Inefficiency and Nonlinear Pricing in the Optimal Multi-Unit Auction*

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Abstract

Until recently, most theoretical treatments of auctions have assumed that buyers have unit demand. In this paper we characterize the efficiency properties of the optimal auction when buyers have multi-unit demand. The relationship between the elasticity of demand and type is shown to be very important in determining the types of inefficiencies that may arise. Although the optimal auction is inefficient, the efficient Vickrey auction is more profitable than auctions in which the goods are shared more evenly than is efficient, such as the uniform price auction.
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*This paper is an abbreviated version of Chapter 1 in my thesis.

1 Introduction

The majority of studies on auctions has focussed on the situation in which only a single good is for sale. Likewise, the analysis of optimal auctions has been largely constrained to environments where each buyer desires to buy only one of a number of goods offered. While important results have been established in this context, it is important to realize that in many situations the assumption of unit demand is unrealistic. Hence, in this paper we aim to extend the theory of optimal auctions to include buyers with multi-unit demand. In particular, we will focus on the efficiency properties of the optimal auction.

Throughout, we restrict attention to the independent private values model and assume that objects are identical. Buyers demand several units of the same good and marginal utility is assumed to be decreasing.[1] We consider the case where each buyer is described by a one-dimensional type, which has the advantage that we can use Myerson's (1981) approach to design the optimal auction.[2]

When analyzing the efficiency properties of the optimal auction, it will be fruitful to distinguish between two forms of inefficiency, allocative inefficiency and output inefficiency.

The term *allocative inefficiency* is used to describe the fact that sold goods are, in general, not allocated efficiently. That is, sold units can usually be redistributed among consumers to yield a higher social surplus. There are two sources of allocative inefficiency, asymmetry among buyers and multi-unit demand, but we will focus on the latter.

Furthermore, *output inefficiency* occurs when the seller does not meet demand when he has the capacity to do so, or does not exhaust capacity when unable to meet demand.[3]

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[1] Levin (1997) also considers optimal multi-unit auctions. He allows goods to be heterogeneous and explicitly assumes that goods are complements, whereas we follow Maskin and Riley (1989) (see below) in assuming a downwards sloping demand curve.

[2] Engelbrecht-Wiggans (1988) and Krishna and Perry (2000) establish that Myerson's (1981) celebrated Revenue Equivalence Theorem continues to hold when types are multi-dimensional. However, the optimal auction with multi-dimensional types has not been derived. Armstrong (2000) considers auctions for heterogenous objects with additive valuations, and derives the optimal auction when the type space is finite. In a similar model, Jehiel, Meyer-ter-Vehn and Moldovanu (2003) assume the type space is infinite, but they consider only a limited class of auctions.

[3] Armstrong (2000) defines strong and weak efficiency. Strong efficiency describes out-

The paper most closely related to this one is Maskin and Riley (1989). Assuming buyers have multi-unit demand they notice that "the optimal [auction] procedure is thus a nonlinear pricing scheme modified to take account of the supply constraint". This complements Bulow and Roberts' (1989) insight that, when buyers have unit demand, "the analysis of optimal auctions [...] is essentially equivalent to the analysis of standard monopoly third-degree price discrimination".

A central assumption in Maskin and Riley (1989), see also Maskin and Riley (1984), is that the (absolute value of the) elasticity of demand is increasing in type. That is, buyers with higher demand curves are assumed to be more price sensitive than buyers with lower demand curves. We relax this assumption, and show that the elasticity of demand is instrumental in determining the types of inefficiencies that arise in the optimal auction. We present a new result, namely that the probability that all objects are sold may be non-monotonic in the number of objects for sale. In other words, the optimal auction may be more or less likely to be *output inefficient* when the number of objects for sale increases.

Although the optimal auction is generally inefficient, not all inefficient auctions perform well. Focussing on *allocative inefficiency*, we show that auctions in which the units are shared between *more buyers* than is efficient are revenue inferior to efficient auctions. An implication is that the uniform price auction is inferior to the efficient Vickrey auction. This result is intimately linked to the fact that quantity discounts are generally superior to quantity premia. On the other hand, bundling of units, leading to units being shared between *fewer buyers* than is efficient, may be desirable.

The remainder of the paper is organized as follows. Section 2 discusses the design of an incentive compatible auction. In Section 3 the connection between elasticity of demand and output inefficiency in the optimal auction is examined. Section 4 demonstrates that a class of allocatively inefficient auctions are not as profitable as efficient auctions. Section 5 concludes.

2 Auction design

The auctioneer has H homogeneous objects (units), and it is assumed that his valuation (offer cost) of each unit is zero.

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comes that are both allocative and output efficient, while weak efficiency is synonymous with allocative efficiency.

ASSUMPTION A: There are n symmetric buyers.[4] Buyer i, $i = 1, 2, ..., n$, is described by a type, v_i, drawn at random from the interval $[\underline{v}, \overline{v}]$, using the cumulative distribution function $F(\cdot)$ which is strictly increasing and twice continuously differentiable, with $F(\underline{v}) = 0$ and $F(\overline{v}) = 1$. Regardless of type, buyer i demands h units, where $g_j(v_i) \geq 0$ denotes the willingness to pay for the jth unit, with $g_j(v_i) = 0$ for $j > h$. Demand is not upwards sloping, $g_j(v) \geq g_{j+1}(v)$. $g_j(v)$ is twice continuously differentiable with $g_j'(v) > 0$, for $j = 1, ..., h$, implying higher types have higher demand, and demand curves do not cross.

Using the mechanism design approach, it is straightforward to generalize the work of Myerson (1981) to the case of multi-unit demand, and the details will thus be omitted (see Maskin and Riley (1989) or Kirkegaard (2004)).

