in an auction as $MR_{ij}(\cdot)$ plays in monopoly, the functions are also identical (though $F(\cdot)$ and $f(\cdot)$ have different interpretations). In both situations the seller maximizes revenue by maximizing the sum of marginal revenue from the winners, subject to capacity and incentive constraints.

5 Efficiency properties

It is useful to start with a brief discussion of the assumptions implicit in Figure 2. First of all, the marginal revenue curves are downwards sloping. In fact, this is equivalent to $J_j(\cdot)$ being increasing in v and Myerson (1981) calls this the “regular case”.¹⁵

Alternatively, it is possible that the MR curve is non-monotonic. For instance, suppose that the MR curve is shaped like a series of waves. According to (12) the auctioneer would only want to serve the crests of the waves, yet such an auction design would violate incentive compatibility. Hence, in this case the auctioneer would have to employ an “ironing procedure”, in which $q_j(v)$ is constant over some intervals. See e.g. Myerson (1981), Maskin and Riley (1989) or Bulow and Roberts (1989). For simplicity, we will focus on the regular case. This implies that if we maximize the term inside the expectations operator in (12) for every realization of types, the winning probabilities are increasing in type (because $J_j(\cdot)$ is). Hence, the IC condition is satisfied.

Second, the demand specification in Section 2 implies that the marginal revenue curves in Figure 2 intersect the first axis in the same place. This is not due to the assumed form of $F(\cdot)$, but rather to the fact that $g_2(v) = kg_1(v)$. In essence, this section explores what happens when more general specifications are allowed.

We focus on the regular case. That is, we follow Maskin and Riley (1989) in imposing the following two assumptions.

ASSUMPTION B: $J^i(v)$ is strictly increasing, for all i , $i = 1, \dots, n$, where

$$J^i(v) = v - \frac{1 - F_i(v)}{f_i(v)}. \quad (18)$$

¹⁵Marginal revenue is usually defined as a function of quantity, whereas $J_j(\cdot)$ is defined as a function of price, v . The inverse relationship between quantity and price explains why decreasing marginal revenue in Figure 2 corresponds to increasing $J_j(\cdot)$.

ASSUMPTION C: $g''_{ij}(v) \leq 0$, $j = 1, \dots, h_i$, for all i , $i = 1, \dots, n$.¹⁶

To minimize notation the subscript and superscript i , denoting the identity of the buyer, is being suppressed in the following unless it is necessary. Together, Assumptions B and C imply that $J_j(v)$ is strictly increasing since

$$J'_j(v) = g'_j(v)J'(v) - g''_j(v)\frac{1 - F(v)}{f(v)} > 0.$$

In the following, let

$$v_j^0 = \min\{x | J_j(x) \geq 0\}. \quad (19)$$

Notice that v_j^0 is unique, since $J_j(\cdot)$ is strictly increasing. If $v_j^0 = \underline{v}$, marginal revenue on unit j is positive for all types. On the other hand, if $v_j^0 > \underline{v}$, some types, those below v_j^0 , have negative marginal revenue on unit j , and it may be optimal to exclude these from the auction (however, as we will see, this is not necessarily the case).

To describe how the marginal revenue curves for units j and j' relate to each other, it is very useful to consider the elasticity of demand.

Definition 1 *The (numerical value of the price) elasticity of demand is*

$$\varepsilon_j(v) = \frac{1}{j} \frac{g_j(v)}{g_j(v) - g_{j+1}(v)}, \quad j = 1, \dots, h - 1. \quad (20)$$

(20) can be rewritten as

$$\varepsilon_j(v) = -\frac{j - (j + 1)}{j} \frac{g_j(v)}{g_j(v) - g_{j+1}(v)} = -\frac{j - (j + 1)}{g_j(v) - g_{j+1}(v)} \frac{g_j(v)}{j}, \quad (21)$$

where j is quantity, and $g_j(v)$ the price. Hence, it is not hard to see that (20) is indeed an expression of elasticity.

What will turn out to be important is how $\varepsilon_j(v)$ depends on v . Maskin and Riley (1984, 1989) assume that the elasticity of demand is non-decreasing in type, which we will also refer to as price sensitivity being non-decreasing in type. We will assume, purely for convenience, that the derivative of $\varepsilon_j(v)$ has the same sign for all values of v . However, we will, in Section 7, consider the possibility that the sign of this derivative depends on j .

¹⁶This is Assumption B2 in Maskin and Riley (1989). See also Maskin and Riley (1984).

To proceed, compare $\varepsilon_j(v)$ and $\widehat{\varepsilon}_j(v)$. The former captures how price sensitive the buyer is if he has type v and has purchased j units, or the size of the price reduction needed to induce him to purchase $j + 1$ units. The latter measures how price sensitive incremental demand for the j 'th unit is at the point where the buyer of type v is the last to buy. As evident from the discussion in the Introduction, the two are related. The following proposition formalizes the connection.

Proposition 2 *If the elasticity of demand at unit j is increasing (decreasing) in type, incremental demand for unit j is more (less) price sensitive than incremental demand for unit $j + 1$,*

$$\varepsilon'_j(v) \leq 0 \iff \widehat{\varepsilon}_j(v) \leq \widehat{\varepsilon}_{j+1}(v). \quad (22)$$

Proof. The derivative of $\varepsilon_j(v)$ is

$$\varepsilon'_j(v) = \frac{g'_j(v)g'_{j+1}(v)}{j(g_j(v) - g_{j+1}(v))^2} \left[\frac{g_j(v)}{g'_j(v)} - \frac{g_{j+1}(v)}{g'_{j+1}(v)} \right], \quad (23)$$

and so the sign of $\varepsilon'_j(v)$ is determined by the term in brackets. From (16),

$$\widehat{\varepsilon}_j(v) - \widehat{\varepsilon}_{j+1}(v) = \frac{f(v)}{1 - F(v)} \left[\frac{g_j(v)}{g'_j(v)} - \frac{g_{j+1}(v)}{g'_{j+1}(v)} \right], \quad (24)$$

which obviously has the same sign as $\varepsilon'_j(v)$. ■

First, consider the simplest case, $\varepsilon'_j(v) > 0$, which was analyzed by Maskin and Riley (1989). If $\varepsilon_j(v)$ is increasing in v then $\widehat{\varepsilon}_j(v) > \widehat{\varepsilon}_{j+1}(v)$. In other words, when the buyer is more price sensitive the higher his demand curve is, incremental demand is less price sensitive the more units have been consumed. However, since incremental demand for a j 'th unit is more price sensitive than incremental demand for a $(j + 1)$ 'th unit, conventional monopoly theory tells us that the quantity sold on the former market should exceed the quantity sold on the latter. As is standard from monopoly theory, rearranging (11) yields

$$J_j(v) = g_j(v) \left[1 - \frac{1}{\widehat{\varepsilon}_j(v)} \right], \quad (25)$$

and since $\widehat{\varepsilon}_j(v) > \widehat{\varepsilon}_{j+1}(v)$ it must be the case that $J_j(v)$ is positive whenever $J_{j+1}(v)$ is, and it follows that $v_{j+1}^0 \geq v_j^0$. Consequently, the set of types,

$[v_j^0, \bar{v}]$, to which it is desirable to sell j units is larger than the set of types, $[v_{j+1}^0, \bar{v}]$, to which it is desirable to sell $j + 1$ units.

In similar fashion, if $\varepsilon'_j(v) < 0$, $J_{j+1}(v)$ must be positive whenever $J_j(v)$ is positive, and so $v_{j+1}^0 \leq v_j^0$. In the special case where $\varepsilon'_j(v) = 0$, $J_j(v)$ and $J_{j+1}(v)$ have the same sign.

