

Web Appendix - January 2008

“Buy-Out Prices in Auctions: Seller Competition and Multi-Unit Demands”

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Additional Results, Proofs, and Details of the Example

(numbers of equations and figures refer to this web appendix)

1 Results and Proofs

Lemma 1 *The optimal bidding strategy in the first auction can be written as*

$$b^1(v) = \frac{kvF_{1,n-2}(kv) + \int_{kv}^v xf_{1,n-2}(x)dx}{F_{1,n-2}(v)}. \quad (1)$$

Thus, in two straight second-price auctions with n bidders, the expected revenues in the first and second auctions are, respectively,

$$\begin{aligned} ER_1^{SSP} &= \int_0^{\bar{v}} kv \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv \\ &+ \int_0^{\bar{v}} v \cdot \frac{(1-F(v))^2 - (1-F(m(v)))^2}{(1-F(v))^2} \cdot f_{3,n}(v) dv \end{aligned} \quad (2)$$

and

$$\begin{aligned} ER_2^{SSP} &= \int_0^{\bar{v}} v \cdot \frac{1-F(m(v))}{1-F(v)} \cdot f_{2,n}(v) dv \\ &+ \int_0^{\bar{v}} kv \cdot \frac{F(v) - F(kv)}{1-F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv \\ &+ \int_0^{\bar{v}} v \cdot \frac{(F(m(v)) - F(v))^2}{(1-F(v))^2} \cdot f_{3,n}(v) dv \end{aligned} \quad (3)$$

Proof. First note that $b^1(v) = E(\max\{ky_1, y_2\} \mid y_1 = v)$ can be rewritten as $b^1(v) = E(\max\{kv, y_2\} \mid y_1 = v) = E(\max\{kv, y_2\} \mid y_2 < v)$. But, $E(\max\{kv, y_2\} \mid y_2 < v)$ is just

$$\begin{aligned} E(\max\{kv, y_2\} \mid y_2 < v) &= \frac{\int_0^{kv} kv f_{1,n-2}(y_2) dy_2 + \int_{kv}^v y_2 f_{1,n-2}(y_2) dy_2}{F_{1,n-2}(v)} \\ &= \frac{kvF_{1,n-2}(kv) + \int_{kv}^v y_2 f_{1,n-2}(y_2) dy_2}{F_{1,n-2}(v)} \end{aligned}$$

as stated in (1).

If bidders bid according to $b^1(v)$, it follows that expected revenue in the first auction is

$$ER_1^{ssp} = \int_0^{\bar{v}} b^1(v) f_{2,n}(v) dv = \int_0^{\bar{v}} \left(\frac{kvF_{1,n-2}(kv) + \int_{kv}^v y f_{1,n-2}(y) dy}{F_{1,n-2}(v)} \right) f_{2,n}(v) dv$$

or

$$ER_1^{ssp} = \int_0^{\bar{v}} kv \frac{F_{1,n-2}(kv)}{F_{1,n-2}(v)} f_{2,n}(v) dv + \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) \frac{f_{2,n}(v)}{F_{1,n-2}(v)} dv \quad (4)$$

Now, the first term, t_1 , in (4) can be rewritten as

$$t_1 = \int_0^{\bar{v}} kv \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv \quad (5)$$

Noting that

$$\frac{f_{2,n}(v)}{F_{1,n-2}(v)} = \frac{n(n-1)(1-F(v))F^{n-2}(v)f(v)}{F^{n-2}(v)} = n(n-1)(1-F(v))f(v)$$

the second term, t_2 , in (4) can be rewritten as

$$\begin{aligned} t_2 &= n(n-1) \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) (1-F(v))f(v) dv \\ &= n(n-1) \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) f(v) dv - \frac{n(n-1)}{2} \int_0^{\bar{v}} \left(\int_{kv}^v y f_{1,n-2}(y) dy \right) 2F(v)f(v) dv \end{aligned}$$

Changing the order of integration, we can write the last expression as

$$\begin{aligned} &n(n-1) \int_0^{\bar{v}} y f_{1,n-2}(y) \left(\int_y^{m(y)} f(v) dv \right) dy - \frac{n(n-1)}{2} \int_0^{\bar{v}} y f_{1,n-2}(y) \left(\int_y^{m(y)} 2F(v)f(v) dv \right) dy \\ &= n(n-1) \int_0^{\bar{v}} y f_{1,n-2}(y) [F(v)]_y^{m(y)} dy - \frac{n(n-1)}{2} \int_0^{\bar{v}} y f_{1,n-2}(y) [F^2(v)]_y^{m(y)} dy \\ &= n(n-1) \int_0^{k\bar{v}} y f_{1,n-2}(y) \left[F\left(\frac{y}{k}\right) - F(y) \right] dy + n(n-1) \int_{k\bar{v}}^{\bar{v}} y f_{1,n-2}(y) [1 - F(y)] dy \\ &\quad - \frac{n(n-1)}{2} \int_0^{k\bar{v}} y f_{1,n-2}(y) \left[F^2\left(\frac{y}{k}\right) - F^2(y) \right] dy - \frac{n(n-1)}{2} \int_{k\bar{v}}^{\bar{v}} y f_{1,n-2}(y) [1 - F^2(y)] dy \\ &= n(n-1) \int_0^{k\bar{v}} y f_{1,n-2}(y) \left[F\left(\frac{y}{k}\right) - F(y) - \frac{1}{2} \left(F\left(\frac{y}{k}\right) + F(y) \right) \left(F\left(\frac{y}{k}\right) - F(y) \right) \right] dy \\ &\quad + n(n-1) \int_{k\bar{v}}^{\bar{v}} y f_{1,n-2}(y) \left[1 - F(y) - \frac{1}{2} (1 + F(y))(1 - F(y)) \right] dy \\ &= n(n-1) \int_0^{k\bar{v}} y(n-2)F^{n-3}(y)f(y) \frac{1}{2} \left(F\left(\frac{y}{k}\right) - F(y) \right) \left(2 - F\left(\frac{y}{k}\right) - F(y) \right) dy \\ &\quad + n(n-1) \int_{k\bar{v}}^{\bar{v}} y(n-2)F^{n-3}(y)f(y) \frac{1}{2} (1 - F(y))^2 dy \\ &= \int_0^{k\bar{v}} y \frac{\left(F\left(\frac{y}{k}\right) - F(y) \right) \left(2 - F\left(\frac{y}{k}\right) - F(y) \right)}{(1 - F(y))^2} \frac{n(n-1)(n-2)}{2} (1 - F(y))^2 F^{n-3}(y) f(y) dy \\ &\quad + \int_{k\bar{v}}^{\bar{v}} y \frac{n(n-1)(n-2)}{2} (1 - F(y))^2 F^{n-3}(y) f(y) dy \\ &= \int_0^{\bar{v}} y \frac{\left(F(m(y)) - F(y) \right) \left(2 - F(m(y)) - F(y) \right)}{(1 - F(y))^2} \frac{n(n-1)(n-2)}{2} (1 - F(y))^2 F^{n-3}(y) f(y) dy \end{aligned}$$

where $m(y) = \min\{\frac{y}{k}, \bar{v}\}$, which implies that we can write

$$t_2 = \int_0^{\bar{v}} y \cdot \frac{(1 - F(y))^2 - (1 - F(m(y)))^2}{(1 - F(y))^2} \cdot f_{3,n}(y) dy \quad (6)$$

From (5) and (6) we get

$$\begin{aligned} ER_1^{ssp} &= t_1 + t_2 \\ &= \int_0^{\bar{v}} kv \frac{F^{n-2}(kv)}{F^{n-2}(v)} f_{2,n}(v) dv + \int_0^{\bar{v}} v \cdot \frac{(1 - F(v))^2 - (1 - F(m(v)))^2}{(1 - F(v))^2} \cdot f_{3,n}(v) dv \end{aligned}$$

as stated in (2).

To intuitively understand the revenue in (3), recall that bidding strategies are simple in stage two, in the sense that all bidders will simply bid their *relevant* valuations. Thus, bidder i will bid $b^2(v_i) = kv_i$, if he won stage one, and $b^2(v_i) = v_i$ otherwise. This will allow us to write ER_2^{ssp} as the sum of three terms following from three types of events: (i) the winner of stage one wins again, (ii) the winner of stage one does not win again but is the runner-up and (iii) the winner of stage one does not win again nor is the runner up. Let x denote the valuation of the winner of stage one, while y_1 and y_2 denote the valuations of his strongest and second-strongest rivals, respectively. Thus, $x \geq y_1 \geq y_2$. Then, associated with the three events is a stage-two price, p_2 , of y_1 , kx and y_2 , resp.

i) $p_2 = y_1$. The winner of stage one also wins stage two at a price y_1 if $x \geq kx \geq y_1 \geq y_2$, in particular, if $x \geq \frac{y_1}{k}$. Writing the first term in (3) as

$$\int_0^{k\bar{v}} y_1 \cdot \frac{1 - F(\frac{y_1}{k})}{1 - F(y_1)} \cdot f_{2,n}(y_1) dy_1$$

captures the revenue contribution when the winner of the first auction also wins the second auction at a price of y_1 .

ii) $p_2 = kx$. The winner of stage one is the runner-up in stage two if $x \geq y_1 \geq kx \geq y_2$, in which case the bidder with valuation y_1 wins stage two at a price of kx . The second term in (3) written as

$$\int_0^{\bar{v}} kx \cdot \frac{F(x) - F(kx)}{1 - F(x)} \cdot \frac{F^{n-2}(kx)}{F^{n-2}(x)} f_{2,n}(x) dx$$

captures the revenue contribution in this event.

iii) $p_2 = y_2$. Finally, the winner of stage one is neither the winner nor the runner-up in stage two if $x \geq y_1 \geq y_2 \geq kx$. In this event, the bidder with valuation y_1 wins stage two at a price of y_2 . Then, the third term in (3) written as

$$\int_0^{\bar{v}} y_2 \frac{(F(m(y_2)) - F(y_2))^2}{(1 - F(y_2))^2} f_{3,n}(y_2) dy_2$$

captures the revenue. ■

Proposition 2: Let $B(\hat{v})$ be defined by

$$\begin{aligned} B(\hat{v})(1 - F_{1,n}(\hat{v})) &= \hat{v}[1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v}))F_{1,n-1}(\hat{v}) \\ &\quad - \frac{1}{2}n(n-1)(1 - F(\hat{v}))(F(m(\hat{v})) - F(\hat{v}))F_{1,n-2}(\hat{v})] \\ &\quad + n(n-1)(1 - F(\hat{v})) \int_0^{\hat{v}} \left(kx F_{1,n-2}(kx) + \int_{kx}^x y f_{1,n-2}(y) dy \right) f(x) dx \\ &\quad + \frac{1}{2}n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} \left(kx F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} y f_{1,n-2}(y) dy \right) f(x) dx \end{aligned} \quad (7)$$

Then, it is an equilibrium for bidders with $v \in [\hat{v}, \bar{v}]$ to take the buy-out price $B(\hat{v})$ in stage 1 and for bidders with $v \in [\underline{v}, \hat{v})$ not to.