Let $q_{ij}(x_i, x_{-i})$ denote the allocation of unit j to buyer i, given buyer i reports to be of type x_i and the rivals report types x_{-i}. $q_{ij}(x_i, v_{-i})$ is one if buyer i receives at least j units. Using Myerson's (1981) terminology, define $J_j(\cdot)$ as

$$J_j(v) = g_j(v) - g_j'(v)\frac{1 - F(v)}{f(v)}, \tag{1}$$

where $J_j(v)$ is the virtual valuation of type v on unit j.

Assuming there exists a feasible and incentive compatible mechanism to implement the allocation, it is a straightforward matter using mechanism design to show that expected revenue is

$$E\left[\sum_{i=1}^{n}\sum_{j=1}^{h} J_j(v_i)q_{ij}(v_i, v_{-i})\right] + \sum_{i=1}^{n} E\pi_i(\underline{v}, \underline{v}), \tag{2}$$

where $E\pi_i(\underline{v}, \underline{v})$ is the expected payoff to buyer i, provided his type is $\underline{v}$ (and his report is $\underline{v}$). Since the functions $g_j(\cdot)$ and $F(\cdot)$ are exogenous to the seller, he must maximize expected revenue by determining $E\pi_i(\underline{v}, \underline{v})$ and the allocation. The fact that (2) depends only on $E\pi_i(\underline{v}, \underline{v})$ and on the allocation gives rise to the well known Revenue Equivalence Theorem.[5]

[4] The assumption of symmetry is unimportant in Sections 2 and 3, but it serves to simplify the exposition.

[5] See Myerson (1981) and Riley and Samuelson (1981) for the first Revenue Equivalence Theorems, in which unit demand is assumed. The most general Revenue Equivalence Theorem, encompassing Proposition 1, is to be found in Engelbrecht-Wiggans (1988). See also Krishna and Perry (2000).

Proposition 1 (Revenue Equivalence) *Any two mechanisms that allocate the goods in the same way, and between which each buyer would be indifferent if he had the lowest possible type, generate the same expected revenue.*

However, the seller is constrained in the types of mechanisms he can implement. First, any allocation must satisfy the feasibility constraints,

$$0 \leq q_{ij}(x_i, x_{-i}) \leq 1, \tag{3}$$

$$q_{ij}(x_i, x_{-i}) \geq q_{ij+1}(x_i, x_{-i}) \tag{4}$$

and

$$\sum_{i=1}^{n} \sum_{j=1}^{h_i} q_{ij}(x_i, x_{-i}) \leq H. \tag{5}$$

Second, the mechanism must be incentive compatible. The following result, due to Maskin and Riley (1989), is sufficient for our needs.

Lemma 1 (Maskin and Riley (1989)) *There exists an incentive compatible mechanism that implements the allocation if the allocation is feasible and if, for all i and j, the probability that buyer i wins at least j units is nondecreasing in type.*

Notice that there exists some incentive compatible mechanism that always allocates the objects efficiently.

In addition to the incentive compatibility condition, we also require that the auction satisfies the participation constraint, i.e. expected payoff from participating must exceed that from ignoring the auction, or $E\pi(v, v) \geq 0$. Since any rent the seller permits the type $\underline{v}$ buyer to extract decreases expected revenue, the optimal auction is designed in such a way that $E\pi(\underline{v}, \underline{v}) = 0$.[6] The seller then has one instrument left to maximize profit, namely the rules that govern the probabilities of winning, subject to (3), (4), (5) as well as incentive compatibility.

To proceed, notice that (2) with $E\pi_i(\underline{v}, \underline{v}) = 0$ lends itself to pointwise maximization. That is, maximizing the term inside the expectations operator

[6] Since $g_j'(\cdot)$ and $q_{ij}(\cdot)$ are both non-negative, the former by assumption, $E\pi_i(v, v) \geq E\pi_i(\underline{v}, \underline{v})$, cf. Lemma 1. The participation constraint is therefore fulfilled for all types.

for every realization of types, subject to (3) and (5), and then taking the expectation over the possible types yields an obvious candidate for maximal revenue.

However, to be optimal the allocation resulting from pointwise maximization must also satisfy the last remaining feasibility constraint, (4). Maskin and Riley's (1989) assumption on preferences, namely that a buyer is more price sensitive the higher his type is, is sufficient to ensure that this condition is satisfied. On the other hand, when buyers are less price sensitive the higher their type is, (4) may turn out to be a binding constraint. We explore this further in the next section.

3 Efficiency properties

In this section we examine the efficiency properties of the optimal auction. For simplicity, we will focus on the so-called regular case. That is, $J_j(\cdot)$ is assumed to be strictly increasing in v. This implies that if we maximize the term inside the expectations operator in (2) for every realization of types, the winning probabilities are increasing in type (because $J_j(\cdot)$ is). Hence, the mechanism is incentive compatible, by Lemma 1. To examine the regular case we follow Maskin and Riley (1989) in imposing the following two assumptions.

Assumption B: $J(v)$ is strictly increasing, where

$$J(v) = v - \frac{1 - F(v)}{f(v)}. \tag{6}$$

Assumption C: $g_j''(v) \leq 0$, $j = 1, ..., h$.[7]

Together, Assumptions B and C imply that $J_j(v)$ is strictly increasing since

$$J_j'(v) = g_j'(v) J'(v) - g_j''(v) \frac{1 - F(v)}{f(v)} > 0.$$

As mentioned in the Introduction, the elasticity of demand plays an important role in the optimal auction.

[7] This is Assumption B2 in Maskin and Riley (1989). See also Maskin and Riley (1984).

Definition 1 *The (numerical value of the price) elasticity of demand is*

$$\varepsilon_j(v) = \frac{1}{j}\frac{g_j(v)}{g_j(v) - g_{j+1}(v)},\ \ j = 1, ..., h-1. \tag{7}$$

Notice that (7) can be rewritten as

$$\varepsilon_j(v) = -\frac{j-(j+1)}{j}\frac{g_j(v)}{g_j(v) - g_{j+1}(v)} = -\frac{j-(j+1)}{g_j(v) - g_{j+1}(v)}\frac{g_j(v)}{j}, \tag{8}$$

where j is quantity, and $g_j(v)$ the price. Hence, it is not hard to see that (7) is indeed an expression of elasticity.