Lemma 1 *The way in which the elasticity of demand depends upon type has consequences for the relationship between $J_j(v)$ and $J_{j+1}(v)$, $j = 1, \dots, h - 1$.*

- (i) *If $\varepsilon'_j(v) = 0 \forall v$, it follows that $v_{j+1}^0 = v_j^0$ and $J_j(v) > J_{j+1}(v) > 0$ for $v > v_j^0$ if $g_j(v) > g_{j+1}(v)$. If the demand curve is flat ($g_j(v) = g_{j+1}(v)$), then $\varepsilon'_j(v) = 0$ and $J_j(v) = J_{j+1}(v)$, $\forall v$.*
- (ii) *$\varepsilon'_j(v) > 0 \forall v$ implies that $v_{j+1}^0 > v_j^0$ if $v_j^0 > \underline{v}$ and $v_{j+1}^0 \geq v_j^0$ if $v_j^0 = \underline{v}$. Furthermore, $J_j(v) > J_{j+1}(v)$ for $v > v_j^0$.*
- (iii) *If $\varepsilon'_j(v) < 0 \forall v$, then $v_{j+1}^0 < v_j^0$ if $v_j^0 > \underline{v}$ and $v_{j+1}^0 = v_j^0$ if $v_j^0 = \underline{v}$. If $v_j^0 > \underline{v}$, $J_j(v)$ and $J_{j+1}(v)$ will cross as v increases from v_j^0 to \bar{v} . If $J_j(v) > 0$, $J_{j+1}(v)$ will also be positive.*

Proof. The relationship between v_j^0 and v_{j+1}^0 was proven in the text preceding the lemma. The fact that $J_j(v) > J_{j+1}(v) > 0$ for $v > v_j^0$ if $\varepsilon'_j(v) \geq 0$ is established by noticing that each term in (25) is higher for $J_j(v)$ than for $J_{j+1}(v)$. If $\varepsilon'_j(v) < 0$, we know that $J_{j+1}(v_j^0) > J_j(v_j^0)$ and that $J_{j+1}(\bar{v}) = g_{j+1}(\bar{v}) < g_j(\bar{v}) = J_j(\bar{v})$, and so they must cross. ■

These properties are illustrated graphically in Figure 3, in the case where $\varepsilon'_1(v) > 0$. Figure 3a depicts the $J_1(v)$ and the $J_2(v)$ function. In Figure 3b these are translated into *type specific* marginal revenue schedules, $MR(j, v)$, $j = 1, 2$. For a given type, v , these give marginal revenue from the j 'th unit, i.e. $MR(j, v) = J_j(v)$. Notice that the $MR(j, v)$ curves in Figure 3b have negative slope as long as $MR(1, v) \geq 0$ and that $MR(2, v)$ is negative whenever $MR(1, v)$ is negative.

Since the type specific marginal revenue curves are well behaved when $\varepsilon_j(v)$ is increasing in v , it is straightforward to describe the allocation in the optimal auction.

Proposition 3 *Whenever $\varepsilon'_{ij}(v) \geq 0 \forall v_i \in [\underline{v}_i, \bar{v}_i]$, for all j , $j = 1, \dots, h_i - 1$, and all i , $i = 1, \dots, n$, the optimal auction implements the following allocation. Given the vector of realized types, calculate for each buyer $MR_i(j, v)$, $j =$*

$1, \dots, h_i$, yielding a total of $\sum h_i$ numbers. Let H^* be the set in which these numbers have been arranged in descending order. If $H > \sum h_i$, include $H - \sum h_i$ zeros in H^* . buyer i then receives h_i^* units, where h_i^* is the highest number for which $MR_i(h_i^*, v_i)$ is among the H first elements in H^* and for which $MR_i(h_i^*, v_i) \geq 0$.

Proof. From Lemma 1, or the preceding discussion, we know that if $MR_i(j, v_i) < 0$ then $MR_i(j + 1, v_i)$ will also be negative. Furthermore as long as the type specific marginal revenue curve is positive, it will have negative slope, implying that the contribution to profit diminishes as the buyer is awarded more and more units. Clearly, the proposed allocation maximizes the term inside the expectation operator in (12) for every realization of types. Since $MR_i(j, v_i)$ is increasing in v_i , $q_{ij}(v_i)$ is increasing. That is, the IC condition is satisfied. Finally, the allocation is feasible, $\sum_{i=1}^n \sum_{j=1}^{h_i} q_{ij}(v_i, v_{-i}) \leq H$ and $q_{ij}(v_i, v_{-i}) \geq q_{ij+1}(v_i, v_{-i})$. Hence, the allocation is optimal. ■

Of course, if $\varepsilon'_j(v) < 0$ the line of argument preceding Proposition 3 seems to lead to the paradoxical recommendation that more types should be sold $j + 1$ units than should be sold j units. In this case, the feasibility constraint that $q_{ij}(v_i, v_{-i}) \geq q_{ij+1}(v_i, v_{-i})$ becomes binding. Consequently, the procedure in Proposition 3 does not work.

Figure 4 is the counterpart of Figure 3, when $\varepsilon'_j(v) < 0$. The problem is that the type specific marginal revenue curve may cross the first axis from below. That is, a buyer of type v may contribute less to revenue on the incremental market for a first unit than on the incremental market for a second unit.

Then, in order to maximize the sum of marginal revenues while satisfying feasibility, the auctioneer has to accept selling a first unit to some types that have negative marginal revenue on the first unit. This opens the market for a second unit, allowing the seller to take advantage of the higher marginal revenues.

While it is difficult to concisely present the optimal auction when $\varepsilon'_j(v) < 0$, it is possible to do so when buyers are symmetric and $h = 2$. In this case Figure 4 describes marginal revenue.

Proposition 4 *Assume buyers are symmetric, $H \geq h = 2$, $\varepsilon'_1(v) < 0$, $v_1^0 > v_2^0 > \underline{v}$, and assume there is a unique type, \hat{v} , for which $J_1(\hat{v}) = J_2(\hat{v}) > 0$ (by Lemma 1, one exists). Define x_1^0 as the unique value for which $J_1(x_1^0) + J_2(x_1^0) = 0$. Then, the optimal auction implements the following allocation.*

First, among all buyers with type exceeding \hat{v} , allocate units based on highest marginal revenues (as in Proposition 3). Second, if capacity is not exhausted, allocate units in bundles of 2 among buyers with types between x_1^0 and \hat{v} based on highest type, or equivalently highest $J_1(\cdot) + J_2(\cdot)$ or $g_1(\cdot) + g_2(\cdot)$. Third, if precisely one unit remains hereafter, allocate it to the highest unsurplied buyer, if this buyer has type above v_1^0 .

Proof. It is clear by inspection that for every realization of types, the term inside the expectation operator in (12) is maximized, subject to the feasibility constraint that $q_{ij}(v_i, v_{-i}) \geq q_{ij+1}(v_i, v_{-i})$. Since the probabilities of winning are increasing in type, the auction is incentive compatible. ■

When $\varepsilon'_j(v) < 0$, the probability that the optimal auction is output efficient may increase with H . This is easiest to illustrate with symmetric buyers.

Proposition 5 (Output inefficiency) *Assume buyers are symmetric.*

- (i) *If $\varepsilon'_j(v) \geq 0$, for all $j = 1, \dots, h - 1$, the same set of types is excluded from the optimal auction regardless of the size of H . If $\varepsilon'_j(v) > 0$, for all $j = 1, \dots, h - 1$, the probability that all units are sold **decreases** as H increases, $H < nh$.*
- (ii) *If $\varepsilon'_j(v) < 0$, for all $j = 1, \dots, h - 1$, then, if H increases, the set of types with a possibility of winning units in the optimal auction expands if $H < h$ and is constant if $H \geq h$. Either zero units are sold, H units are sold, or an integer multiple of h is sold.*
- (iii) *If $\varepsilon'_j(v) < 0$, for all $j = 1, \dots, h - 1$, the probability that all units are sold is non-monotonic in H , $H < nh$. The probability that all units are sold if $H = h$ exceeds the same probability if $H = 2h$, which in turn exceeds the same probability if $H = 3h$, and so on. However, if there is individual excess demand, $H < h$, or if H is not an integer multiple of h , the probability that all units are sold **increases** if H increases by one.*

Proof. (i) From Lemma 1 we know that $J_j(v) < 0$ for $v < v_1^0$, $j = 1, \dots, h$, if $\varepsilon'_j(v) \geq 0$ for all $j = 1, \dots, h - 1$. Hence, if $v < v_1^0$, type specific marginal revenues are negative, and it follows that buyers of this type are never offered units, irrespective of the size of H . When $\varepsilon'_j(v) > 0$ a buyer who is offered

h' units may not be offered $h' + 1$ units, $h' < h$, since type specific marginal revenue has negative slope and may cross the first axis (from above) for these buyers. Hence, when H increases from H' to $H' + 1$, it is not certain that the buyers who together bought H' units will be allowed to buy the additional unit. In that case, the extra unit will be left unsold unless there is another buyer with type $v \geq v_1^0$ who was not among the buyers of the H' units.