Proof. The expected payoff to a bidder with valuation \hat{v} from accepting or not accepting the buy-out price, $EU(B, \hat{v})$ and $EU(NB, \hat{v})$ respectively, were derived in the main appendix to the paper. As we are looking for an equilibrium in which $B(\hat{v})$ is accepted if, and only if, the bidder has valuation above \hat{v} , it must be the case that the bidder with valuation \hat{v} is indifferent between accepting and not accepting, or $EU(B, \hat{v}) = EU(NB, \hat{v})$. This can be written as

$$\begin{aligned}
& B(\hat{v})(1 - F_{1,n}(\hat{v})) \\
&= \hat{v}(1 - F_{1,n}(\hat{v})) - n(1 - F(\hat{v})) \int_0^{\hat{v}} (\hat{v} - b^1(x)) f_{1,n-1}(x) dx \\
&\quad - \frac{1}{2} n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} [(\hat{v} - kx) F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} (\hat{v} - y) f_{1,n-2}(y) dy] f(x) dx \\
&= \hat{v}[1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v})) F_{1,n-1}(\hat{v}) - \frac{1}{2} n(n-1)(1 - F(\hat{v})) F_{1,n-2}(\hat{v}) F(m(\hat{v})) - F(\hat{v})] \\
&\quad + \int_0^{\hat{v}} b^1(x) n(n-1)(1 - F(\hat{v})) F_{1,n-2}(x) f(x) dx \\
&\quad + \int_{\hat{v}}^{m(\hat{v})} \left(kx F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} y f_{1,n-2}(y) dy \right) \frac{1}{2} n(1 - F(\hat{v})) (n-1) f(x) dx.
\end{aligned}$$

Substituting for $b^1(x)$, we get

$$\begin{aligned}
& B(\hat{v})(1 - F_{1,n}(\hat{v})) \\
&= \hat{v}[1 - F_{1,n}(\hat{v}) - n(1 - F(\hat{v})) F_{1,n-1}(\hat{v}) - \frac{1}{2} n(n-1)(1 - F(\hat{v})) F_{1,n-2}(\hat{v}) (F(m(\hat{v})) - F(\hat{v}))] \\
&\quad + n(n-1)(1 - F(\hat{v})) \int_0^{\hat{v}} \left(kx F_{1,n-2}(kx) + \int_{kx}^x y f_{1,n-2}(y) dy \right) f(x) dx \\
&\quad + \frac{1}{2} n(n-1)(1 - F(\hat{v})) \int_{\hat{v}}^{m(\hat{v})} \left(kx F_{1,n-2}(kx) + \int_{kx}^{\hat{v}} y f_{1,n-2}(y) dy \right) f(x) dx
\end{aligned}$$

as stated in the proposition.

Notice that the right-hand-side is strictly positive whenever $\hat{v} \in (0, \bar{v})$.¹ The left-hand-side is zero if $B = 0$, but increases without bound as B increases (when $\hat{v} < \bar{v}$). Hence, for any $\hat{v} \in (0, \bar{v})$ there exists a unique $B > 0$ which satisfies the indifference condition. $B = 0$ when $\hat{v} = 0$. Any strictly negative B would support an equilibrium where all bidders accept the buy-out price regardless of v . In the remark to the Proposition following this proof, we characterize B for $\hat{v} \rightarrow \bar{v}$. Any B in excess of this value supports an equilibrium where the buy-out price is never accepted. Finally, since (7) is continuous in \hat{v} , $B(\hat{v})$ must be continuous as well. As a consequence, there exists one or more equilibria for any $B \in (-\infty, \infty)$.

Now, what remains is to verify that buyers with valuation $v \neq \hat{v}$ has no incentive to deviate. First, consider $v > \hat{v}$. Expected payoff by accepting B is

$$\begin{aligned}
EU(B, v) &\geq \Pr(W|\hat{v})(v - B) + \int_0^{\min\{kv, \hat{v}\}} (kv - x) f_{1,n-1}(x) dx \\
&\quad + \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \left(1 - \frac{1}{i+1} \right) EU_2(v|L, \hat{v}, i)
\end{aligned} \tag{8}$$

¹Below, in the discussion of expression (10), we show that the first term is non-negative.

where the first term comes from the fact that the buyer may win stage one at the buy-out price, and the second term from the fact that he wins both stages with probability one if the highest rival type is lower than $\min\{kv, \hat{v}\}$. In the third term, $EU_2(v|L, \hat{v}, i)$ denotes the expected payoff in stage 2, given the buyer *lost* stage 1 and there are i rivals with type above \hat{v} . The third term is then explained by noticing that $1 - \frac{1}{i+1}$ is the probability that the buyer loses stage one when faced with i rivals who are also willing to accept B . Finally, the inequality in (8) derives from the possibility that the buyer may win both stages even if his highest rival has type above \hat{v} . Naturally, this requires that $kv > \hat{v}$, so this possibility does not exist for type \hat{v} buyers, for whom $ER(B, v)$ therefore equals the right hand side of (8).

If the buyer rejects B , on the other hand, we have already established that he will choose to outbid anybody with valuation below \hat{v} in stage one, if the buy-out price is not accepted by a rival. This means that expected payoff is

$$\begin{aligned} EU(NB, v) &= \int_0^{\hat{v}} (v - b(x)) f_{1, n-1}(x) dx + \int_0^{\min\{kv, \hat{v}\}} (kv - x) f_{1, n-1}(x) dx \\ &\quad + \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} EU_2(v|L, \hat{v}, i) \end{aligned}$$

The first two terms are self-explanatory, and the third captures the fact that the buyer is bound to lose stage one if there is one or more rivals with type above \hat{v} . By construction, $EU(B, \hat{v}) - EU(NB, \hat{v}) = 0$, so we need only show that

$$\begin{aligned} EU(B, v) - EU(NB, v) &\geq \Pr(W|\hat{v})(v - B) - \int_0^{\hat{v}} (v - b^1(x)) f_{1, n-1}(x) dx \\ &\quad - \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} EU_2(v|L, \hat{v}, i) \end{aligned}$$

is increasing in v . The derivative equals

$$\Pr(W|\hat{v}) - F^{n-1}(\hat{v}) - \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} EU_2'(v|L, \hat{v}, i)$$

Assuming for the moment that $EU_2'(v|L, \hat{v}, i) \leq 1$, this is at least

$$\Pr(W|\hat{v}) - F^{n-1}(\hat{v}) - \sum_{i=1}^{n-1} \binom{n-1}{i} (1 - F(\hat{v}))^i F(\hat{v})^{n-1-i} \frac{1}{i+1} = 0$$

implying that $EU(B, v) - EU(NB, v) \geq 0$ for all $v > \hat{v}$. To see that $EU_2'(v|L, \hat{v}, i) \leq 1$, notice that $EU_2(v|L, \hat{v}, i)$ can be written in the form

$$EU_2(v|L, \hat{v}, i) = \int_{k\hat{v}}^v (v - x) h(x|L, \hat{v}, i) dx$$

where $h(\cdot|\cdot)$ is the density of the highest rival bid in stage two, given the buyer lost stage one to one of i rivals with type above \hat{v} . This density is positive on $[k\hat{v}, \max\{\hat{v}, k\bar{v}\}]$ for $i = 1$ and on $[\hat{v}, \bar{v}]$ for $i > 1$.

By Leibniz' rule, the derivative of $EU_2(v|L, \hat{v}, i)$ is

$$EU_2'(v|L, \hat{v}, i) = \int_{k\hat{v}}^v h(x|L, \hat{v}, i) dx.$$

Since $h(\cdot|\cdot)$ is a density, it integrates to at most 1 on a subset of its domain, implying $EU_2'(v|L, \hat{v}, i) \leq 1$.

Next, consider $v < \widehat{v}$. If the buyer chooses not to accept the buy-out price, we know that the best response in the remainder of stage 1 is to bid $b^1(v)$. Hence, we need only show that it is preferable to reject B , and subsequently bid $b^1(v)$, than to accept B . The latter strategy yields expected payoff of

$$\begin{aligned} EU(B, v) &= (v - B) \Pr(W|\widehat{v}) + \int_0^{kv} (kv - x) f_{1,n-1}(x) dx \\ &\quad + \frac{1}{2}(n-1) \int_{\widehat{v}}^{\max\{\widehat{v}, m(v)\}} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \end{aligned}$$

The third term derives from the fact that if the buyer loses stage 1, there must be at least one rival with type above \widehat{v} . If there is more than one, the buyer will lose stage two as well (he will be outbid). However, if there is precisely one rival with type above \widehat{v} , the buyer loses stage 1 with probability 0.5, but in that event he has a chance of winning stage 2, provided $v > k\widehat{v}$, or $\frac{v}{k} > \widehat{v}$.