Given the prominent role of the elasticity of demand in the results to follow, it is worthwhile to examine it in some detail before proceeding. To do so, it is instructive to take a detour through monopoly pricing (recalling Bulow and Roberts' (1989) observation that auctions and monopoly are similar). Hence, consider the *incremental demand* for a j'th unit, i.e. the imaginary aggregate demand for an extra unit given $j-1$ units have been purchased already. The function $g_j(v)$ can be thought of as the unit specific (inverse) demand curve for unit j. If the price is $g_j(v)$, a mass of $Q(v) = 1 - F(v)$ consumers buy a jth unit.

Now, profit attributable to the sale of the incremental jth unit is $g_j(v)Q(v)$. Marginal revenue can thus be derived,

$$\begin{aligned} MR_j(v) &= \frac{d}{dQ}(g_j(v)Q(v)) = g_j(v) + g_j'(v)\frac{Q}{dQ/dv} \\ &= g_j(v) - g_j'(v)\frac{1-F(v)}{f(v)} = J_j(v), \end{aligned} \tag{9}$$

which confirms Bulow and Roberts' (1989) finding that $J_j(v)$ can be thought of as marginal revenue.

In the following, let

$$v_j^0 = \min\{v | v \in [\underline{v}, \overline{v}],\ J_j(v) \geq 0\}. \tag{10}$$

Notice that v_j^0 is unique, since $J_j(\cdot)$ is strictly increasing. If $v_j^0 = \underline{v}$, marginal revenue on unit j is positive for all types. On the other hand, if $v_j^0 > \underline{v}$, some types, those below v_j^0, have negative marginal revenue on unit j, and it may be optimal to exclude these from the auction.

The demand curve for a j'th unit has slope

$$\frac{dg_j(v)}{dQ} = \frac{g_j'(v)}{dQ/dv} = -\frac{g_j'(v)}{f(v)}, \tag{11}$$

and the (numerical value of the) price elasticity on this incremental market is

$$\widehat{\varepsilon}_j(v) = -\frac{dQ}{dg_j(v)}\frac{g_j(v)}{Q} = \frac{f(v)}{1-F(v)}\frac{g_j(v)}{g_j'(v)}. \tag{12}$$

To proceed, compare $\varepsilon_j(v)$ and $\widehat{\varepsilon}_j(v)$. The former captures how price sensitive the buyer is if he has type v and has purchased j units, or the size of the price reduction needed to induce him to purchase unit $j+1$. The latter measures how price sensitive incremental demand for the j'th unit is at the point where the buyer of type v is the last to buy. The two are related.

Proposition 2 *If the elasticity of demand at unit j is increasing (decreasing) in type, incremental demand for unit j is more (less) price sensitive than incremental demand for unit $j+1$,*

$$\varepsilon_j'(v) \lesseqgtr 0 \Longleftrightarrow \widehat{\varepsilon}_j(v) \lesseqgtr \widehat{\varepsilon}_{j+1}(v). \tag{13}$$

Proof. The derivative of $\varepsilon_j(v)$ is

$$\varepsilon_j'(v) = \frac{g_j'(v)g_{j+1}'(v)}{j(g_j(v)-g_{j+1}(v))^2}\left[\frac{g_j(v)}{g_j'(v)} - \frac{g_{j+1}(v)}{g_{j+1}'(v)}\right], \tag{14}$$

and so the sign of $\varepsilon_j'(v)$ is determined by the term in brackets. From (12),

$$\widehat{\varepsilon}_j(v) - \widehat{\varepsilon}_{j+1}(v) = \frac{f(v)}{1-F(v)}\left[\frac{g_j(v)}{g_j'(v)} - \frac{g_{j+1}(v)}{g_{j+1}'(v)}\right], \tag{15}$$

which obviously has the same sign as $\varepsilon_j'(v)$. ■

Figure 1 sketches two polar assumptions. In the monopoly interpretation we have drawn the inverse demand curves for three different consumers in a given market, although we imagine there is a continuum.[8] In Figure 1a the

[8] Alternatively, in the auction interpretation, the figure concentrates on one buyer (rather than one market). This buyer has private information about his demand curve, or type. There is a continuum of possible types (rather than a continuum of consumers), but we have drawn only three.

consumer with the highest (inverse) demand curve is the most price sensitive, as in Maskin and Riley (1989). This can be seen by noting that the relative price reduction needed to induce this consumer to purchase more units is small relative to the other consumers. In contrast, Figure 1b depicts the possibility that price sensitivity is smaller for consumers with high demand.

Consider now the aggregate demand for one unit, and the imaginary aggregate demand for a second unit, given one unit has already been consumed. In Figure 1a the aggregate demand for one unit is dense, in the sense that the willingness to pay for the first unit does not vary much across consumers. Therefore, a small reduction in the price for the first unit purchased will imply a large increase in sales. Thus, the aggregate demand for one unit is price sensitive. On the other hand, the aggregate demand for a second unit is price insensitive, because a large price reduction is needed to significantly increase sales of a second unit to consumers who have already bought one unit.

We now have two imaginary aggregate demand curves, with different price sensitivities. However, the more price sensitive demand is, the more it is profitable to sell. Applying this to our imaginary demands, it is clear that the monopolist maximizes profits by satisfying more of the demand for one unit than the demand for a second unit. Consequently, the outcome is that the monopolist designs the nonlinear pricing scheme in such a way that some of the consumers who buy one unit are discouraged from buying a second unit.

More formally, as standard in monopoly theory, rearranging (1) yields

$$J_j(v) = g_j(v)\left[1 - \frac{1}{\widehat{\varepsilon}_j(v)}\right], \tag{16}$$

and since $\widehat{\varepsilon}_j(v) > \widehat{\varepsilon}_{j+1}(v)$ it must be the case that $J_j(v)$ is positive whenever $J_{j+1}(v)$ is, and it follows that $v^0_{j+1} \geq v^0_j$. Therefore, the set of types, $[v^0_j, \overline{v}]$, to which it is desirable to sell j units is larger than the set of types, $[v^0_{j+1}, \overline{v}]$, to which it is desirable to sell $j+1$ units.