(ii) When $\varepsilon'_j(v) < 0$ we define x_j^0 as the unique value of v which satisfies

$$x_j^0 = \min \left\{ x \mid \sum_{m=1}^j J_m(x_j^0) \geq 0 \right\}, \quad (26)$$

for $j = 1, 2, \dots, h$. Importantly, x_j^0 decreases as j increases. This can be seen by noting that $x_j^0 \geq v_j^0$ and that $J_{j+1}(v_j^0) \geq 0$. Hence, adding $J_{j+1}(v_j^0)$ to the sum above increases it, and it follows that x_{j+1}^0 must be smaller than x_j^0 to offset this effect. When $H = j < h$ it is evident that a buyer with type $v \in [x_{j+1}^0, x_j^0]$ has no chance of winning any goods in the optimal auction, cf. (12). However, if H increases to $j + 1$ units it becomes profitable for the seller to allocate *all* goods to the buyer, given that he is the buyer with the highest type.

To show that either zero units are sold, H units are sold, or an integer multiple of h is sold, assume to the contrary that this is not the case in the optimal auction. Hence, there must be at least one buyer who is allocated a number of units, say k , but fewer than h , $0 < k < h$, while there is excess capacity at the same time. If the buyer's type is below x_k^0 , revenue would increase by denying him any units, while if the buyer's type is higher than x_j^0 , revenue would increase by giving him more units. In either case, the auction is not optimal.

(iii) Assume that $H = h$. By (ii), either zero or H units are sold. In the former case, this reveals that there is no buyer with type above x_h^0 , and it follows that no sale is made regardless of H . In the latter case, there must be at least one buyer included in the auction (i.e. with type above x_h^0). If two or more buyers are included, it remains true that all units are sold if capacity increases to $2h$. However, if only one buyer is included, increasing the capacity to $2h$ leaves h units unsold. Hence, the probability that all units are sold decreases. This argument is easily extended to any integer multiple of h .

Next, assume that H is not an integer multiple of h . If fewer than H units are sold, it must be the case that all included buyers get h units, by the

argument in (ii). Furthermore, if more than h units are unsold, the excluded buyers must all be of type below x_h^0 , in which case increasing the capacity does not change the allocation. However, if fewer than h units, say k units, are unsold, the excluded buyers have type below x_k^0 , but it is possible that one of them has a type in the interval (x_{k+1}^0, x_k^0) . In this case, increasing the capacity by one unit makes it profitable to sell the remaining $k + 1$ units to the buyer in question. Hence, by increasing the capacity, the probability that the capacity is not exhausted decreases. That is, the probability that all units are sold must increase. ■

As long as $H \leq nh$, the actual allocation of units is by definition output inefficient if some units remain unsold. Proposition 5 implies that *output inefficiency* does not necessarily become more likely when the number of units increases, although aggregate demand as a proportion of H diminishes. However the optimal auction is generally *allocatively inefficient*, because the allocation is based on marginal revenue rather than willingness to pay.¹⁷

To illustrate the third part of Proposition 5, consider again the example in Figure 4 and Proposition 4. If $H = 2$, both objects are sold if the highest type is above x_1^0 . However, if $H = 4$, capacity is exhausted only if the *two* highest types exceeds x_1^0 . Hence, the probability that all units are sold *decreases* if H jumps from two to four. On the other hand, if $H = 1$, the unit is sold only if the highest type is higher than v_1^0 . However, if $H = 2$, both objects are, as mentioned, sold whenever the highest type exceeds x_1^0 , where $x_1^0 < v_1^0$. Consequently, the probability that all units are sold *increases* if H jumps from one to two, or from three to four.

6 Revenue and inefficiency

When buyers have multi-unit demand, the optimal auction is typically inefficient. However, not all forms of inefficiency are desirable for a seller seeking to maximize revenue. In this section we first investigate the role of allocative inefficiency, and show that any auction in which the units are shared between

¹⁷There are exceptions. For instance, assume buyer are symmetric, $n \geq H \geq h = 2$ and that the marginal utility is much higher on the first than on the second unit consumed. Specifically, assume that $J_1(v) > J_2(\bar{v}) = g_2(\bar{v}) \geq 0$. If $J_1(v)$ and $J_2(v)$ are regular, the optimal allocation will give one unit each to those H agents who value it the most. But since $J_1(v) > J_2(\bar{v})$ it must be the case that $g_1(v) > g_2(\bar{v})$ and it follows that the allocation is efficient. Likewise, when buyers are symmetric and demand curves are horizontal ($g_{j+1}(v) = g_j(v)$ for all v), the optimal auction is allocative efficient.

too many buyers, relative to what is efficient, yields lower revenue than an efficient auction. This leads to the conclusion that the efficient Vickrey auction is revenue superior to the uniform price auction.¹⁸

Then, we examine the role of output inefficiency, absent allocative inefficiency. Specifically, we consider the allocatively efficient Vickrey auction, and compare reserve prices and entry fees. With unit demand, these are revenue equivalent, but this is not the case in the presence of multi-unit demand. Although no unambiguous revenue ranking can be obtained, it is possible to get a good understanding of when one might be preferable to the other.

6.1 Allocative inefficiency and Vickrey auctions

We assume that buyers are symmetric and that there is excess demand. Since buyers are symmetric, we can write $g_{ij}(v_i)$ and $J_{ij}(v_i)$ simply as $g_j(v_i)$ and $J_j(v_i)$, respectively. Without loss of generality the buyers are ordered such that $v_1 \geq v_2 \geq \dots \geq v_n$. To focus on the role of allocative inefficiency we disregard output inefficiency for now. That is, we consider auctions where all the units are sold with probability one. Later, we will return to the issue of output inefficiency.

There are many different types of auctions in which the units are shared among too many buyers. At one extreme, consider auctions where it is impossible to win more than one unit. Such auctions are revenue inferior to efficient auctions.

Proposition 6 *Assuming that $J_1(v)$ is strictly increasing in v and that $n > H$, any output efficient auction in which buyers can win at most one unit is revenue inferior to an efficient auction with $E\pi(\underline{v}, \underline{v}) = 0$.¹⁹*

Proof. If $J_1(v)$ is strictly increasing an auction which allocates one unit to each of the first H buyers (the buyers with highest type) and in which $E\pi(\underline{v}, \underline{v}) = 0$ maximizes revenue among auctions where buyers win at most one unit each. One way to implement this allocation is to use the uniform price auction where each bidder can submit at most one bid and where the

¹⁸Krishna (2002) contains a thorough discussion of auctions with multiple objects and describes several different auction formats, including the Vickrey and uniform price auctions.

¹⁹Since $n > H$ a buyer with type \underline{v} never wins a unit in an efficient auction. Hence, $E\pi(\underline{v}, \underline{v}) = 0$ as long as losers never receive a positive transfer.

winners are the H highest bidders, who each pays the highest losing bid. It is a dominant strategy to bid one's true valuation in such an auction, implying that revenue is $Hg_1(v_{H+1})$. On the other hand, Vickrey (1961) has proposed another auction which efficiently allocates all the units (this auction is now known as the Vickrey auction). In the Vickrey auction a buyer who wins k units pays the sum of the k highest losing bids submitted by his rivals. For a buyer with valuation v it is a dominant strategy to submit h bids equalling the buyer's marginal valuations, i.e. the set of bids is $\{g_1(v), g_2(v), \dots, g_h(v)\}$, from which it follows that the auction is efficient. Notice that buyer i has no chance of winning any objects if $i = H + 1, \dots, n$. Now, if buyer i , $i \leq H$, wins k objects he must pay at least $g_1(v_{H+2-k}) + \dots + g_1(v_{H+1}) \geq kg_1(v_{H+1})$ because if he wins k objects buyers with higher index (that is, with lower type) are excluded. In other words, the unit price is at least $g_1(v_{H+1})$. Hence, the (efficient) Vickrey auction is more profitable than the best auction among auctions where each buyer wins at most one unit. ■

To proceed to more general auctions, we focus in the following on the most simple multi-unit case, with $h = 2$ and $n, H \geq 2$ as well as $nh > H$ (excess demand). Furthermore, assume that $g_1(\underline{v}) < g_2(\bar{v})$. Finally, we impose the following assumption.

ASSUMPTION D: $J_2(v_1) - J_1(v_2) \geq 0$ whenever $g_2(v_1) = g_1(v_2)$.