By rejecting B and bidding $b^1(v)$ in stage 1, expected payoff is

$$\begin{aligned} EU(NB, v) &= \int_0^v (v - b^1(x)) f_{1,n-1}(x) dx + \int_0^{kv} (kv - x) f_{1,n-1}(x) dx \\ &\quad + (n-1) \int_v^{m(v)} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \end{aligned}$$

Again, the third term derives from the fact that if the highest rival has type above v , the buyer loses stage 1 with probability one, but nevertheless has a chance of winning stage two, provided there is only one rival with a type exceeding v .

The difference can be written as

$$\begin{aligned} &EU(NB, v) - EU(B, v) \\ &= \int_0^v (v - b^1(x)) f_{1,n-1}(x) dx - (v - B) \Pr(W|\widehat{v}) \\ &\quad + (n-1) \int_v^{m(v)} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \\ &\quad - \frac{1}{2}(n-1) \int_{\widehat{v}}^{\max\{\widehat{v}, m(v)\}} \left((v - kx) F_{1,n-2}(kx) + \int_{kx}^v (v - y) f_{1,n-2}(y) dy \right) f(x) dx \end{aligned}$$

which by construction is zero for $v = \widehat{v}$. Differentiating with respect to v gives

$$\begin{aligned} &(v - b^1(v)) f_{1,n-1}(v) + F_{1,n-1}(v) - \Pr(W|\widehat{v}) \\ &\quad - (n-1) \left((v - kv) F_{1,n-2}(kv) + \int_{kv}^v (v - y) f_{1,n-2}(y) dy \right) f(v) \\ &\quad + (n-1) \int_v^{m(v)} F_{1,n-2}(v) f(x) dx - \frac{1}{2}(n-1) \int_{\widehat{v}}^{\max\{\widehat{v}, m(v)\}} F_{1,n-2}(v) f(x) dx \end{aligned}$$

which, after inserting $b^1(v)$, reduces to

$$(n-1) F^{n-2}(v) \left(F(m(v)) - F(v) - \frac{1}{2}(F(\max\{\widehat{v}, m(v)\}) - F(\widehat{v})) \right) + F^{n-1}(v) - \Pr(W|\widehat{v}) \quad (9)$$

For $v = \widehat{v}$, this is

$$(n-1) F^{n-2}(\widehat{v}) \frac{1}{2} (F(m(\widehat{v})) - F(\widehat{v})) + F^{n-1}(\widehat{v}) - \frac{1 - F^n(\widehat{v})}{n(1 - F(\widehat{v}))}$$

which is smaller than

$$(n-1)F^{n-2}(\hat{v})\frac{1}{2}(1-F(\hat{v})) + F^{n-1}(\hat{v}) - \frac{1-F^n(\hat{v})}{n(1-F(\hat{v}))}$$

Rewriting, we get

$$-\frac{1}{n(1-F(\hat{v}))} \left[1 - F^n(\hat{v}) - n(1-F(\hat{v}))F^{n-1}(\hat{v}) - n(n-1)\frac{1}{2}(1-F(\hat{v}))^2F^{n-2}(\hat{v}) \right] \leq 0 \quad (10)$$

where the inequality follows from the fact that the term in brackets equals

$$1 - \sum_{i=0}^2 \binom{n}{i} (1-F(\hat{v}))^i F^{n-i}(\hat{v}) \geq 0.$$

The second derivative of $EU(NB, v) - EU(B, v)$ is obtained by differentiating (9), yielding

$$(n-1)(n-2)F^{n-3}(v)f(v) \left(F(m(v)) - F(v) - \frac{1}{2}(F(\max\{\hat{v}, m(v)\}) - F(\hat{v})) \right) \\ + (n-1)F^{n-2}(v) \frac{\partial}{\partial v} \left(F(m(v)) - \frac{1}{2}F(\max\{\hat{v}, m(v)\}) \right).$$

If $\max\{\hat{v}, m(v)\} = \hat{v}$ the first term in parentheses reduces to $F(m(v)) - F(v) > 0$, and the derivative in the second term reduces to the derivative of $F(m(v))$, which is non-negative. If $\max\{\hat{v}, m(v)\} = m(v)$, the first term is positive since $v < \hat{v}$, by assumption, while the derivative in the second term reduces to the derivative of $\frac{1}{2}F(m(v))$. In either case, both terms are positive, implying that $EU(NB, v) - EU(B, v)$ is convex. As it is decreasing at $v = \hat{v}$, it must be decreasing for any $v \leq \hat{v}$, meaning that $EU(NB, v) - EU(B, v)$ is non-negative on $v \in [0, \hat{v}]$. This completes the proof of Proposition 2. ■

Remark to Proposition 2: We note that with $n = 2$, (7) can be restated as

$$B(\hat{v})(1+F(\hat{v})) = \hat{v}(1+F(\hat{v}) - 2F(\hat{v}) - (F(m(\hat{v})) - F(\hat{v}))) + \int_0^{\hat{v}} 2kxf(x)dx + \int_{\hat{v}}^{m(\hat{v})} kxf(x)dx \\ = \hat{v}(1-F(m(\hat{v}))) + \int_0^{\hat{v}} kxf(x)dx + \int_0^{m(\hat{v})} kxf(x)dx$$

from which it is readily seen that $B(\hat{v}) \rightarrow E(kv)$ for $\hat{v} \rightarrow \bar{v}$. However, this is just $E(b^1(v)|v = y_1)$.

In the general case with $n > 2$, applying L'Hôpital's rule to (7) for $\hat{v} \rightarrow \bar{v}$ yields

$$B(\bar{v}) = (n-1) \int_0^{\bar{v}} \left(kxF_{1,n-2}(kx) + \int_{kx}^x yf_{1,n-2}(y)dy \right) f(x)dx \\ = \int_0^{\bar{v}} \left(\frac{kxF_{1,n-2}(kx) + \int_{kx}^x yf_{1,n-2}(y)dy}{F_{1,n-2}(x)} \right) (n-1)F_{1,n-2}f(x)dx \\ = \int_0^{\bar{v}} b^1(x)f_{1,n-1}(x)dx = E(b^1(v)|v = y_1).$$

Hence, for $\hat{v} \rightarrow \bar{v}$, $B(\hat{v})$ approaches the expected value of the highest bid among the $n-1$ rivals in the game without a buy-out price.

Remark to Proposition 3: Expected revenue in the second auction can also be derived. It is

$$ER_2(\hat{v}) = E_1 + E_2 + E_3 \quad (11)$$

where

$$\begin{aligned}
E_1 &= \int_0^{\widehat{v}} v \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot f_{2,n}(v) dv \\
&+ \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot \frac{1}{i+2} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\widehat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv \\
&+ \sum_{i=1}^{n-2} \binom{n-2}{i} \cdot \frac{i}{i+2} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\widehat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv \\
E_2 &= \int_0^{\widehat{v}} kv \cdot \frac{F(v) - F(kv)}{1 - F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} \cdot f_{2,n}(v) dv \\
&+ \int_{\widehat{v}}^{m(\widehat{v})} kv \left(\frac{1}{2} \cdot \frac{F(\widehat{v}) - F(kv)}{1 - F(v)} + \frac{1}{2} \cdot \frac{1 - F(kv)}{1 - F(v)} \right) \frac{F^{n-2}(kv)}{F^{n-2}(v)} \cdot f_{2,n}(v) dv \\
&+ \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot \frac{1}{i+2} \cdot \int_{m(\widehat{v})}^{\bar{v}} kv \cdot \frac{1 - F(kv)}{1 - F(v)} \cdot \frac{(F(kv) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\widehat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv
\end{aligned}$$

and

$$\begin{aligned}
E_3 &= \int_0^{\widehat{v}} v \cdot \frac{(F(m(v)) - F(v))^2}{(1 - F(v))^2} \cdot f_{3,n}(v) dv \\
&+ \int_{k\widehat{v}}^{\widehat{v}} v \cdot \frac{F(m(v)) - F(\widehat{v})}{1 - F(v)} \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot f_{3,n}(v) dv \\
&+ \sum_{i=0}^{n-3} \binom{n-3}{i} \cdot \frac{2}{i+3} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{F(m(v)) - F(v)}{1 - F(v)} \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-3-i}(\widehat{v})}{F^{n-3-i}(v)} \cdot f_{3,n}(v) dv
\end{aligned}$$

Proof. We arrange bidders in descending order, $v_1 > v_2 > \dots > v_n$, and note that the price in the second round, p_2 , can be either v_2 , kv_1 , kv_2 or v_3 . Then, the four possibilities contribute the following to expected revenue in stage two.

i) *The price in stage two is v_2 .* This requires that bidder 2 does not win stage one, and that she is the runner-up to bidder 1 in stage two. This occurs if either A) bidder 1 wins stage one and $kv_1 > v_2$, or B) someone other than bidder 1 or bidder 2 wins stage one, which in turn means that $v_2 > v_3 > \widehat{v}$. Thus, we get the contribution

$$\begin{aligned}
E_1 &= \int_0^{\widehat{v}} v \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot f_{2,n}(v) dv \\
&+ \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot \frac{1}{i+2} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\widehat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv \\
&+ \sum_{i=1}^{n-2} \binom{n-2}{i} \cdot \frac{i}{i+2} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\widehat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv
\end{aligned}$$

where v may be thought of as v_2 .