Figure 2 is another way of illustrating the problem. Figure 2a depicts the $J_1(v)$ and $J_2(v)$ functions.[9] In Figure 2b these are translated into *type specific* marginal revenue schedules, $MR(j,v)$, $j = 1,2$. For a given type, v, these give marginal revenue from the j'th unit, i.e. $MR(j,v) = J_j(v)$.

[9] The fact that $J_j(v) > J_{j+1}(v) > 0$ for $v > v^0_j$ is established by noticing that each term on the right in (16) is higher for $J_j(v)$ than for $J_{j+1}(v)$.

Notice that the $MR(j,v)$ curves in Figure 2b have negative slope as long as $MR(1,v) \geq 0$ and that $MR(2,v)$ is negative whenever $MR(1,v)$ is negative.

Since the type specific marginal revenue curves are well behaved when $\varepsilon_j(v)$ is increasing in v, it is straightforward to describe the allocation in the optimal auction. The mechanism is a standard mechanism – the objects should simply by allocated on the basis of highest marginal revenue, as long as marginal revenue is positive. See Maskin and Riley (1989) or Kirkegaard (2004) for a formal statement and proof.

On the other hand, when demand is described by the demand curves in Figure 1b, the aggregate demand for a second unit is very price sensitive. Hence, the seller would want to sell two units to some consumers that he would never sell only one unit to. Thus, one unit is sold to what appears to be too many consumers, in order to take advantage of the more profitable aggregate demand for a second unit.

Figure 3 is the counterpart of Figure 2, when $\varepsilon_j'(v) < 0$. To generate Figure 3, the following Lemma is useful.

Lemma 2 *If $\varepsilon_j'(v) < 0$ for all $v \in [\underline{v}, \overline{v}]$, then $v_{j+1}^0 < v_j^0$ if $v_j^0 > \underline{v}$ and $v_{j+1}^0 = v_j^0$ if $v_j^0 = \underline{v}$. If $v_j^0 > \underline{v}$, $J_j(v)$ and $J_{j+1}(v)$ will cross as v increases from v_j^0 to $\overline{v}$. If $J_j(v) > 0$, $J_{j+1}(v)$ will also be positive.*

Proof. Using (16) and Proposition 2, if $\varepsilon_j'(v) < 0$, $J_{j+1}(v)$ must be positive whenever $J_j(v)$ is positive, and so $v_{j+1}^0 \leq v_j^0$. Since $J_{j+1}(v_j^0) > J_j(v_j^0)$ and $J_{j+1}(\overline{v}) = g_{j+1}(\overline{v}) < g_j(\overline{v}) = J_j(\overline{v})$ the two marginal revenue curves must cross. ■

When $\varepsilon_j'(v) < 0$ the problem is that the type specific marginal revenue curve may cross the first axis from below (Figure 3b). In this case, the feasibility constraint that $q_{ij}(v_i, v_{-i}) \geq q_{ij+1}(v_i, v_{-i})$ is binding, and the auctioneer has to accept selling a first unit to some types that have negative marginal revenue on the first unit, in order to open the market for a second unit. If he can find a buyer who he would be willing to sell one unit to, it will always be profitable to sell two units to this buyer. Indeed, in these circumstances the probability that objects are sold at all in the optimal auction are higher if the auctioneer has two units to sell, rather than just one.

To illustrate the differences between the optimal auction when $\varepsilon_j'(v) > 0$ (Maskin and Riley (1989)) and $\varepsilon_j'(v) < 0$, it is sufficient to consider the simple case where $h = 2$ (see Figure 3).

Proposition 3 *Assume* $H \geq h = 2$, $\varepsilon_1'(v) < 0$, $v_1^0 > v_2^0 > \underline{v}$, *and assume there is a unique type,* $\widehat{v}$, *for which* $J_1(\widehat{v}) = J_2(\widehat{v}) > 0$ *(by Lemma 2, one exists). Define* x_1^0 *as the unique value for which* $J_1(x_1^0) + J_2(x_1^0) = 0$. *Then, the optimal auction implements the following allocation. First, among all buyers with type exceeding* $\widehat{v}$, *allocate units based on highest marginal revenues. Second, if capacity is not exhausted, allocate units in bundles of 2 among buyers with types between* x_1^0 *and* $\widehat{v}$ *based on highest type, or equivalently highest* $J_1(\cdot) + J_2(\cdot)$ *or* $g_1(\cdot) + g_2(\cdot)$. *Third, if precisely one unit remains hereafter, allocate it to the highest unsurplied buyer, if this buyer has type above* v_1^0.

Proof. It is clear by inspection that for every realization of types, the term inside the expectation operator in (2) is maximized, subject to the feasibility constraint that $q_{ij}(v_i, v_{-i}) \geq q_{ij+1}(v_i, v_{-i})$. Since the probabilities of winning are increasing in type, the auction is incentive compatible. ■

The structure of the optimal auction depends on the relationship between type and elasticity of demand. As a consequence, the efficiency properties of the optimal auction depends on this relationship. This is especially clear when it comes to *output inefficiency*. Recall that as long as $H \leq nh$, the allocation of units is by definition output inefficient if some units remain unsold.

Proposition 4 (Output Efficiency) *If* $\varepsilon_j'(v) \geq 0$, *for all* $j = 1, ..., h-1$, *the probability that all units are sold* ***decreases*** *as* H *increases,* $H < nh$. *If* $\varepsilon_j'(v) < 0$, *for all* $j = 1, ..., h-1$, *the probability that all units are sold is* ***non-monotonic*** *in* H, $H < nh$.