The consequence of Assumption D is that any auction in which the units are shared between too many buyers is revenue inferior to an efficient auction. Essentially, the reason is that buyer 1's marginal revenue on the second unit exceeds buyer 2's marginal revenue on the first unit, when the two buyers derive the same marginal utility from the unit. Consequently, changing the rules of the auction to favor buyer 2 at the expense of buyer 1 lowers revenue.

Assumption D has an appealing interpretation in the monopoly context. *Specifically, it is equivalent to assuming that, for any given capacity, the monopolist prefers not to use quantity premia.* To see this, assume the monopolist is charging a constant unit price, p , and that $p = g_2(v_1) = g_1(v_2)$. The quantity sold in a given market is $Q_p = 1 - F(g_1^{-1}(p)) + (1 - F(g_2^{-1}(p)))$, or $Q_p = 1 - F(v_1) + 1 - F(v_2)$ since buyers with type above v_2 will buy one unit, and buyers with type above v_1 will buy two units. However, since $J_2(v_1) > J_1(v_2)$, revenue can be increased by stipulating a price of p_1 , marginally higher than p , for the first unit and a price p_2 , marginally below p , for the second unit purchased, where (p_1, p_2) leads to the same quantity, Q_p ,

being sold. By offering a *quantity discount*, the new price schedule results in units being “transferred” from buyers with low marginal revenue (on the first unit) to buyers with high marginal revenue (on the second unit).

Notice that when $g_2(v_1) = g_1(v_2)$,

$$J_2(v_1) - J_1(v_2) = g_1'(v_2) \frac{1 - F(v_2)}{f(v_2)} - g_2'(v_1) \frac{1 - F(v_1)}{f(v_1)}. \quad (27)$$

Hence, in order to ensure that Assumption D is satisfied, it is sufficient that

$$(i) \quad g_1'(v) - g_2'(v) \geq 0, v \in [\underline{v}, \bar{v}]$$

and

$$(ii) \quad \frac{d}{dv} \left(g_j'(v) \frac{1 - F(v)}{f(v)} \right) \leq 0 \text{ for } j = 1, 2.$$

Condition (i) is equivalent to requiring that the aggregate (inverse) demand curve for the first unit is steeper than the aggregate (inverse) demand curve for the second unit, see (15).²⁰ Condition (ii) is satisfied if the inverse hazard rate,

$$\frac{1 - F(v)}{f(v)},$$

is decreasing in v and if assumption C is satisfied.²¹ Note that if the inverse hazard rate is decreasing, $J(\cdot)$ is increasing, see (18). Though these conditions are sufficient, they are obviously not necessary. When v_1 is sufficiently close to \bar{v} and $g_2(v_1) = g_1(v_2)$, $J_2(v_1) - J_1(v_2) > 0$ since the last term in (27) is close to zero.

Proposition 7 *Assume buyers are symmetric, $h = 2$ and $n, H \geq 2$ as well as $nh > H$. Furthermore, assume that $g_1(\underline{v}) < g_2(\bar{v})$, that $J_j(\cdot)$ is strictly increasing, $j = 1, 2$, and that Assumption D is satisfied. Then, any output efficient (but potentially allocatively inefficient) auction in which a buyer never wins two units when this is inefficient is revenue inferior to an efficient auction with $E\pi(\underline{v}, \underline{v}) = 0$.*

²⁰The condition is satisfied if $\varepsilon_1(v)$ is decreasing and may be satisfied if it is increasing. Notice that the example in Section 2, with $g_2(v) = kg_1(v)$, satisfies the condition, and that $\varepsilon_1(v)$ is constant. This specification is commonly used in the literature. See, for instance, Black and de Meza (1992) and Février, Roos and Visser (2003) for models of repeated auctions with this assumption.

²¹A decreasing inverse hazard rate is equivalent to log-concavity of $1 - F(v)$. See Bagnoli and Bergstrom (1989) for examples. Condition (ii) also implies that $\hat{\varepsilon}_j(\cdot)$ is monotonic.

Proof. The proposition follows from (12). Let the set of buyers who wins at least one unit in an efficient auction be denoted by W and let the set of buyers who does not win a unit be denoted by L . Assume buyer w wins fewer units in the inefficient auction than he would have won in an efficient auction, $w \in W$. If buyer w would have won one unit in an efficient auction but wins none, some buyer l , $l \in L$ must now win one unit (since the inefficient auction is assumed to be output efficiency). However, this causes revenue, (12), to be lowered since $J_1(v_w) \geq J_1(v_l)$, which follows from the fact that $J_1(\cdot)$ is increasing and that $v_w \geq v_l$ (since $w \in W$, $l \in L$). A similar argument applies if buyer w would have won two units in an efficient auction but wins fewer units in the inefficient auction. In this case, $g_2(v_w) \geq g_1(v_l)$ and it follows that $J_1(v_w), J_2(v_w) \geq J_1(v_l)$ by the assumptions in the proposition. Hence, taking one unit (or two) from a buyer in W and “transferring” it, by changing the auction rules, to a buyer (or two) in L decreases the term inside the expectations operator in (12) and thus lowers revenue. ■

To illustrate the potential usefulness of results of this type, consider the uniform price auction in which each buyer submits $h = 2$ bids and where the winner of k units, $k = 1, 2$, pays k times the highest losing bid. Since there is a chance that a buyer’s lowest bid turns out to be the highest losing bid, or the unit price, buyers have an incentive to submit a low second bid in order to obtain one unit cheaply. In fact, in any equilibrium in undominated strategies, buyer i submits one bid of $g_1(v_i)$, and one bid lower than $g_2(v_i)$.²²

When $H = 2$, the fact that the buyer shades his second bid below his marginal utility may result in him winning only one unit, when he would have won two units in an efficient auction. However, it is impossible to win two units, when he would have won fewer units in an efficient auction, because there is no bid shading on the highest bid submitted by the rival buyers. Hence, Proposition 7 applies.

On the other hand, when $H > 2$, it is possible that a buyer wins two units, when he would have won fewer in an efficient auction, because rival buyers also engage in bid shading. For instance, if $H = 3$ and $g_1(v_1) > g_1(v_2) > g_2(v_1) > g_2(v_2) > g_1(v_3)$, it is efficient for buyer 1 to get two units,

²²See Vickrey (1961), Engelbrecht-Wiggans and Kahn (1998) or Krishna (2002) and the references therein. There are other equilibria as well, however. For instance, if $H = h = 2$, it is an equilibrium for buyer 1 to submit two bids of $g_1(\bar{v})$ and for everybody else to submit bids of zero. Revenue is zero in this equilibrium. While this is certainly lower than revenue in any efficient auction with $E\pi(\underline{v}, \underline{v}) = 0$, this equilibrium is in fact not encompassed by the following result.

and buyer 2 to get one unit. However, if buyer 1 shades his second bid more aggressively than buyer 2, it is possible that the allocation is reversed. Although Proposition 7 does not apply in this case, notice that the sum of marginal revenues has decreased, simply because marginal revenues are increasing in type.

Using arguments of this kind, we can conclude that the uniform price auction is revenue inferior to an efficient auction, such as the Vickrey auction, without having to derive the equilibrium strategies in the former auction.²³

Proposition 8 *Assume buyers are symmetric, $h = 2$ and $n, H \geq 2$ as well as $nh > H$. Furthermore, assume that $g_1(\underline{v}) < g_2(\bar{v})$, that $J_j(\cdot)$ is strictly increasing, $j = 1, 2$, and that Assumption D is satisfied. Then, in any equilibrium in undominated strategies of a uniform price auction, expected revenue is lower than in (the truth-telling equilibrium of) a Vickrey auction.²⁴*

Proof. We start by showing that $E\pi(\underline{v}, \underline{v})$ is higher in the uniform price auction than in the Vickrey auction. If $n > H$, a buyer of type \underline{v} wins zero units in the efficient Vickrey auction, so $E\pi(\underline{v}, \underline{v}) = 0$. If $nh > H \geq n$, a buyer of type \underline{v} may win one unit in a Vickrey auction, but never two units. If he wins one unit, the price he pays is $g_2(z)$, where z is the highest type among the rival buyers who win precisely one unit, $z \geq \underline{v}$. Now, if he won zero units in the Vickrey auction, he may win one unit in the uniform price auction, which leaves him better off. If he won one unit in the Vickrey auction, he may win one or two units in the uniform price auction. If he wins two units, the unit price must be lower than $g_2(\underline{v})$, which in turn is lower than $g_2(z)$, because we know that his second bid, which is no higher than $g_2(\underline{v})$, is higher than the highest losing bid. If he wins one unit, the price paid must be no higher than $g_2(z)$, for the simple reason that any bid above $g_2(z)$ must be a winning bid, and thus cannot determine the unit price. Hence, in