The first two terms capture the possibility that bidder 1 wins both stages, which requires that his valuation, v_1 , exceeds $m(v_2)$. Notice that if there are $i+2$ bidders with valuations above \widehat{v} , then the probability that bidder 1 wins stage one is $\frac{1}{i+2}$. The third term captures the cases in which neither bidder

1 nor bidder 2 win stage one, which clearly requires not only that v_1 and v_2 are above \hat{v} , but also that at least one other bidder has a valuation in excess of \hat{v} . If there are i such rivals, $i > 0$, the probability that neither bidder 1 nor bidder 2 win stage one is $\frac{i}{i+2}$.

ii) *The price in stage two is kv_1 .* This requires that bidder 1 wins stage one (which may involve accepting the buy-out price) and is the runner-up to bidder 2 in stage two, that is, $v_2 > kv_1 > v_3$. Hence, we get a revenue contribution corresponding to

$$\begin{aligned}\hat{E}_2 &= \int_0^{\hat{v}} kv \cdot \frac{F(v) - F(kv)}{1 - F(v)} \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} \cdot f_{2,n}(v) dv \\ &+ \int_{\hat{v}}^{m(\hat{v})} kv \left(\frac{1}{2} \cdot \frac{F(\hat{v}) - F(kv)}{1 - F(v)} + \frac{1}{2} \cdot \frac{F(v) - F(kv)}{1 - F(v)} \right) \frac{F^{n-2}(kv)}{F^{n-2}(v)} \cdot f_{2,n}(v) dv \\ &+ \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv \cdot \frac{F(v) - F(kv)}{1 - F(v)} \cdot \frac{(F(kv) - F(\hat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\hat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv\end{aligned}$$

where v may be thought of as v_1 .

If $v_1 = v$ is less than \hat{v} , bidder 1 will win stage one, and the price in stage two is kv if $v_2 > kv_1 > v_3$, which explains the first term. If $v_1 > \hat{v}$ but $kv_1 < \hat{v}$, the price in stage two is kv_1 only if there is at most one rival with valuation between \hat{v} and v_1 (and the rest have valuations below \hat{v}), and that rival lost the lottery for the buy-out price. This explains the second term. The third comes from the fact that if $kv_1 > \hat{v}$, it is possible that the price in stage two is kv_1 even if several rivals have valuations above \hat{v} . However, if $i+1$ rivals have valuations above \hat{v} , bidder 1 only wins stage one with probability $\frac{1}{i+2}$. This explains the last term.

iii) *The price in stage two is kv_2 .* This requires that bidder 2 wins stage one (implying that $v_2 > \hat{v}$) and is the runner-up to bidder 1 in stage two ($kv_2 > v_3$). Consequently, we get the contribution

$$\begin{aligned}\tilde{E}_2 &= \frac{1}{2} \cdot \int_{\hat{v}}^{m(\hat{v})} kv \cdot \frac{F^{n-2}(kv)}{F^{n-2}(v)} \cdot f_{2,n}(v) dv \\ &+ \sum_{i=0}^{n-2} \binom{n-2}{i} \cdot \frac{1}{i+2} \cdot \int_{m(\hat{v})}^{\bar{v}} kv \cdot \frac{(F(kv) - F(\hat{v}))^i}{F^i(v)} \cdot \frac{F^{n-2-i}(\hat{v})}{F^{n-2-i}(v)} \cdot f_{2,n}(v) dv\end{aligned}$$

where v may be thought of as v_2 .

If $v_2 > \hat{v}$, but $kv_2 < \hat{v}$, then the price in stage two can be kv_2 only if bidder 1 is the only rival with valuation above \hat{v} , in which case stage one is won by bidder 2 with probability 1/2 (the first term). On the other hand, if $kv_2 > \hat{v}$, the price in stage two may be kv_2 even if several rivals have valuations above \hat{v} .

Adding \hat{E}_2 and \tilde{E}_2 gives E_2 . This collects the revenue contributions when $p_2 = kv_1$ and $p_2 = kv_2$.

iv) *The price in stage two is v_3 .* For the price to be v_3 , bidder 3 must lose stage one and be the runner-up in stage two. Furthermore, if bidder 3 is the runner up in stage two, either bidder 1 or bidder 2 must have won stage one. However, notice that if $v_1, v_2 > m(v_3)$, both would be willing to pay more for the second unit than v_3 . Hence, to get a price of v_3 , it must be the case that either a) both bidder 1 and bidder 2 have valuations between v_3 and $m(v_3)$, in which case whoever wins stage one has a willingness to pay below v_3 , or b) bidder 1's valuation exceeds $m(v_3)$, but bidder 2 wins stage one and has a valuation

between v_3 and $m(v_3)$, implying that his bid in stage two is below v_3 . We get the contribution

$$\begin{aligned}
E_3 &= \int_0^{\widehat{v}} v \cdot \frac{(F(m(v)) - F(v))^2}{(1 - F(v))^2} \cdot f_{3,n}(v) dv \\
&+ \sum_{i=0}^{n-3} \binom{n-3}{i} \cdot \frac{2}{i+3} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{(F(m(v)) - F(v))^2}{(1 - F(v))^2} \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-3-i}(\widehat{v})}{F^{n-3-i}(v)} \cdot f_{3,n}(v) dv \\
&+ \int_{k\widehat{v}}^{\widehat{v}} v \cdot \frac{F(m(v)) - F(\widehat{v})}{1 - F(v)} \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot f_{3,n}(v) dv \\
&+ \sum_{i=0}^{n-3} \binom{n-3}{i} \cdot \frac{2}{i+3} \cdot \int_{\widehat{v}}^{\bar{v}} v \cdot \frac{1 - F(m(v))}{1 - F(v)} \cdot \frac{F(m(v)) - F(v)}{1 - F(v)} \cdot \frac{(F(v) - F(\widehat{v}))^i}{F^i(v)} \cdot \frac{F^{n-3-i}(\widehat{v})}{F^{n-3-i}(v)} \cdot f_{3,n}(v) dv
\end{aligned}$$

where v may be thought of as v_3 .

The first two terms in E_3 are related to the first possibility described,² while the last two relate to the second possibility. In particular, if $v_3 = v$, and both v_1 and v_2 are between v and $m(v)$, the price in stage two must be v_3 if either bidder 1 or bidder 2 won stage 1 (the first two terms). A price of $v_3 = v$ is also possible if v_1 is above $m(v)$, but then it requires that v_2 is between v and $m(v)$, and that bidder 2 wins stage one (the last two terms). Collecting the second and the fourth term allows us to write E_3 .

Note how the three terms in (11) correspond to the three terms in (3) of Lemma 1. ■

Proposition 5: *If demand is regular, any $\widehat{v} < \bar{v}$ leads to a strict loss in (expected) overall revenue, when $n > 2$. When $n = 2$, any $\widehat{v} < k\bar{v}$ (including the optimal \widehat{v}) leads to a strict loss in (expected) overall revenue, but overall revenue is unaffected if $\widehat{v} \geq k\bar{v}$.*

Proof. Given the complexity of the expressions of expected revenue, Proposition 5 is most easily proven by using mechanism design techniques. In auctions with unit demand it is well-known that the virtual valuation of a bidder with valuation v is $J(v) = v - \frac{1-F(v)}{f(v)}$, and that the virtual valuation is useful in computing revenues.

Following the approach in Myerson (1981) it is easy to show that in auctions with multi-unit demand the virtual valuation associated with the first unit is $J(v)$, while the virtual valuation associated with the second unit is $kJ(v)$.³ We let v_i denote bidder i 's valuation, $i = 1, 2, \dots, n$, while $\mathbf{v} = (v_1, v_2, \dots, v_n)$ denotes the vector of valuations. Given valuations are \mathbf{v} , let $q_{i1}(\mathbf{v})$ and $q_{i2}(\mathbf{v})$ denote the probability that bidder i wins one unit (or more) and the probability that he wins both units, respectively, in the mechanism that is being considered (with or without the buy-out price). As in Myerson (1981), expected revenue in any mechanism where a bidder with valuation 0 earns zero surplus can be then be written as

$$E \left[\sum_{i=1}^n (J(v_i)q_{i1}(\mathbf{v}) + kJ(v_i)q_{i2}(\mathbf{v})) \right], \tag{12}$$

where the expectation is over \mathbf{v} .⁴ In words, expected revenue is the expected value of the sum of the virtual valuations of the winners.

We will show that the sum of virtual valuations will never increase, but may decrease, when a buy-out price is introduced. Hence, in expectation, revenue will decrease as well. Without loss of generality, assume that $v_1 \geq v_2 \geq \dots \geq v_n$. The regularity assumption implies that $J(v)$ is strictly increasing in v , meaning that $J(v_1) \geq J(v_2) \geq \dots \geq J(v_n)$. Likewise, if $kv_1 \geq v_2$, i.e. it is efficient for bidder 1 to win

²Note that the probability that either bidder 1 or bidder 2 wins when $i+3$ bidders accept the buy-out price is $\frac{2}{i+3}$.

³Maskin and Riley (1989) consider the case of a perfectly divisible good.

⁴As long as $B \geq 0$, any bidder with valuation 0 will earn zero expected payoff.

both units, we notice that

$$kJ(v_1) - J(v_2) = (kv_1 - v_2) + \left(\frac{1 - F(v_2)}{f(v_2)} - \frac{1 - F(v_1)}{f(v_1)} \right) > 0, \quad (13)$$

by the regularity assumption.

A buy-out price may affect the allocation in the following cases (all other cases either occurs with probability zero or the buy-out price will have no impact on the allocation):

1. $n > 2$, $v_1 > v_2 > kv_1$, $v_2 > v_3 \geq \hat{v}$: Bidder 1 and bidder 2 each win exactly one unit in the sequence of auctions without a buy-out price ($q_{11}(\mathbf{v}) = q_{21}(\mathbf{v}) = 1$), and the term in brackets in (12) reduces to $J(v_1) + J(v_2)$. Once the buy-out price is introduced, bidder 1 may lose stage 1 but in that case he would win stage two. Hence, bidder 1 will win one unit with probability one ($q_{11}(\mathbf{v}) = 1$). However it is possible that neither bidder 1 nor bidder 2 win stage one, in which case bidder 2 will also lose stage two (to bidder 1). Thus, bidder 2 does not win one unit with probability one ($q_{21}(\mathbf{v}) \in (0, 1)$). However, bidder i , $i = 3, 4, \dots, n$, may win one (and only one) unit in the sequence of auctions ($q_{i1}(\mathbf{v}) \in (0, 1)$, for any i with $v_i \geq \hat{v}$, $i = 3, \dots, n$). Since some bidder other than bidder 1 wins a unit it must be the case that $\sum_{j=2}^n q_{j1}(\mathbf{v}) = 1$. Hence, the term in brackets in (12) decreases,

$$J(v_1) + J(v_2) > J(v_1) + \sum_{j=2}^n J(v_j)q_{j1}(\mathbf{v}),$$

since $J(v_2) \geq J(v_3) \geq \dots \geq J(v_n)$.