Proof. First, if $\varepsilon_j'(v) \geq 0$ it becomes less likely that all units are sold in the optimal auction, when the number of units (H) increases. The reason is that it becomes increasingly difficult (or unlikely) to find an agent with a positive marginal revenue on an additional unit.

Second, if $\varepsilon_j'(v) < 0$ for all $j = 1, ..., h-1$, the initial number of units may, or may not, be an integer multiple of h. If it is, then an integer multiple of h is sold, because if the seller is willing to sell j units, $j < h$, to a given buyer, then he would be willing to sell h units to this buyer, since $v_{j+1}^0 < v_j^0$. Then, if less than all units were sold, all buyers that were served must have been served in full, and the seller must have been unable to find another buyer to whom he would be willing to sell h units. Getting an additional unit is of no help, and the auction will *remain* output inefficient. Likewise, if all units were sold, the buyers served were either served in full, or some were not served in full. In the latter case, the seller would be willing to supply one of these buyers with the additional unit, again because $v^0_{j+1} < v^0_j$, and the auction *remains* output efficient. However, if all the buyers that were served were served in full, then the seller needs to find an additional buyer to whom he would be willing to sell. Since this is not a probability one event, the auction may *become* output inefficient, and the probability that all units are sold decreases.

On the other hand, if the initial number of units is not an integer multiple of h, an additional unit makes it *more likely* that the auction is output efficient. If all units were sold initially, some buyer must have been allocated fewer than h units, but in that case the seller would be willing to give the additional unit to such a buyer, and the auction *remains* output efficient. If all units were not sold initially, it must be because there was no buyer to whom the seller would sell the remaining quantity. However, with an additional unit the condition on buyer type is less stringent, because there are more types to whom it is profitable to sell $j+1$ units than j units. Hence, the auction may *become* output efficient. ■

Since the optimal allocation is based on marginal revenue rather than willingness to pay, the optimal auction is generally also *allocatively inefficient.* The next section demonstrates that the mechanism design approach can be utilized to dismiss certain mechanisms as being unprofitable based on their efficiency properties.

4 Allocative inefficiency and Vickrey auctions

In this section we investigate the role of allocative inefficiency, and show that any auction in which the units are shared between too many buyers, relative to what is efficient, yields lower revenue than an efficient auction. This leads to the conclusion that the efficient Vickrey auction is revenue superior to the uniform price auction.[10]

Without loss of generality buyers are ordered such that $v_1 \geq v_2 \geq ... \geq v_n$. To focus on the role of allocative inefficiency we disregard output inefficiency. That is, we consider auctions where all units are sold with probability one.

[10]Krishna (2002) contains a thorough discussion of auctions with multiple objects and describes several different auction formats, including the Vickrey and uniform price auctions.

There are many different types of auctions in which the units are shared among too many buyers. At one extreme, consider auctions where it is impossible to win more than one unit. Such auctions are revenue inferior to efficient auctions.

Proposition 5 *Assuming that $J_1(v)$ is strictly increasing in v and that $n > H$, any output efficient auction in which buyers can win at most one unit is revenue inferior to an efficient auction with $E\pi(\underline{v}, \underline{v}) = 0$.*[11]

Proof. If $J_1(v)$ is strictly increasing an auction which allocates one unit to each of the first H buyers (the buyers with highest type) and in which $E\pi(\underline{v}, \underline{v}) = 0$ maximizes revenue among auctions where buyers win at most one unit each. One way to implement this allocation is to use the uniform price auction where each bidder can submit at most one bid and where the winners are the H highest bidders, who each pays the highest losing bid. It is a dominant strategy to bid one's true valuation in such an auction, implying that revenue is $Hg_1(v_{H+1})$. On the other hand, Vickrey (1961) has proposed another auction which efficiently allocates all the units. In the Vickrey auction a buyer who wins k units pays the sum of the k highest losing bids submitted by his rivals. For a buyer with valuation v it is a dominant strategy to submit h bids equalling the buyer's marginal valuations, i.e. the set of bids is $\{g_1(v), g_2(v), ..., g_h(v)\}$, from which it follows that the auction is efficient. Notice that buyer i has no chance of winning any objects if $i = H+1, ..., n$ (implying that $E\pi(\underline{v}, \underline{v}) = 0$). Now, if buyer i, $i \leq H$, wins k objects he must pay at least $g_1(v_{H+2-k}) + ... + g_1(v_{H+1}) \geq kg_1(v_{H+1})$ because if he wins k objects buyers with higher index (that is, with lower type) are excluded. In other words, the unit price it as least $g_1(v_{H+1})$. Hence, the (efficient) Vickrey auction is more profitable than the best auction among auctions where each buyer wins at most one unit. ■

To proceed to more general auctions, we focus on the simplest multi-unit case, with $h = 2$, and impose the following assumption in the remainder of the paper.

Assumption D: (i) $h = 2$ and $n, H \geq 2$ as well as $nh > H$ (excess demand), (ii) $g_1(\underline{v}) < g_2(\overline{v})$ (effective multi-unit demand) and (iii) $J_2(v_1) - J_1(v_2) \geq 0$ whenever $g_2(v_1) = g_1(v_2)$.

[11] Since $n > H$ a buyer with type $\underline{v}$ never wins a unit in an efficient auction. Hence, $E\pi(\underline{v}, \underline{v}) = 0$ as long as losers never receive a positive transfer.

The consequence of the last part of Assumption D is that any auction in which the units are shared between too many buyers is revenue inferior to an efficient auction. Essentially, the reason is that buyer 1's marginal revenue on the second unit exceeds buyer 2's marginal revenue on the first unit, when the two buyers derive the same marginal utility from the unit. Consequently, changing the rules of the auction to favor buyer 2 at the expense of buyer 1 lowers revenue.