²³While revenue is lower in expectation in the uniform price auction, realized revenue may at times be higher. For example, assume that $v \in [1, 4]$ and that $g_1(v) = v$, $g_2(v) = .5v$ and assume the realized types are $v_1 = 4$, $v_2 \in (1, 2)$, $v_i = 1$ for $i = 3, 4, \dots, n$. Assume $h = H = 2$. In the efficient Vickrey auction buyer 1 wins both units and revenue is $1 + v_2$. In the inefficient uniform price auction, on the other hand, buyer 1's second bid, b_1^2 , is below 2 (bid shading) and above 1 (a bid below 1 will be a losing bid but it will never be the highest losing bid). Then, if $v_2 > b_1^2$ buyer 1 and buyer 2 each wins one unit, and revenue is $2b_1^2$. If v_2 is sufficiently close to b_1^2 then $2b_1^2 > 1 + v_2$.

²⁴If $H \geq nh$ (excess supply) all buyers get h units in both auctions, and the price is zero. Hence, they are revenue equivalent.

any event, the type \underline{v} buyer is better off in the uniform price auction than in the Vickrey auction.

Next, we show that the term inside the expectation operator in (12) is higher for the Vickrey auction than for the uniform price auction. First, notice that anybody who won at least one unit in the Vickrey auction must also win at least one unit in the uniform price auction, because the first bid is not shaded below marginal utility. Hence, if the allocation changes as we switch from the Vickrey auction to the uniform price auction, it must be the case that (at least one) buyer wins one unit instead of two, and that some other buyer, or buyers, wins more units than before. If one buyer wins one unit rather than two, the unit freed up moves to someone who would win one unit in the Vickrey auction, or to someone who would win none. In both cases, the sum of marginal revenue declines, in the former case because $J_2(\cdot)$ is increasing in type and the buyer losing a unit has the higher type, and in the latter case because the buyer losing a unit has higher willingness to pay for a second unit than the new winner has for the first unit and so, by Assumption D, his marginal contribution to revenue is higher. If two (or more) buyers win one unit rather than two, the freed units can either be split among two (or more) buyers, or end up with one buyer who would then win two units rather than zero units. In the former case, the same argument as before reveals that the sum of marginal revenues declines. In the latter case, lost marginal revenue is $J_2(v_a) + J_2(v_b)$, where a and b are the buyers losing a unit, and the gained marginal revenue is $J_1(v_c) + J_2(v_c)$, where c is the new winner. It must be the case that $g_2(v_a), g_2(v_b) > g_1(v_c)$. Hence, $J_2(v_a) > J_2(v_c)$, since $v_a > v_c$, and $J_2(v_b) > J_1(v_c)$, since $g_2(v_b) > g_1(v_c)$ enables us to invoke Assumption D. Thus, lost marginal revenue exceeds gained marginal revenue. ■

In situations in which $h > 2$, Assumption D simply becomes $J_{j+1}(v_1) - J_j(v_2) \geq 0$ whenever $g_{j+1}(v_1) = g_j(v_2)$, $j = 1, 2, \dots, h - 1$. Again, its interpretation is that a monopolist would want to offer a price schedule that is non-increasing. Generalizing Proposition 8, however, is not quite as straightforward, because the allocation may change significantly when switching from a Vickrey auction to a uniform price auction, especially in asymmetric equilibria.

However, focusing on symmetric equilibria yields some useful insights. First, in a symmetric equilibrium, the buyer with the highest type wins at

least as many units as the buyer with the second highest type, and so on.²⁵ With $h = 2$, this implies that the allocation changes only if a buyer, buyer 1 say, who was supposed to win two units only wins one, and that a buyer, buyer 2, who was supposed to win zero units wins one unit. From the outcome of the Vickrey auction we know that $g_2(v_1) > g_1(v_2)$. Assumption D then implies that revenue decreases if the two buyers share two units.

Now, without deriving the equilibrium, Ausubel and Cramton (2002) notice that there is an incentive for what they term “differential shading”, i.e. higher bid shading on unit 3 than on unit 2, and so on. Recall that the reason that bid shading is attractive is that the bid may end up being the highest losing bid, and therefore the unit price. Then, if the k 'th bid submitted by a buyer turns out to be the purchase price, he benefits from a lower unit price on all the $k - 1$ units he wins. The higher k is, the bigger is the gain from bid shading. This explains the incentive for differential bid shading.

However, symmetric and differential bid shading implies that the H units are shared more “evenly” among buyers than is efficient. That is, the buyer with the highest type wins fewer units than in the Vickrey auction (but still more than anybody else), and buyers with lower types win more units than in the Vickrey auction. Again, if a unit changes hands, it moves from someone with a high willingness to pay on that unit to someone with a lower willingness to pay who wins fewer units. But then, Assumption D reveals that the marginal revenue lost exceeds the marginal revenue gained.

This argument suggests that the Vickrey auction dominates the uniform price auction in terms of revenue when $h \geq 2$, at least in a symmetric equilibrium with differential bid shading, provided one exists.

Another restriction in Proposition 8 is the fact that buyers are assumed to be symmetric. Symmetry allows us to compare marginal revenue across buyers, and to establish that type \underline{v} is better off in the uniform price auction than in the Vickrey auction. The next example illustrates that the second property does not necessarily hold when buyers are asymmetric. Consequently, the uniform price auction may dominate the Vickrey auction, when buyers are asymmetric.

²⁵This depends on there being zero probability of ties in bids. If bids are monotone in the sense that the bid for unit j is strictly higher than the bid for unit $j + 1$, and that bids are strictly increasing in type, this is the case. Engelbrecht-Wiggans and Kahn (1998) show that when $h = 2$, any symmetric equilibrium has monotone bids, whenever bids have a positive probability of winning. Pooling of bids (at zero) may occur, but only if these bids have no chance of being winning bids.

Assume $H = 2$, and that there are two asymmetric buyers. Buyer 2 is known to value the first unit consumed at 2, and the second unit at 1.²⁶ Regardless of type, buyer 1 values the first as well as the second unit consumed above 2. In the Vickrey auction, buyer 1 wins both units, and pays 3. In the uniform price auction, his second bid is either above or below 2. If it exceeds 2, he again wins both units, but pays 4. If below 2, he will win one unit, and pay at least 0. Hence, if his valuation for a second unit is higher than 4 regardless of type, he always prefers submitting two bids above 2, and revenue is 4. Though the allocation is the same in the two auctions, revenue equivalence does not obtain because buyer 1 of type \underline{v}_1 is better off in the Vickrey auction (more rent is extracted in the uniform price auction). Notice that with symmetric buyers, a buyer of type \underline{v} never wins two units when there is excess demand.

With symmetric buyers, auctions in which the units are shared between too many buyers produce low expected revenue. In the optimal auction, the same line of argument can be used to show that the units must necessarily be concentrated in the hands of fewer buyers than is efficient. If $J_2(v) - J_1(v_2) > 0$ when $v = g_2^{-1}(g_1(v_2))$ it must also hold that $J_2(v_1) - J_1(v_2) > 0$ when v_1 is lower than, but close to, v . Then, buyer 1 is awarded two units in the optimal auction, but only one unit in an efficient auction, at least if $H = 2$.

6.2 Output inefficiency: Reserve prices vs. entry fees

In the following, we take a closer look at the Vickrey auction with symmetric buyers. Specifically, the focus is on the use of reserve prices and entry fees. When $h = 1$, it is well known that the two are revenue equivalent. For any reserve price, r , buyers with valuations above r participates. But for any r , there exists an entry fee, $c(r)$, such that, in a symmetric equilibrium, buyers with valuations above r participates. In either case, the object is won by the buyer with the highest valuation, provided this exceeds r . Since the allocation is the same, expected revenue is also the same.