2. $n \geq 2$, $kv_1 > v_2 \geq \hat{v}$: This case is possible only if $\hat{v} < k\bar{v}$. Bidder 1 wins both units when there is no buy-out price ($q_{11}(\mathbf{v}) = q_{12}(\mathbf{v}) = 1$), and the term in brackets in (12) is $J(v_1) + kJ(v_1)$. If the introduction of a buy-out price affects the allocation, it must be because some bidder other than bidder 1, say bidder i , won the first auction, $i = 2, 3, \dots, n$. In any case, bidder 1 will win stage 2, meaning that he will win at least one unit ($q_{11}(\mathbf{v}) = 1$, $q_{12}(\mathbf{v}) \in (0, 1)$). On the other hand, bidder i , $i = 2, \dots, n$, wins one unit with positive probability if $v_i \geq \hat{v}$ ($q_{i1}(\mathbf{v}) \in (0, 1)$). As before, the term in brackets in (12) decreases,

$$J(v_1) + kJ(v_1) > J(v_1) + kJ(v_1) \left(1 - \sum_{j=2}^n q_{j1}(\mathbf{v}) \right) + \sum_{j=2}^n J(v_j)q_{j1}(\mathbf{v}),$$

which follows from (13).

In conclusion, for any realization of valuations for which the buy-out price may influence the allocation, the term over which the expectation is taken in (12) decreases. Hence, the introduction of a buy-out price strictly decreases total expected revenue when $\hat{v} < \bar{v}$ and $n > 2$. When $n = 2$, the first case does not apply, and the second case is relevant only if $\hat{v} < k\bar{v}$. Hence, any $\hat{v} < k\bar{v}$ leads to a strict loss in expected overall revenue, but expected revenue is unaffected if $\hat{v} \geq k\bar{v}$ and $n = 2$, simply because the final allocation is unaffected. ■

Proposition 6 (Multiple Equilibria): *If $B(x) = B(y)$ and $0 < x < y \leq \bar{v}$ then $ER_1(x) > ER_1(y)$. That is, if there are multiple equilibria associated with any given B , the first seller prefers the equilibrium in which the buy-out price is accepted more often.*

Proof. As a first step, we prove that if $\hat{v} \in (0, \bar{v})$, then

$$B(\hat{v}) > \int_0^{\hat{v}} b^1(x) \frac{f_{1,n-1}(x)}{F_{1,n-1}(\hat{v})} dx. \quad (14)$$

Recall that if a buyer with valuation at or above \hat{v} rejects the buy-out price, he should follow up by outbidding all other bidders in the first stage (assuming no other bidder accepted the buy-out price).

The interpretation of (14) then is that B exceeds what such a buyer would expect to pay if he rejects the buy-out price, contingent on all other bidders also rejecting it (i.e. having valuations below \hat{v}).

Now, consider a buyer with valuation \hat{v} , and assume the claim is false. If (14) does not hold, and if he is the buyer with the highest valuation he would win stage one whether or not he accepts the buy-out price, but he would pay less in expectation by accepting.

Assume now that he does not have the highest valuation. If he rejects the buy-out price he will then lose stage one, but he may win stage two. However, he can win stage two only if there is exactly one rival with a valuation in excess of \hat{v} . Now, if (14) is false it must necessarily be the case that $B < b^1(\hat{v})$. Thus, if the buyer in question loses the first auction the highest rival bid in the second auction must in expectation be higher than $b^1(\hat{v})$ (since the winner's valuation is above \hat{v}) and therefore higher than B . Consequently, the buyer is better off accepting the buy-out price in this case as well. If he wins stage one, the price he pays, B , is less than what it would take to win the second auction if he rejects.

Finally, if there are more than one rival with valuation above \hat{v} the buyer is also better off accepting the buy-out price, since otherwise he will have no chance of winning either auction.

In conclusion, if (14) is false, it is *strictly* better to accept than reject the buy-out price for a buyer with valuation \hat{v} . However, this contradicts the equilibrium condition that he be indifferent.

Turn now to the proposition. Expected revenue in the first auction is

$$ER_1(\hat{v}) = \int_0^{\hat{v}} b^1(x)n(F(\hat{v}) - F(x))f_{1,n-1}(x)dx + B(\hat{v})(1 - F_{1,n}(\hat{v})),$$

which may be rewritten as

$$ER_1(\hat{v}) = \int_0^{\hat{v}} \left(\int_0^x b^1(y) \frac{f_{1,n-1}(y)}{F_{1,n-1}(x)} dy \right) f_{1,n}(x) dx + \int_{\hat{v}}^{\bar{v}} B(\hat{v}) f_{1,n}(x) dx.$$

This has the advantage of making explicit how expected revenue depends on the highest valuation (the density of which is $f_{1,n}$). If the highest valuation is below \hat{v} , meaning the buy-out price is rejected, the price will equal the bid of the runner-up. If the highest valuation is above \hat{v} , the buy-out price will be accepted. To verify the two expressions are equal, change the order of integration in the latter.

Assuming for the moment that B is constant (independent of \hat{v}), expected revenue is simply

$$ER_1(\hat{v}) = \int_0^{\hat{v}} \left(\int_0^x b^1(y) \frac{f_{1,n-1}(y)}{F_{1,n-1}(x)} dy \right) f_{1,n}(x) dx + \int_{\hat{v}}^{\bar{v}} B f_{1,n}(x) dx,$$

the derivative of which with respect to \hat{v} is

$$ER'_1(\hat{v}) = \left[\int_0^{\hat{v}} b^1(y) \frac{f_{1,n-1}(y)}{F_{1,n-1}(\hat{v})} dy - B \right] f_{1,n}(\hat{v}).$$

Although B is not constant in general, it is possible that there are several values of \hat{v} which generates the same value of B . The implication of the above derivative is that among such a set of cutoffs, the expected revenue in the first auction is highest for the *smallest* cut-off, which follows from the fact that (14) implies that the derivative is negative. This completes the proof of Proposition 6. ■

2 Example in Detail

To add some further insights into the general results of the paper, we present the details as they apply to the *uniform* case with $v \in [0, 1]$, that is, $\bar{v} = 1$, $f(v) = 1$ and $F(v) = v$.⁶ A condensed version of the example is contained in the paper itself.

⁶This example certainly satisfies the notion of regularity alluded to above (monotonicity of $J(v) = v - \frac{1-F(v)}{f(v)}$), as well as the stronger notion of regularity referred to above as log-concavity of $1 - F(v)$.

2.1 No buy-out

Assume first that there is no buy-out price. Bidding in the first round as captured by (1) reduces to

$$b^1(v) = \frac{n-2+k^{n-1}}{n-1}v = v - \frac{1-k^{n-1}}{n-1}v$$

where we note that $b^1(v) \rightarrow \frac{n-2}{n-1}v$ for $k \rightarrow 0$ (*single-unit demand*),⁷ $b^1(v) \rightarrow v$ for $k \rightarrow 1$ (*horizontal demand*) and $b^1(v) \rightarrow v$ for $n \rightarrow \infty$ (*unlimited competition*). The expected revenues in the two auctions reduce to

$$ER_1^{SSP} = \frac{n-2+k^{n-1}}{n+1}$$

and

$$ER_2^{SSP} = \frac{n-2+k^{n-1}}{n+1} + \frac{n-1}{n+1}k^{n-1}(1-k) = ER_1^{SSP} + \frac{n-1}{n+1}k^{n-1}(1-k)$$

while total revenue is

$$\sum_{t=1}^2 ER_t^{ssp} = \frac{2(n-2) + (n+1 - (n-1)k)k^{n-1}}{n+1}$$

We plot ER_1^{SSP} and ER_2^{SSP} against k for different n in Fig. 1 - 4, where ER_2^{SSP} is the *fat* line, while ER_1^{SSP} is *thin*.

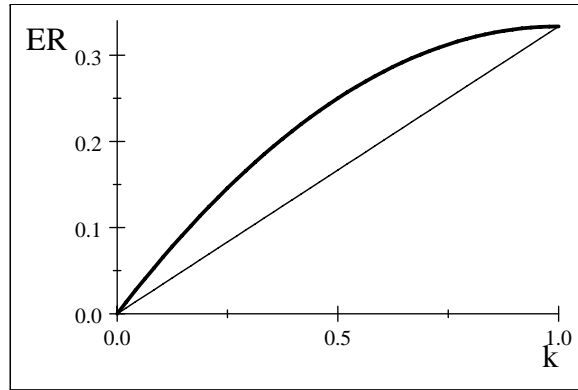


Fig. 1: SSP auctions with two bidders

⁷This is a special case of an example developed by Krishna (2002, Example 15.2, p. 219).