Assumption D has an appealing interpretation in the monopoly context. Specifically, it is equivalent to assuming that, for any given capacity, *the monopolist prefers quantity discounts to uniform prices or quantity premia.* By offering a quantity discount, the price schedule results in units being "transferred" from buyers with low marginal revenue (but high willingness to pay) on the first unit to buyers with high marginal revenue (but low willingness to pay) on the second unit when compared to the allocative efficient allocation arising from uniform pricing.

Given Assumption C, the last part of Assumption D is satisfied if $g_1'(v) - g_2'(v) \geq 0$, $v \in [\underline{v}, \overline{v}]$, and the inverse hazard rate, $(1 - F(v))/f(v)$, is decreasing in v.[12,13]

Proposition 6 *Any output efficient (but potentially allocatively inefficient) auction in which a buyer never wins two units when this is inefficient is revenue inferior to an efficient auction with* $E\pi(\underline{v}, \underline{v}) = 0$.

Proof. The proposition follows from (2). Let the set of buyers who wins at least one unit in an efficient auction be denoted by W and let the set of buyers who does not win a unit be denoted by L. Assume buyer w wins fewer units in the inefficient auction than he would have won in an efficient auction, $w \in W$. If buyer w would have won one unit in an efficient auction but wins none, some buyer l, $l \in L$ must now win *exactly one* unit (since buyers, by assumption, do not win more units than is efficient). However,

[12] The first condition is satisfied if $\varepsilon_1(v)$ is decreasing and may be satisfied if it is increasing (in fact, Maskin and Riley (1984) mention the same sufficient conditions but at the same time assume that the elasticity of demand is increasing in type). For instance, if $g_2(v) = kg_1(v)$ the condition is satisfied and $\varepsilon_1(v)$ is constant. This specification is commonly used in the literature. See, for instance, Black and de Meza (1992) and Février, Roos and Visser (2003) for models of repeated auctions with this assumption.

[13] A decreasing inverse hazard rate is equivalent to log-concavity of $1 - F(v)$, and implies Assumption B. See Bagnoli and Bergstrom (2005) for examples. The condition also implies that $\widehat{\varepsilon}_j(\cdot)$ is monotonic (both terms are increasing).

this causes revenue, (2), to be lowered since $J_1(v_w) \geq J_1(v_l)$, which follows from the fact that $J_1(\cdot)$ is increasing and that $v_w \geq v_l$ (since $w \in W$, $l \in L$). A similar argument applies if buyer w would have won two units in an efficient auction but wins fewer units in the inefficient auction. In this case, $g_2(v_w) \geq g_1(v_l)$ and it follows that $J_1(v_w), J_2(v_w) \geq J_1(v_l)$ by Assumption D. Hence, taking one unit (or two) from a buyer in W and "transferring" it, by changing the auction rules, to a buyer (or two) in L decreases the term inside the expectations operator in (2) and thus lowers revenue. ■

To illustrate the potential usefulness of results of this type, consider the uniform price auction in which each buyer submits $h = 2$ bids and where the winner of k units, $k = 1, 2$, pays k times the highest losing bid. Since there is a chance that a buyer's lowest bid turns out to be the highest losing bid, or the unit price, buyers have an incentive to submit a low second bid in order to obtain one unit cheaply. In fact, in any equilibrium in undominated strategies, buyer i submits one bid of $g_1(v_i)$, and one bid lower than $g_2(v_i)$.[14]

Then, when $H = 2$, it is impossible for a buyer to win two units, when he would have won fewer units in an efficient auction, because there is no bid shading on the highest bid submitted by the rival buyers.

However, when $H > 2$, it is possible that a buyer wins two units, when he would have won fewer in an efficient auction since bid shading may be asymmetric across buyers. Although Proposition 6 does not apply in this case, we can nevertheless strengthen it to conclude that the uniform price auction is revenue inferior to an efficient auction, such as the Vickrey auction, without having to derive the equilibrium strategies in the former auction.[15]

Proposition 7 *In any equilibrium in undominated strategies of a uniform price auction, expected revenue is lower than in (the truthtelling equilibrium of) a Vickrey auction.*[16,17]

[14] See Vickrey (1961), Engelbrecht-Wiggans and Kahn (1998) or Krishna (2002) and the references therein.

[15] When types are multi-dimensional, Krishna and Perry (2000) show that the Vickrey auction is optimal among all efficient auctions.

[16] If $H \geq nh$ (excess supply) all buyers get h units in both auctions, and the price is zero. Hence, they are revenue equivalent.

[17] A buyer with type $\underline{v}$ may earn positive expected payoff in the Vickrey auction. To drive $E\pi(\underline{v}, \underline{v})$ to zero, the auctioneer could for instance charge a participation fee, or insist on a set of reserve prices. However, this would require knowledge of $F(\cdot)$ and $g_j(\cdot)$, $j = 1, 2$. Proposition 7 is stronger, implying that no such knowledge is needed to rank the Vickrey auction ahead of the uniform price auction.

Proof. We start by showing that $E\pi(\underline{v},\underline{v})$ is higher in the uniform price auction than in the Vickrey auction. If $n > H$, a buyer of type $\underline{v}$ wins zero units in the efficient Vickrey auction, so $E\pi(\underline{v},\underline{v}) = 0$. If $nh > H \geq n$, a buyer of type $\underline{v}$ may win one unit in a Vickrey auction, but never two units. If he wins one unit, the price he pays is $g_2(z)$, where z is the highest type among the rival buyers who win precisely one unit, $z \geq \underline{v}$. Now, if he won zero units in the Vickrey auction, he may win one unit in the uniform price auction, which leaves him better off. If he won one unit in the Vickrey auction, he may win one or two units in the uniform price auction. If he wins two units, the unit price must be lower than $g_2(\underline{v})$, which in turn is lower than $g_2(z)$, because we know that his second bid, which is no higher than $g_2(\underline{v})$, is higher than the highest losing bid. If he wins one unit, the price paid must be no higher than $g_2(z)$, for the simple reason that any bid above $g_2(z)$ must be a winning bid, and thus cannot determine the unit price. Hence, in any event, the type $\underline{v}$ buyer is better off in the uniform price auction than in the Vickrey auction.