This argument, however, does not apply when buyers have multi-unit demand. With a *uniform reserve price*, i.e. a reserve price that is the same for each unit, a buyer participates in the auction if his valuation for the first unit exceeds the reserve price.²⁷ For a given reserve price, let x denote

²⁶The important feature of the example is that the range of buyers' valuations do not overlap, not that buyer 2's distribution function is degenerate.

²⁷The Vickrey auction with a uniform reserve of r works the following way. Only bids

the *exclusion level*, such that a buyer participates if, and only if, his type is above x (the reserve price is $g_1(x)$). Notice that even if a buyer is the only one to participate, he may elect to buy only one unit, since the reserve may be higher than his willingness to pay for a second unit.

As before, we can again choose an entry fee, $c(x)$, such that a buyer participates if, and only if, his type exceeds x . *However, though the exclusion level is the same, the allocation is not necessarily the same.* The reason is that if a given buyer is the only one to enter the auction in the presence of an entry fee, he will certainly buy h units (when $H \geq h$), since the unit price will be zero in the absence of competition. With a uniform reserve, he will only buy one unit if his type is close to x .²⁸

To illustrate the consequences of this difference, consider the special case with $n \geq H = h = 2$. Then, for a constant exclusion level, the allocation differs only if precisely one buyer enters. Assume that $x > \underline{v}$, implying that $E\pi(\underline{v}, \underline{v}) = 0$ since \underline{v} never participates, and that $g_1(x) < g_2(\bar{v})$, such that a buyer with type \bar{v} would be willing to buy two units when the reserve is $g_1(x)$.²⁹ The difference in expected revenue between the Vickrey auction with an entry fee of $c(x)$ and the Vickrey auction with a uniform reserve price of $g_1(x)$ is then

$$ER_c(x) - ER_r(x) = nF(x)^{n-1} \int_x^{g_2^{-1}(g_1(x))} J_2(z)f(z)dz. \quad (28)$$

To understand (28), notice again that the allocation is different if $n - 1$ buyers have type below x , and that the remaining buyer has type above x , thus participates, and below $g_2^{-1}(g_1(x))$, such that he buys two units with an entry fee, but only one unit with a reserve of $g_1(x)$. Integrating under the

above r are considered by the seller, and the price paid for the j 'th unit won is the maximum of r and the j 'th highest losing bid submitted by the competitors. Alternative, the seller could himself submit H (shill) bids of r to augment the standard Vickrey auctions and achieve the same result.

²⁸For entry fees as well as uniform reserve prices, the units sold are allocated efficiently among the participating buyers. Strictly speaking, though the auction with a reserve price is allocatively efficient, the auction with an entry fee is not. The reason is that the lone participant may win two units, even though some non-participant may have a higher valuation for a first unit than the participant has for a second unit.

²⁹If $g_1(x) \geq g_2(\bar{v})$, it is easily shown that the auction with an entry fee of $c(x)$ is more profitable than the auction with a uniform reserve price of $g_1(x)$.

marginal revenue curve yields

$$ER_c(x) - ER_r(x) = nF(x)^{n-1} \times [g_2(x)(1 - F(x)) - g_1(x)(1 - F(g_2^{-1}(g_1(x))))]. \quad (29)$$

As an aside, consider the following thought experiment involving a monopolist selling in one market. If he decides to sell in *bundles* of two units charging $g_1(x) + g_2(x)$ for the bundle, buyers with type above x will decide to buy. On the other hand, if he decides to employ *uniform pricing*, and charge $g_1(x)$ per unit, he will sell two units to buyers of type above x , and one unit to buyers with type above $g_2^{-1}(g_1(x))$. The difference in revenue from the two policies is

$$(g_1(x) + g_2(x))(1 - F(x)) - g_1(x)(1 + F(x) + 1 - F(g_2^{-1}(g_1(x))))),$$

which reduces to the term in brackets in (29).

Hence, for a given exclusion level, *entry fees are more profitable than uniform reserve prices if, and only if, a monopolist would prefer bundling to uniform pricing.*³⁰

Of course, the optimal exclusion level depends on whether entry fees or reserve prices are used. Even taking this into account, it is impossible to unambiguously rank entry fees and reserve prices in terms of revenue. This is most easily demonstrated by considering the case with one player, $n = 1$. If $v_1^0 = v_2^0$, for instance because elasticity of demand is independent of type, a take-it-or-leave it price, or fee, of $g_1(v_1^0) + g_2(v_2^0)$ for the bundle of both units implements the optimal allocation (but only as long as $n = 1$), and so a Vickrey auction with an entry fee dominates any other auction, including auctions with uniform reserve prices.

On the other hand, if $v_2^0 = g_2^{-1}(g_1(v_1^0))$, the discussion following Assumption D reveals that it is optimal to charge a constant unit price. In this case, with $n = 1$, uniform reserve prices implement the optimal allocation, and are therefore superior to entry fees.

Proposition 9 *In a Vickrey auction with multi-unit demand, entry fees and uniform reserve prices are not revenue equivalent, and an unambiguous revenue ranking cannot be obtained.*

³⁰This explains why an entry fee is preferable to a uniform reserve price if $g_1(x) \geq g_2(\bar{v})$. With a reserve price, the set of participants buy exactly one unit. However, any buyer in this set would be willing to pay more for a bundle of two units.

While reserve prices and entry fees are revenue equivalent when buyers demand only one unit, this is no longer the case with multi-unit demand. Furthermore, with unit demand, there is no reason to combine reserve prices and entry fees, but this is not true with multi-unit demand. Again, for $n = 1$, combining an entry fee of c and a uniform reserve price of r in effect implies that the price of the first unit is $c + r$, and the price of the second unit r . Hence, the combination leads to quantity discounts.

Another feature of the model with unit demand is that the optimal reserve price is independent of the number of bidders. The optimal reserve price equals $g_1(v_1^0)$, and therefore excludes everybody with negative marginal revenue. However, with multi-unit demand, the optimal uniform reserve price depends on the number of buyers. The reason is that as the number of buyers increases, participation in the auction increases, for a given exclusion level. Hence, it becomes less likely that someone will win two units in the Vickrey auction, and so multi-unit demand becomes less relevant in the design problem.

Proposition 10 *Assume buyers are symmetric, marginal revenues are strictly increasing, that $n \geq H = h = 2$ and $g_2(\bar{v}) > g_1(v_1^0)$ with $v_1^0 > \underline{v}$, and that Assumption D holds with a strict inequality. Then, in a Vickrey auction with uniform reserve prices, the optimal exclusion level, x_r , and reserve price, $g_1(x_r)$, are strictly lower than if buyers had unit demand. Furthermore, if the optimal exclusion level is a unique local maximum and strictly above \underline{v} , it is strictly increasing in n , as is the reserve price.*

Proof. Consider the case where the exclusion level and reserve price is low enough to afford the buyer a chance to win two units, $g_1(x) < g_2(\bar{v})$, but that $x > \underline{v}$. It can be show that the condition $g_2(\bar{v}) > g_1(v_1^0)$ implies that the optimal exclusion level indeed satisfies $g_1(x) < g_2(\bar{v})$. Then, a buyer with type above x may win one or more units, and a buyer with type above $g_2^{-1}(g_1(x))$ has a chance of winning two. Expected revenue per buyer is, using (10),

$$ER_r(x) = \int_x^{\bar{v}} J_1(z)q_1(z)f(z)dz + \int_{g_2^{-1}(g_1(x))}^{\bar{v}} J_2(z)q_2(z)f(z)dz, \quad (30)$$

where q_j is the probability that the buyer wins at least j units. $E\pi(\underline{v}, \underline{v}) = 0$ because buyers with type \underline{v} are excluded, since $x > \underline{v}$. In the following,

let $g_2^{-1}(y)$ be the inverse of $g_2(\cdot)$ if $y < g_2(\bar{v})$ and let $g_2^{-1}(y)$ be equal to \bar{v} otherwise. Then, the probability of winning at least one unit, $q_1(\cdot)$, equals

$$q_1(z) = F(z)^{n-1} + (n-1)F(z)^{n-2}(F(g_2^{-1}(g_1(z))) - F(z)),$$

since the buyer of type z , $z \geq x$, wins one unit if he is the buyer with the highest type, or if he is the buyer with the second highest type and second highest marginal valuation. The probability that he wins two units is

$$q_2(z) = F(g_1^{-1}(g_2(z)))^{n-1},$$

since it requires that his valuation for the second unit, $g_2(z)$, exceeds the highest rival valuation.