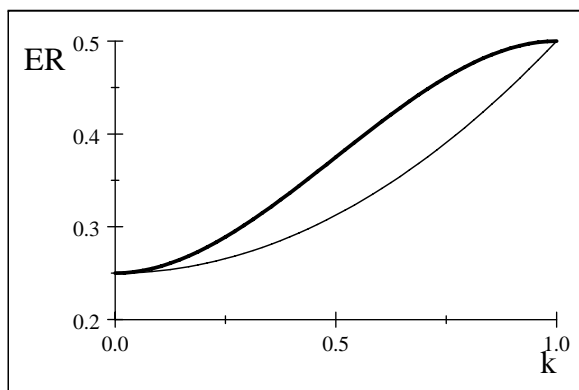


Fig. 2: SSP auctions with three bidders

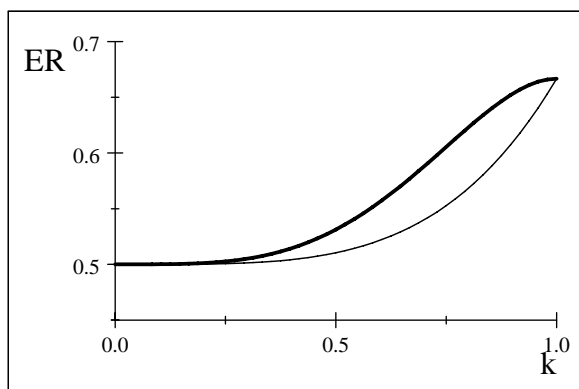


Fig. 3: SSP auctions with five bidders

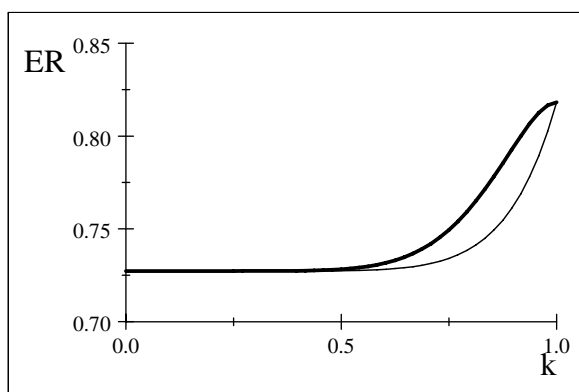


Fig. 4: SSP auctions with ten bidders

The *ratio* between expected revenues in the first and second auction

$$RR = \frac{ER_1^{SSP}}{ER_2^{SSP}} = \frac{n - 2 + (k)^{n-1}}{n - 2 + (k)^{n-1} + (n - 1)(k)^{n-1}(1 - k)}$$

is illustrated in Fig. 5.

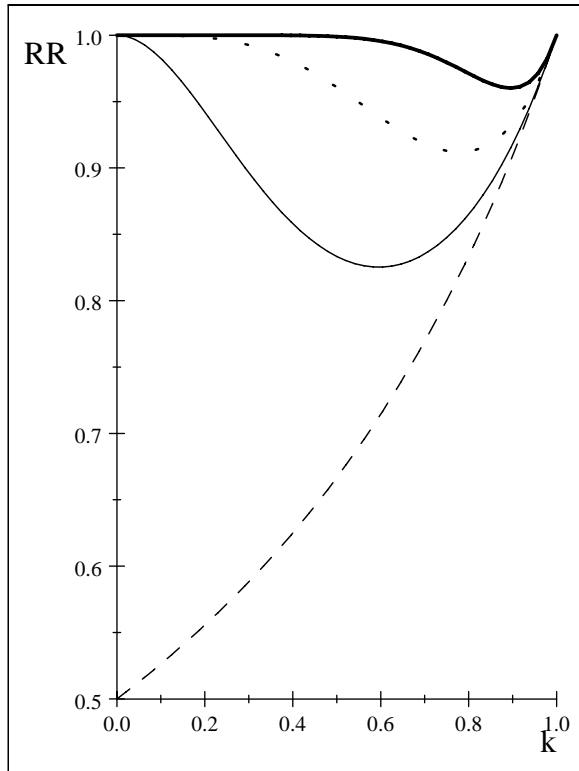


Fig. 5: The revenue-ratio in two SSP auctions
($n = 2$ - dashed, $n = 3$ - thin, $n = 5$ - dotted, $n = 10$ - fat)

For $n = 2$, note the discontinuity at $k = 0$. When $k = 0$, both sellers earn nothing, that is, the same. However, when k is small, but strictly positive, we observe that the winner of the first auction is very unlikely also to be the winner of the second auction. Hence, the expected revenue in the first auction is k times (the expected value of) the second highest valuation, while the expected revenue in the second auction is *approximately* k times (the expected value of) the highest valuation. For the uniform case with $n = 2$, the ratio between the expected value of the highest ($2/3$) and the expected value of the second highest valuation ($1/3$) is exactly $1/2$. For $n > 2$, there are no such discontinuities.

From this example it is immediate that the difference in expected revenues can be significant unless k is close to one (demands are near-horizontal) or, for $n > 2$, if k is close to zero (near unit-demands). Any difference, of course, disappears as the number of bidders increases without bound.

2.2 Buy-out

Now, consider introducing a buy-out price in the uniform example. Given \hat{v} , the stage-one revenues reduce to

$$ER_1(\hat{v}) = \begin{cases} \left. \begin{aligned} &\hat{v} - \frac{n}{2}\hat{v}^{n-1} + (n-2)\hat{v}^n - \frac{(n-1)(n-2)}{2(n+1)}\hat{v}^{n+1} \\ &+ \frac{k^{n-1}}{2}(1-\hat{v} + \hat{v}^n - \frac{n-1}{n+1}\hat{v}^{n+1}) \end{aligned} \right\} & \text{if } \hat{v} \geq k \\ \left. \begin{aligned} &\hat{v} - \frac{n}{2}\hat{v}^{n-1} + (n-2)\hat{v}^n - \frac{(n-1)(n-2)}{2(n+1)}\hat{v}^{n+1} \\ &+ \frac{k^{n-1}}{2}(1-\hat{v} + \hat{v}^n - \frac{n-1}{n+1}\hat{v}^{n+1}) \\ &- \frac{1}{2k}(1-\hat{v})(k^n - \hat{v}^n - n\hat{v}^{n-1}(k-\hat{v})) \end{aligned} \right\} & \text{if } \hat{v} \leq k \end{cases}$$

First, for $\hat{v} \geq k$, we have

$$\begin{aligned} ER_1'(\hat{v}) &= 1 - \frac{n(n-1)}{2}\hat{v}^{n-2} + (n-2)n\hat{v}^{n-1} - \frac{(n-1)(n-2)}{2}\hat{v}^n + \frac{k^{n-1}}{2}(-1 + n\hat{v}^{n-1} - (n-1)\hat{v}^n), \\ &= 1 - \frac{n(n-1)}{2}\hat{v}^{n-2} + (n-2)n\hat{v}^{n-1} - \frac{(n-1)(n-2)}{2}\hat{v}^n + \frac{k^{n-1}}{2}(\hat{v}^n + n\hat{v}^{n-1}(1-\hat{v}) - 1) \\ &= A_1(\hat{v}) + \frac{k^{n-1}}{2}A_2(\hat{v}) \end{aligned}$$

where $A_1(\hat{v}) \equiv 1 - \frac{n(n-1)}{2}\hat{v}^{n-2} + (n-2)n\hat{v}^{n-1} - \frac{(n-1)(n-2)}{2}\hat{v}^n$ and $A_2(\hat{v}) \equiv \hat{v}^n + n\hat{v}^{n-1}(1-\hat{v}) - 1$. It can easily be shown that $A_1(\hat{v})$ is positive and decreasing, while $A_2(\hat{v})$ is negative and increasing. The latter implies that the higher is k , the smaller is $ER_1'(\hat{v})$, for any \hat{v} .

If the optimal cut-off, v^* , is larger than k , for some k , then the first-order condition is $A_1(v^*) + A_2(v^*)\frac{k^{n-1}}{2} = 0$. By implicit differentiation this yields

$$\frac{dv^*}{dk} = -\frac{A_2(v^*)\frac{(n-1)k^{n-2}}{2}}{A_1'(v^*) + A_2'(v^*)\frac{k^{n-1}}{2}}.$$

However, since $A_2(\hat{v}) < 0$, this has the same sign as $A_1'(v^*) + A_2'(v^*)\frac{k^{n-1}}{2}$, which, in turn, must be negative, since v^* is the optimal cut-off point (this is the second-order condition). We conclude that when the optimal cut-off is above k (which never happens for $n = 2$), the cut-off valuation is *decreasing* in k .

Second, if $\hat{v} \leq k$, then $ER_1(\hat{v})$ is as before, but with the additional term $-\frac{1}{2k}(1-\hat{v})(k^n - \hat{v}^n - n\hat{v}^{n-1}(k-\hat{v}))$. This additional term is zero for $k - \hat{v} = 0$, and we conclude that $ER_1(\hat{v})$ is continuous. The derivative of the last term w.r.t. \hat{v} is

$$\begin{aligned} &\frac{1}{2k}(k^n - \hat{v}^n - n\hat{v}^{n-1}(k-\hat{v})) - \frac{1}{2k}(1-\hat{v})(-n\hat{v}^{n-1} + n\hat{v}^{n-1} - n(n-1)\hat{v}^{n-2}(k-\hat{v})) \\ &= \frac{1}{2k}(k^n - \hat{v}^n - n\hat{v}^{n-1}(k-\hat{v})) + \frac{1}{2k}(1-\hat{v})n(n-1)\hat{v}^{n-2}(k-\hat{v}), \end{aligned}$$

which is also zero for $k - \hat{v} = 0$. Hence, $ER_1(\hat{v})$ is differentiable everywhere.

With these remarks in mind, we turn to consider the two cases $n = 2$ and $n = 3$ to get some further insights into the results of Section 3 and Section 4.