Next, we show that the term inside the expectation operator in (2) is higher for the Vickrey auction than for the uniform price auction. First, notice that anybody who won at least one unit in the Vickrey auction must also win at least one unit in the uniform price auction, because the first bid is not shaded below marginal utility. Hence, if the allocation changes as we switch from the Vickrey auction to the uniform price auction, it must be the case that (at least one) buyer wins one unit instead of two, and that some other buyer, or buyers, wins more units than before. If one buyer wins one unit rather than two, the unit freed up moves to someone who would win one unit in the Vickrey auction, or to someone who would win none. In both cases, the sum of marginal revenue declines, in the former case because $J_2(\cdot)$ is increasing in type and the buyer losing a unit has the higher type, and in the latter case because the buyer losing a unit has higher willingness to pay for a second unit than the new winner has for the first unit and so, by Assumption D, his marginal contribution to revenue is higher. If two (or more) buyers win one unit rather than two, the freed units can either be split among two (or more) buyers, or end up with one buyer who would then win two units rather than zero units. In the former case, the same argument as before reveals that the sum of marginal revenues declines. In the latter case, lost marginal revenue is $J_2(v_a) + J_2(v_b)$, where a and b are the buyers losing a unit, and the gained marginal revenue is $J_1(v_c) + J_2(v_c)$, where c is the new winner. It must be the case that $g_2(v_a), g_2(v_b) > g_1(v_c)$. Hence,

$J_2(v_a) > J_2(v_c)$, since $v_a > v_c$, and $J_2(v_b) > J_1(v_c)$, since $g_2(v_b) > g_1(v_c)$ enables us to invoke Assumption D. Thus, lost marginal revenue exceeds gained marginal revenue. ■

In situations in which $h > 2$, the last part of Assumption D simply becomes $J_{j+1}(v_1) - J_j(v_2) \geq 0$ whenever $g_{j+1}(v_1) = g_j(v_2)$, $j = 1, 2, .., h-1$. Generalizing Proposition 7, however, is not quite as straightforward, because the allocation may change significantly when switching from a Vickrey auction to a uniform price auction, especially in asymmetric equilibria.

However, focusing on symmetric equilibria yields some useful insights. First, in a symmetric equilibrium, the buyer with the highest type wins at least as many units as the buyer with the second highest type, and so on.[18]

Second, without deriving the equilibrium, Ausubel and Cramton (2002) notice that there is an incentive for what they term "differential shading", i.e. higher bid shading on unit 3 than on unit 2, and so on. Recall that the reason that bid shading is attractive is that the bid may end up being the highest losing bid, and therefore the unit price. Then, if the $k'th$ bid submitted by a buyer turns out to be the purchase price, he benefits from a lower unit price on all the $k-1$ units he wins. The higher k is, the bigger is the gain from bid shading. This explains the incentive for differential bid shading.

However, symmetric and differential bid shading implies that the H units are shared more "evenly" among buyers than is efficient. That is, the buyer with the highest type wins fewer units than in the Vickrey auction (but still more than anybody else), and buyers with lower types win more units than in the Vickrey auction. Again, if a unit changes hands, it moves from someone with a high willingness to pay on that unit to someone with a lower willingness to pay who wins fewer units. But then, Assumption D reveals that the marginal revenue lost exceeds the marginal revenue gained.

This argument suggests that the Vickrey auction dominates the uniform price auction in terms of revenue when $h \geq 2$, at least in a symmetric equilibrium with differential bid shading, provided one exists.

However, there may be other types of equilibria as well, and the conclusion

[18] This depends on there being zero probability of ties in bids. If bids are monotone in the sense that the bid for unit j is strictly higher than the bid for unit $j+1$, and that bids are strictly increasing in type, this is the case. Engelbrecht-Wiggans and Kahn (1998) show that when $h = 2$, any symmetric equilibrium has monotone bids, whenever bids have a positive probability of winning. Pooling of bids (at zero) may occur, but only if these bids have no chance of being winning bids.

in Proposition 7 may not be valid in all equilibria of the uniform price auction. For instance, assume that $n = h = H = 2$. Then, it is an equilibrium for both buyers to submit the two bids $g_1(\overline{v})$ and $g_1(\underline{v})$. Each buyer will win one unit, and the price will be $g_1(\underline{v})$. It is easily seen that there is no incentive to deviate, yet buyers are using a weakly dominated strategy. Revenue will obviously be $2g_1(\underline{v})$. Now, if $g_2(\overline{v}) < g_1(\underline{v})$, revenue in the Vickrey auction would be $g_2(v_1) + g_2(v_2) < 2g_1(\underline{v})$, and so the uniform price auction is more profitable with probability one. Then, if $g_2(\overline{v})$ is slightly larger than $g_1(\underline{v})$, the uniform price auction must remain more profitable, in expectation, than the Vickrey auction.

Another restriction in Proposition 7 is the fact that buyers are assumed to be symmetric. Symmetry allows us to compare marginal revenue across buyers, and to establish that type $\underline{v}$ is better off in the uniform price auction than in the Vickrey auction. See Kirkegaard (2004) for an example with asymmetric buyers in which the uniform price auction dominates the Vickrey auction.

With symmetric buyers, auctions in which the units are shared between too many buyers produce low expected revenue. In the optimal auction, the same line of argument can be used to show that the units must necessarily be *concentrated in the hands of fewer buyers than is efficient.* If $J_2(v) - J_1(v_2) > 0$ when $v = g_2^{-1}(g_1(v_2))$ it must also hold that $J_2(v_1) - J_1(v_2) > 0$ when v_1 is lower than, but close to, v. Then, buyer 1 is awarded two units in the optimal auction, but only one unit in an efficient auction, at least if $H = 2$.