Assuming the exclusion level is above \underline{v} and by invoking Assumption D, it can be seen from inspecting (30) (or from the first order condition) that $J_1(x_r) < 0 < J_2(g_2^{-1}(g_1(x_r)))$, implying that $x_r < v_1^0$ and that the reserve price is below $g_1(v_1^0)$. The derivative of (30) with respect to x for a given n can be written as

$$ER'_r(x, n) = F(x)^{n-1} [-J_1(x)A(x, n) - J_2(g_2^{-1}(g_1(x)))B(x)],$$

where $A(x, n)$ is strictly positive and strictly increasing in n , and $B(x) > 0$. Hence, on the interval in which $J_1(x) < 0 < J_2(g_2^{-1}(g_1(x)))$, $ER'_r(x, \cdot)$ starts positive and ends negative, and $ER'_r(x, n+1)$ is strictly positive whenever $ER'_r(x, n)$ is non-negative. Since there is a unique interior maximum for a given n , it follows that the optimal exclusion level strictly increases when n increases. ■

7 Optimal bundling in monopoly

Lemma 1 provides insight that turn out to be useful for a monopolist. Hence, in this section we consider the optimal selling strategy of a monopolist without quantity constraints. As before, marginal costs are normalized to zero.

When we used Figure 3 to study the auctioneer's problem, the only type specific marginal revenue curve we used were the one associated with the actual realized type of the buyer in question. The monopolist, on the other hand, faces a number, $f_i(v_i)$, of consumers of type v_i , $\forall v_i \in [\underline{v}_i, \bar{v}_i]$, in market i , so to him every single type specific marginal revenue curve that can be extracted from Figure 3a is relevant. Since it is assumed that there is no

quantity constraint, consumers of type v_i in Figure 3 are given units as long as marginal revenue is positive. Since the monopolist can enforce third degree price discrimination and is not capacity constrained, the selling strategy in one market is independent of the strategy in another market. Hence, we suppress subscript i , and summarize the results in the following proposition.

Proposition 11 *Assume the monopolist employs nonlinear pricing to maximize profit. Then*

- (i) *If $\varepsilon'_j(v) > 0$ for all $j, j = 1, \dots, h$, and $J_h(\underline{v}) \geq 0$ all consumers will buy h units.*
- (ii) *If $\varepsilon'_j(v) > 0$ for all $j, j = 1, \dots, h$, and $J_h(\underline{v}) < 0$, the smallest number of units that will be bought by any consumer is \underline{j} , where $\underline{j} = \min\{j | J_j(\underline{v}) \geq 0\}$, $\underline{j} = 0$ if $J_1(\underline{v}) < 0$. The number of units bought will be non-decreasing in type, and there will be no "gaps", i.e. if some consumers buy j units, other consumers will buy $j + 1$ units, $j = \underline{j}, \dots, h - 1$.*
- (iii) *If $\varepsilon'_j(v) \leq 0$ for all $j, j = 1, \dots, h$, all the consumers who buy units will buy h units.*

Proof. (i) If $\varepsilon'_j(v) > 0$ for all $j, j = 1, \dots, h$, and $J_h(\underline{v}) \geq 0$, it follows from Lemma 1 that $J_j(\underline{v}) \geq 0$ for all j . By Assumptions B and C marginal revenue is positive for all types and units in this case. Hence, it is profitable to sell as many units as possible, i.e. h , to every single customer.

(ii) If $J_h(\underline{v}) < 0$ the monopolist should not sell h units to type \underline{v} consumers. From Lemma 1 we know that if $J_j(\underline{v}) < 0$, then $J_{j+1}(\underline{v}) < 0$ as long as $\varepsilon'_j(v) > 0$. Hence type \underline{v} consumers should be allowed to purchase \underline{j} units. Since the type specific marginal revenue curve is higher for higher types, all types should be allowed to buy at least \underline{j} units. From (11) it is clear that marginal revenue for type \bar{v} is strictly positive on any unit, so this type, at least, should buy h units. To see that there can be no "gaps", observe that the type specific marginal revenue curves are downward sloping in the first quadrant and increasing in type.

(iii) If $\varepsilon'_j(v) \leq 0$ for all $j, j = 1, \dots, h$, type specific marginal revenue will be positive for unit $j + 1$ if it is positive for unit j . Hence, if it is profitable to sell j units to a specific consumer, profit will increase if he is sold an additional unit. Therefore, all consumers with types above x_h^0 , cf. (26), will end up purchasing h units. ■

When $\varepsilon_j(v)$ is increasing in type, different consumers will in general buy different quantities in light of the profit maximizing pricing policy. By posting nonlinear prices, the monopolist will force customers to select for themselves the appropriate quantity. However, as evident from Proposition 11, there are cases in which all the consumers who decide to buy, will decide to buy the same quantity. This can be achieved either by nonlinear pricing, or simply by bundling units together and charging one price for the entire bundle. In this respect the assumption that all consumers (in a specific market) demand the same quantity of goods is important.

It is clear from Proposition 11 that as long as $\varepsilon'_j(v)$ have the same sign for all j , $j = 1, \dots, h - 1$, there will be no gaps in consumption. Goldman, Leland and Sibley (1984) note the possibility of such gaps, “i.e. ranges of [quantity] over which nobody chooses to consume”. As long as $\varepsilon'_j(v)$ has the same sign for all v , for any given j , these gaps can occur only if $\varepsilon'_j(v)$ does not have the same sign for all j . Figure 5 depicts such a situation in a three unit example, where $\varepsilon'_1(v) > 0$ and $\varepsilon'_2(v) < 0$. Clearly, all consumers should be induced to buy at least one unit. However, if it is profitable to sell two units to a consumer, it is even more profitable to sell three. Hence, consumers of low type buy one units, while consumers of high type buy three units. No one buys precisely two units.

8 Conclusion

The problem facing an auctioneer whose customers have multi-unit demand is essentially equivalent to the problem facing a monopolist who is able to divide his customers into segments and to use nonlinear pricing in each of the segments.

Using this equivalency, it was discussed how allocative and output inefficiency depend on the elasticity of demand and on whether or not the seller is capacity constrained. An example was given in which a monopolist finds it optimal to design the price schedule in such a way that there are gaps in the quantity consumed by different consumers.

Although the optimal auction is generally allocatively inefficient, not all inefficient auctions are revenue superior to efficient auctions. For instance, the uniform price auction yields lower revenue than the Vickrey auction.

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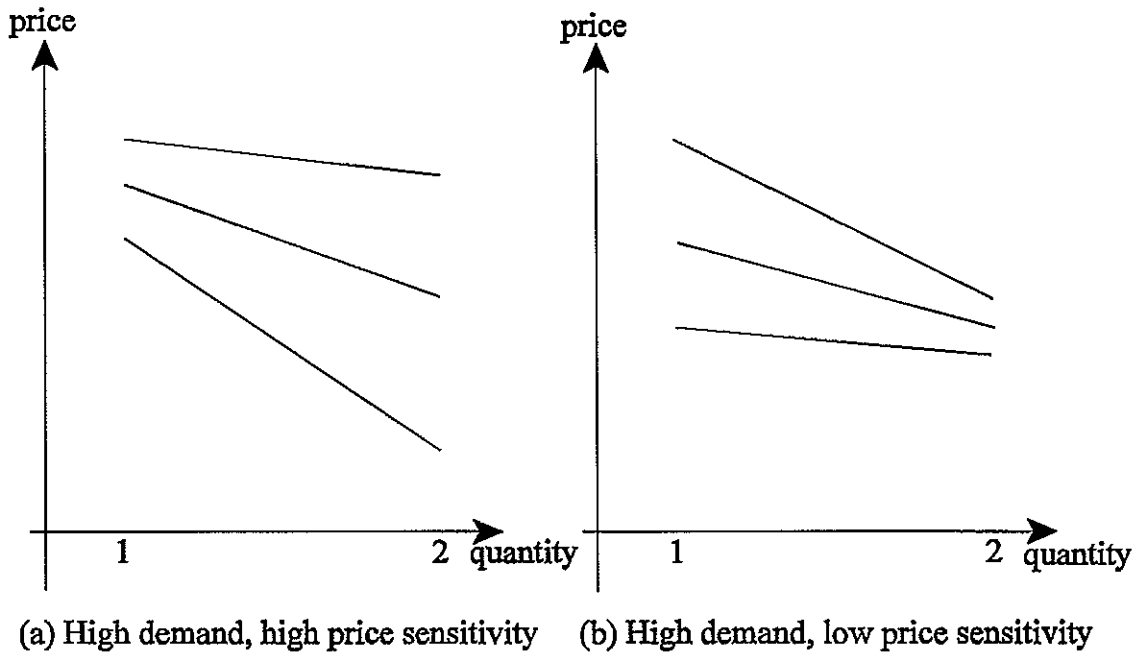