2.2.1 Two bidders

With two bidders, the revenues are

$$ER_1(\hat{v}) = \begin{cases} \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 & \hat{v} \geq k \\ \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 - \frac{1}{2k}(k-\hat{v})^2(1-\hat{v}) & \hat{v} \leq k \end{cases}$$

and

$$ER_2(\hat{v}) = \begin{cases} \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 + \frac{k}{3}((1-k) - (1-\hat{v})^3) & \hat{v} \geq k \\ \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 - \frac{1}{2k}(k-\hat{v})^2(1-\hat{v}) \\ + \frac{k}{3}((1-k) - (1-\hat{v})^3) + \frac{1}{6k}(k-\hat{v})^2(3(1-\hat{v}) + (k-\hat{v})) & \hat{v} \leq k \end{cases}$$

The latter can be written as

$$ER_2(\hat{v}) = \begin{cases} ER_1(\hat{v}) + \frac{k}{3}((1-k) - (1-\hat{v})^3) & \hat{v} \geq k \\ ER_1(\hat{v}) + \frac{k}{3}((1-k) - (1-\hat{v})^3) + \frac{1}{6k}(k-\hat{v})^2(3(1-\hat{v}) + (k-\hat{v})) & \hat{v} \leq k \end{cases}$$

The sum of (expected) revenues is

$$\sum_{t=1}^2 ER_t(\hat{v}) = \begin{cases} \frac{k(3-k)}{3} = \sum_{t=1}^2 ER_t^{ssp} & \hat{v} \geq k \\ \frac{k(3-k)}{3} - \frac{(k-\hat{v})^2(2(1-\hat{v})+1-k)}{6k} < \sum_{t=1}^2 ER_t^{ssp} & \hat{v} < k \end{cases}$$

reflecting that efficiency and “revenue equivalence” is lost, once the buy-out price is set such that the cut-off, \hat{v} , drops below k ($= k\bar{v}$).

To get an initial feel for the dependence of revenues on the buy-out price, consider the special case where $k = \frac{1}{2}$. Fig. 6 plots $ER_1(\hat{v})$ (*thin*), $ER_2(\hat{v})$ (*dots*) and their sum (*fat*) against \hat{v} .

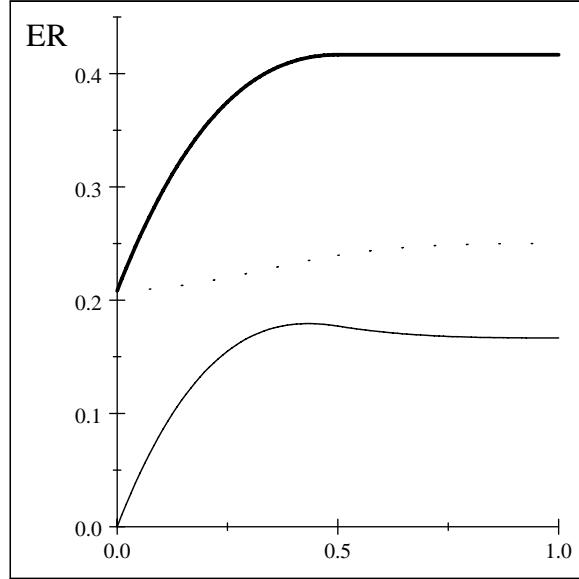


Fig. 6: Revenues against \hat{v}

Next, turn to the optimal value of the cut-off, hence, the first-round buy-out price. From Corollary 1 and Proposition 5 we know that $\hat{v} < k\bar{v} = k$, for any $k \in (0, 1)$. Maximizing

$$ER_1(\hat{v}) = \frac{k}{3} + \frac{k}{6}(1-\hat{v})^3 - \frac{1}{2k}(k-\hat{v})^2(1-\hat{v})$$

with respect to \hat{v} gives the optimal cut-off valuation from the perspective of the first seller

$$v^* = \frac{1+2k-k^2}{3-k^2} - \frac{((1+2k-k^2)^2 - 2k(3-k^2))^{1/2}}{3-k^2} < k = k\bar{v}$$

and the associated, optimal buy-out price, $B(v^*)$ is given by

$$B(v^*) = \frac{k}{2(1+v^*)} \left((1+(v^*)^2) - \left(1 - \frac{v^*}{k}\right)^2 \right)$$

In Fig. 7 v^* and $B(v^*)$ are plotted against k .

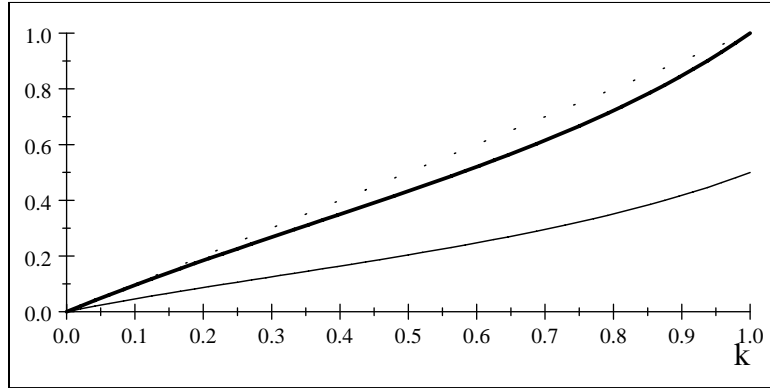


Fig. 7: Optimal cut-off and buy-out
 (v^* - fat, $B(v^*)$ - thin, 45° - dots)

Next, we substitute v^* into the revenue expressions, and Fig. 8 illustrates how $ER_1(v^*)$ (thin) and $ER_2(v^*)$ (fat) vary with k .

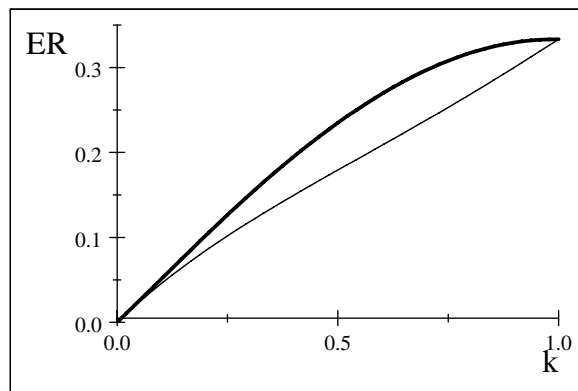


Fig. 8: Revenues with optimal buy-out

The ratio between the expected revenues given an optimally chosen buy-out price, $RR(BO) = \frac{ER_1(v^*)}{ER_2(v^*)}$, is illustrated in the following figure.

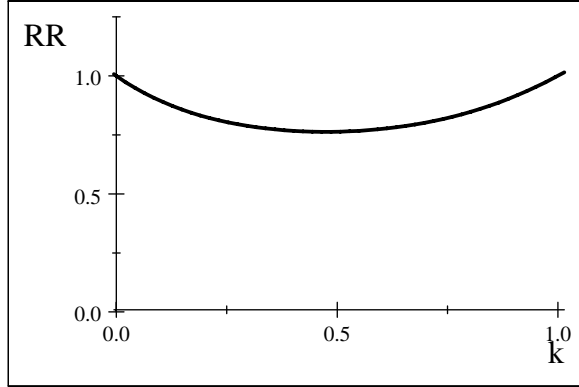


Fig. 9: Revenue ratio with optimal buy-out

We can compare with the case of two straight second-price auctions illustrated in Fig. 1 and Fig. 5. In Fig. 10 we merge the information in Fig. 1 and Fig. 8. The *dashed* lines are for two straight second-price auctions, while the *solid* lines are for the case where the first seller chooses the buy-out price to implement v^* .

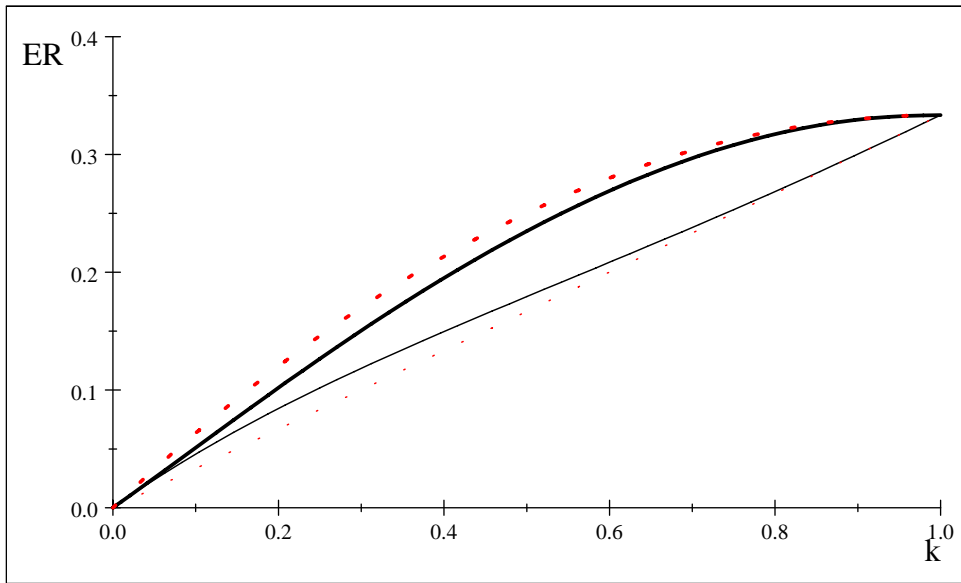


Fig. 10: Comparison of auction revenues

Fig. 11 merges the information from Fig. 5 and Fig. 9, and the *thin* line is for two straight second-price auctions, while the *fat* line is associated with an optimal buy-out price.

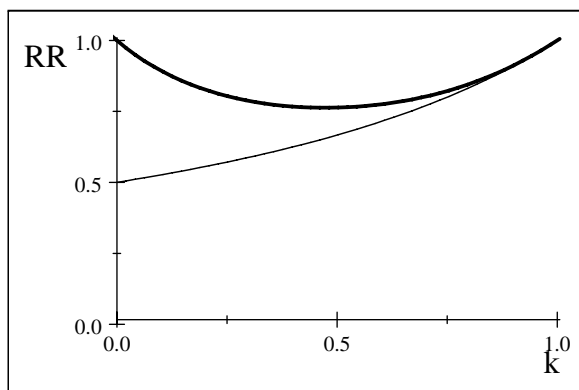


Fig. 11: Revenue Ratios

Finally, in Fig. 12 we plot the percentage gain to the first seller from an optimally chosen buy-out compared to the straight second-price auction, $G = 100 \times \frac{ER_1(v^*) - ER_1^{SSP}}{ER_1^{SSP}}$.