Lebrun and Tremblay (2003) show that bidding in discriminatory or pay-your-bid auctions is, relatively speaking, more aggressive on a second unit than on the first unit. Thus, any buyer will win two units more often than is efficient, suggesting that discriminatory auctions are more profitable than the Vickrey auction. However, the concentration of units among a few buyers may be too extreme, and it is therefore not possible, without further research, to rank the two auctions.

In related research, Palfrey (1983) considers the sale of heterogeneous goods to buyers with multi-dimensional signals and additive utility. Using a Vickrey auction, he shows that the seller is better off bundling goods together than not bundling when there are two buyers, but that this is less likely to be the case when the number of buyers increases. Though the models are different, a similar result obtains in the current paper.[19] Bundling ensures

[19] Palfrey's (1983) proof in the case of 2 buyers easily extends to the model in this paper.

that the goods will be concentrated among fewer buyers than is efficient, which tends to increase revenue. However, when the number of buyers is large, the concentration may be excessive. After all, when the number of buyers increases, the highest and second highest types increase and converge, making it more likely to be profitable to unbundle the goods. The reason is that if v_1 is large and v_2 is sufficiently close to v_1, then $J_1(v_2) > J_2(v_1)$ (see Figures 2 and 3).

5 Conclusion

In auctions with multi-unit demand, the relationship between elasticity of demand and type is important in determining the efficiency properties of the optimal auction. This is especially clear when focussing on output inefficiency. Depending on whether the elasticity of demand is increasing or decreasing in type, the probability that all units are sold is decreasing or non-monotonic, respectively, in the number of units for sale.

While the optimal auction is generally allocative inefficient, some types of inefficiencies are counterproductive from a revenue standpoint. For instance, the efficient Vickrey auction is revenue superior to the inefficient uniform price auction.

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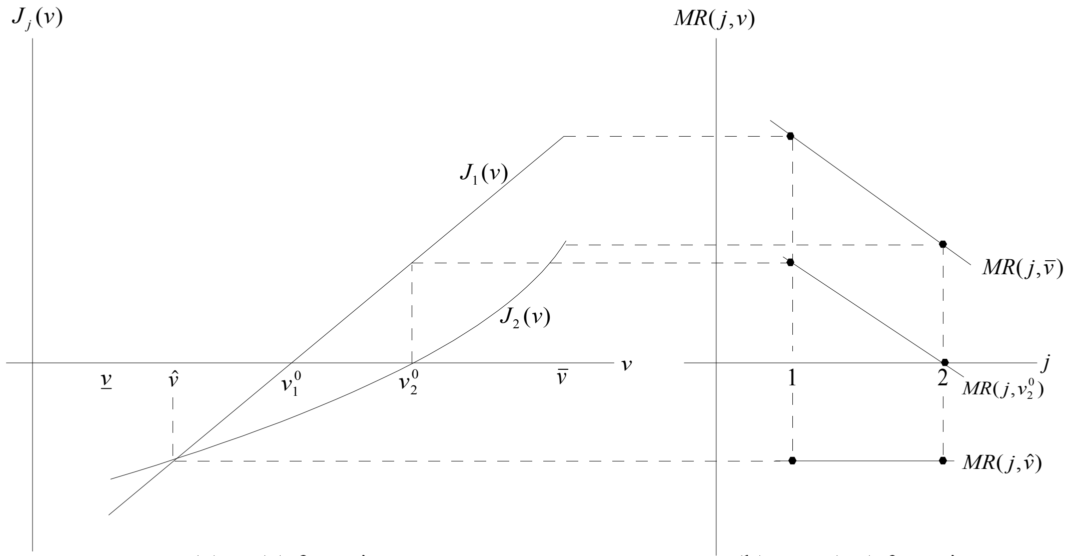
(a) High demand, high price sensitivity (b) High demand, low price sensitivity

Figure 1: Demand and price sensitivity



(a) $J_j(v)$ functions (b) $MR(j,v)$ functions

Figure 2: Increasing price sensitivity

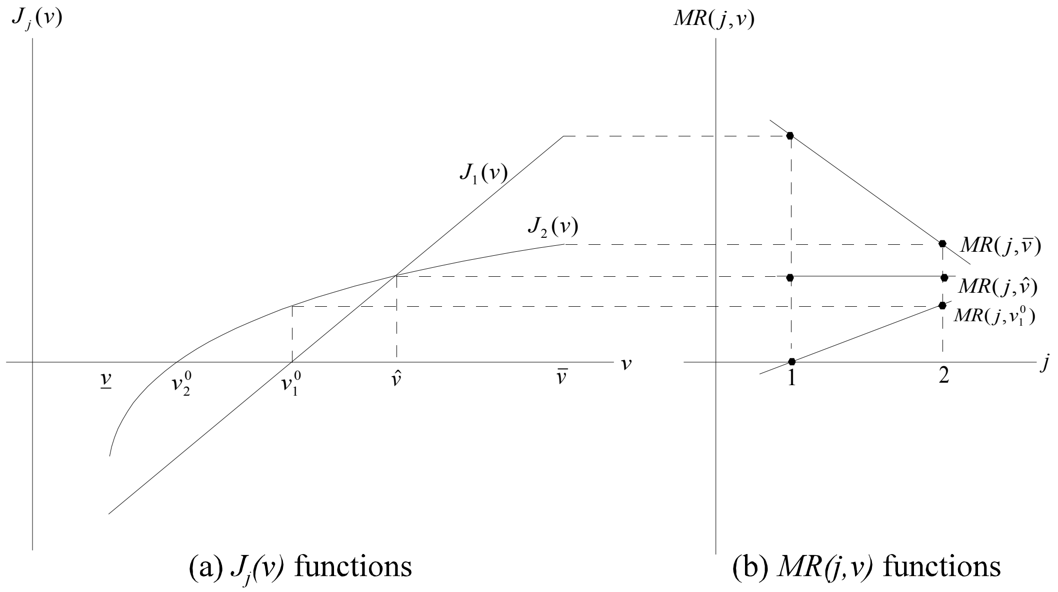


Figure 3: Decreasing price sensitivity