Figure 1: Demand and price sensitivity

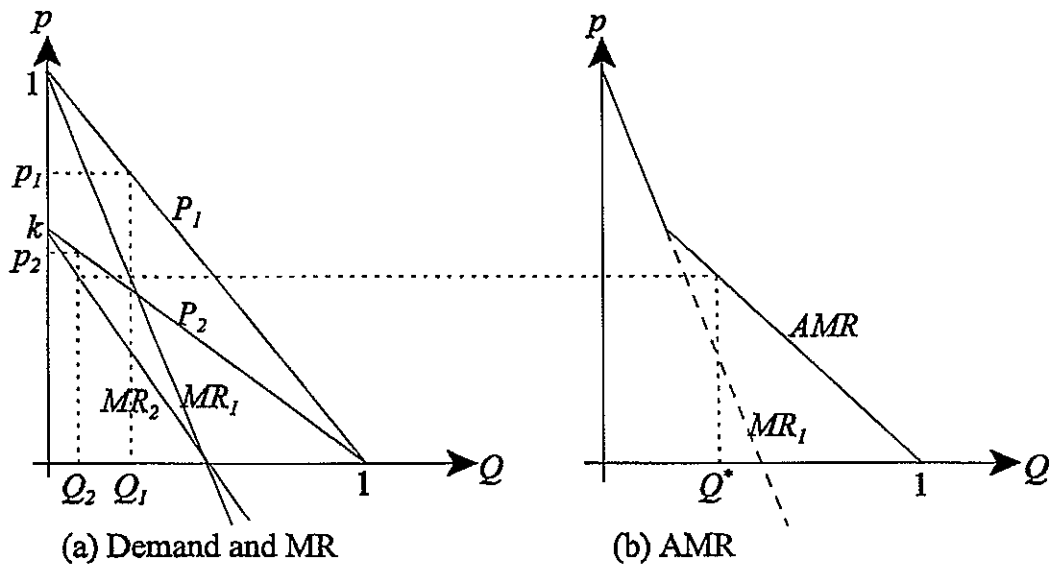


Figure 2: Multi-unit demand

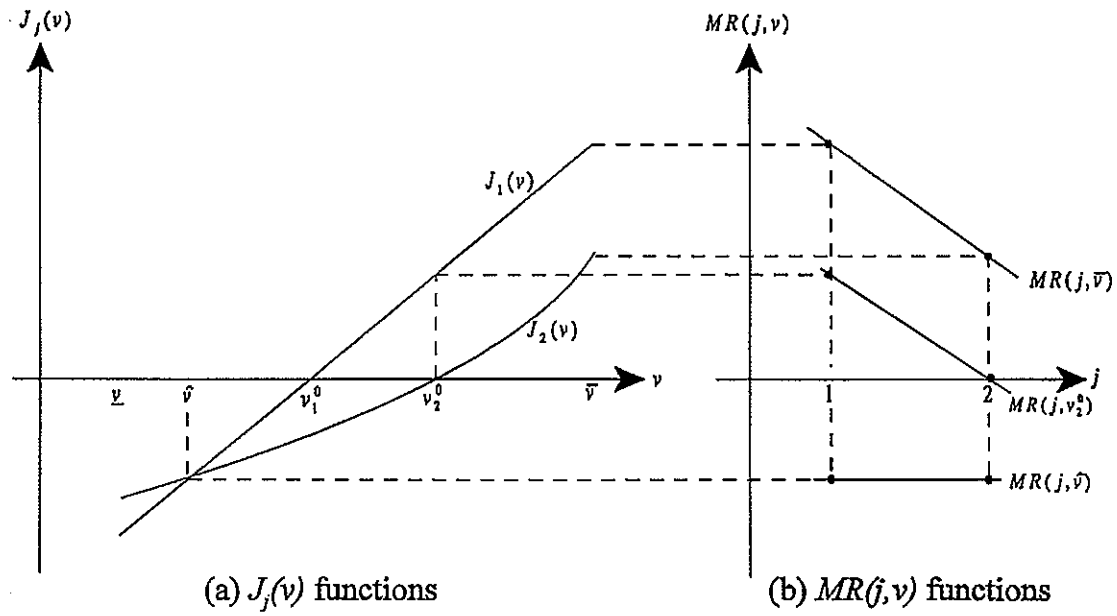


Figure 3: Increasing price sensitivity

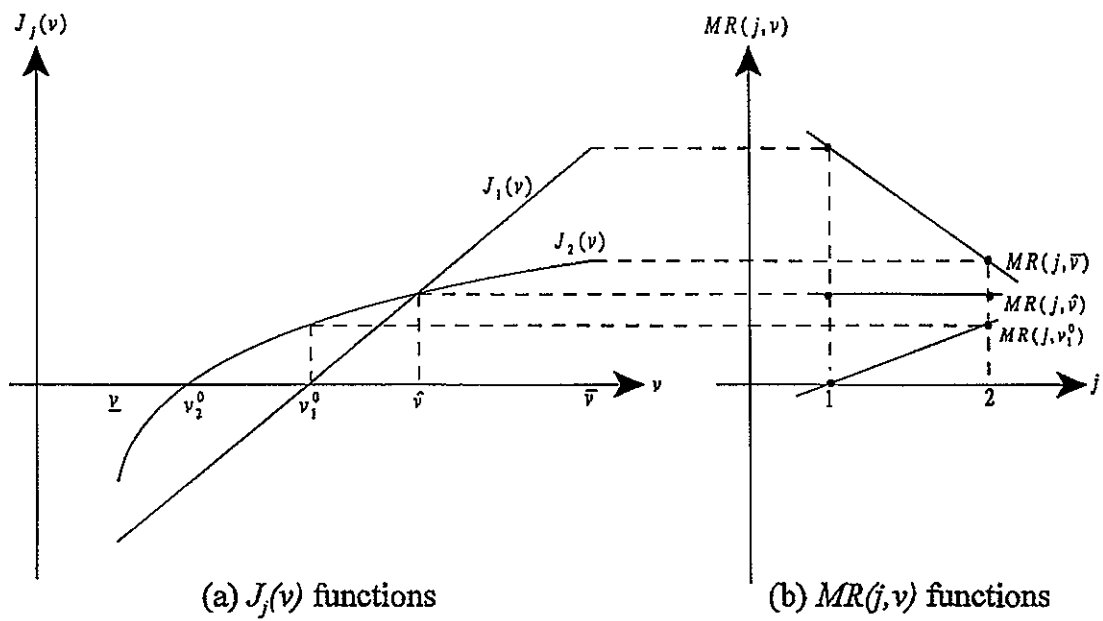


Figure 4: Decreasing price sensitivity

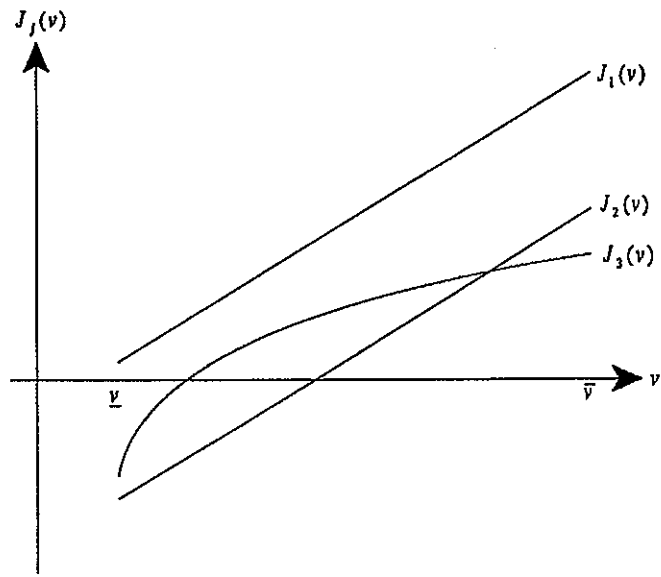


Figure 5: Gaps in consumption