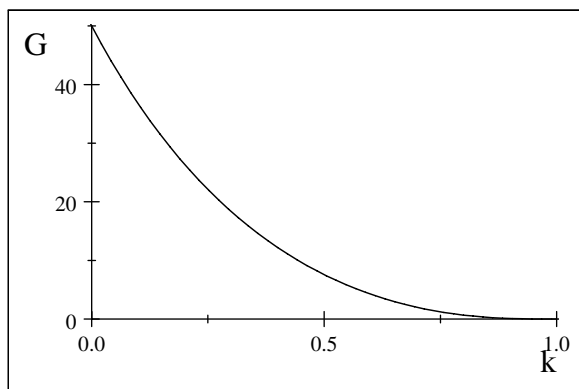


Fig. 12: Percentage gain from buy-out price

The last three figures essentially illustrate that the value from the perspective of the first seller of introducing a buy-out price is substantial when the individual demand functions are relatively steep (k small). When demands are steep, and there are only two bidders, the competition for the first object will be weak. It follows that the first seller has a strong incentive to try to improve his position in this case by introducing a suitably chosen buy-out price. The following table captures central features of the example in an alternative way.

k	ER_1^{SSP}	v^*	$B(v^*)$	$ER_1(v^*)$	G
0.01	0.00333	0.00995	0.00495	0.00495	48.65
0.10	0.03333	0.09549	0.04597	0.04558	36.75
0.25	0.08333	0.22618	0.10623	0.10176	22.12
0.50	0.16667	0.43308	0.20404	0.17931	7.58
0.75	0.25000	0.66667	0.32222	0.25309	1.24

Recall that in this example revenue equivalence and efficiency is lost when \hat{v} is set below $k = k\bar{v}$. Hence, a comparison of the first and third column is indicative of the inefficiency when \hat{v} is set optimally.

For example, when $k = k\bar{v} = \frac{1}{2}$ the optimal \hat{v} is approximately 0.43, which implies that there is a small, but “non-trivial”, probability that the final allocation is inefficient. Note that $k = \frac{1}{2}$ implies that $ER_2^{SSP} - ER_1^{SSP} = \frac{1}{3}k(1 - k)$ is maximized. When the first seller sets the optimal buy-out price $B(v^*) \approx 0.2$, he manages to increase his expected revenue by 7.58%, while total revenue falls by only 0.58%.

Finally, a comment on the *size* of the efficiency loss. When $n = 2$, the sequence of auctions is potentially inefficient only if $\hat{v} < k\bar{v}$. In this case, the expected gains from trade over the auction sequence can be written as⁸

$$\begin{aligned} W_2(\hat{v}) &= 2 \int_0^{\bar{v}} v_1 F(v_1) f(v_1) dv_1 + 2 \int_0^{\frac{\hat{v}}{k}} kv_1 F(kv_1) f(v_1) dv_1 \\ &\quad + 2 \int_{\frac{\hat{v}}{k}}^{\bar{v}} kv_1 \left(F(\hat{v}) + \frac{1}{2}(F(kv_1) - F(\hat{v})) \right) f(v_1) dv_1 \\ &\quad + 2 \int_0^{k\bar{v}} v_2 \left(F\left(\frac{v_2}{k}\right) - F(v_2) \right) f(v_2) dv_2 + 2 \int_{k\bar{v}}^{\bar{v}} v_2 (1 - F(v_2)) f(v_2) dv_2 \\ &\quad + 2 \int_{\hat{v}}^{k\bar{v}} v_2 \frac{1}{2} \left(1 - F\left(\frac{v_2}{k}\right) \right) f(v_2) dv_2 \end{aligned}$$

Notice that $W_2(k\bar{v})$ is (equivalent to) gains from trade given efficiency (no buy-out price).

We can now measure the efficiency loss by introducing a buy-out price, in the uniform example. For $\hat{v} < k\bar{v} = k$, $W_2(\hat{v})$ reduces to

$$W_2(\hat{v}) = \frac{1}{2}k\hat{v} + \frac{1}{6k}k^3 - \frac{1}{2}\hat{v}^2 + \frac{1}{6k}\hat{v}^3 + 1$$

Hence,

$$W_2(k) = \frac{1}{3}k^2 + 1 = \frac{3 + k^2}{3}$$

and we can write the (percentage) loss as

$$l_2(\hat{v}) = \frac{W_2(\hat{v}) - W_2(k)}{W_2(k)} = -\frac{(k - \hat{v})^3}{2k((3 + k^2))}$$

Substituting v^* into this expression, we can illustrate the efficiency loss, when the first seller chooses the buy-out price optimally, as follows.

⁸The *first three* terms relate to the high-valuation bidder and the *last three* to the low-valuation bidder. The first term is due to the fact that the high-valuation bidder always wins at least one unit. The second and third terms derive from the fact that he may win both. Obviously, this necessitates that $kv_1 > v_2$. If $v_2 < \hat{v}$, this is also sufficient. Otherwise, however, there is probability 0.5 that the high-valuation bidder loses stage one, and thus only wins one unit. Regarding the last three terms, observe that the low-valuation bidder wins at most one unit. He is sure to win one unit if $v_1 < \frac{v_2}{k}$, which explains the first two of these terms. Finally, as captured by the last term, he may also win one unit if his type is above \hat{v} and $v_1 > \frac{v_2}{k}$, though this is inefficient. This occurs when he is fortunate enough to win stage one by accepting the buy-out price.

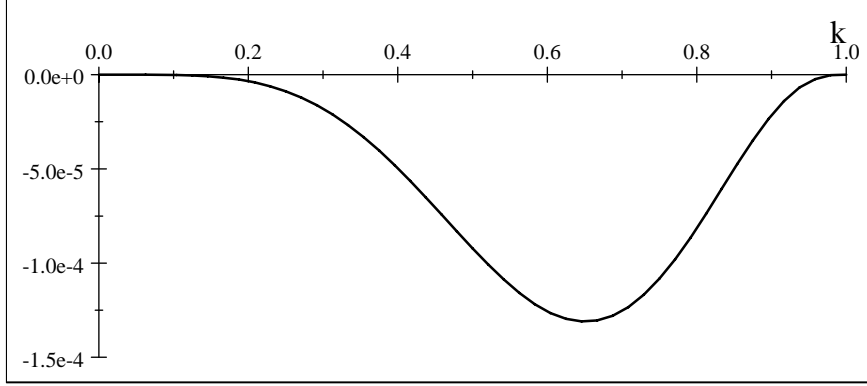


Fig. 13: Optimal buy-out and efficiency loss

Thus, we conclude that with $n = 2$ the maximum efficiency loss is very small, indeed (approximately 0.01%).

2.2.2 Buy-out prices and cut-off valuations

As mentioned earlier, $B(\hat{v})$ need not necessarily be monotonic in \hat{v} . Indeed, when $n = 2$, the buy-out price is globally increasing in \hat{v} when $k > \sqrt{2} - 1$, but *not* when k is smaller. We prove this fact in the following.

First, if the cut-off valuation is small, $\hat{v} \in [0, k]$, the buy-out price is

$$B(\hat{v}) = \frac{2k\hat{v} - \hat{v}^2 + k^2\hat{v}^2}{2k(1 + \hat{v})},$$

and it follows that

$$\begin{aligned} B'(\hat{v}) &= \frac{(1 + \hat{v})(2k - 2\hat{v} + 2k^2\hat{v}) - (2k\hat{v} - \hat{v}^2 + k^2\hat{v}^2)}{2k(1 + \hat{v})^2} \\ &= \frac{2k - 2\hat{v} + 2k^2\hat{v} - \hat{v}^2 + k^2\hat{v}^2}{2k(1 + \hat{v})^2}, \end{aligned}$$

the sign of which is determined by the numerator. The numerator is strictly positive for $\hat{v} = 0$, but it is decreasing in \hat{v} . Hence, if B is decreasing anywhere on $[0, k]$, it must also be decreasing at $\hat{v} = k$. However,

$$B'(k) = \frac{k(2k + k^2 - 1)}{2(1 + k)^2} = \frac{k(k + 1 + \sqrt{2})(k - (\sqrt{2} - 1))}{2(1 + k)^2}.$$

Since $B'(0) > 0$ and $B'(k)$ is negative if $k < \sqrt{2} - 1$, we conclude that $B(\hat{v})$ is non-monotonic when $k < \sqrt{2} - 1$.

However, when $k > \sqrt{2} - 1$, $B(\hat{v})$ is monotonic on $[0, k]$, as $B'(k) > 0$ (implying that $B'(\hat{v}) > 0$ for all $\hat{v} \in [0, k]$). If $\hat{v} \geq k$, the buy-out price is

$$B(\hat{v}) = \frac{k(1 + \hat{v}^2)}{2(1 + \hat{v})},$$

with

$$B'(\hat{v}) = \frac{k(2\hat{v} + \hat{v}^2 - 1)}{2(1 + \hat{v})^2}$$

and we observe that the numerator is increasing in \hat{v} . Hence, if $B(\hat{v})$ is decreasing anywhere on $\hat{v} \in (k, 1]$ then it must also be decreasing at $\hat{v} = k$. However,

$$B'(k) = \frac{k(2k + k^2 - 1)}{2(1 + k)^2},$$

which is positive since $k > \sqrt{2} - 1$. Hence, if $k > \sqrt{2} - 1$, the buy-out price is globally increasing in the cut-off valuation.

2.2.3 Three bidders

With three bidders, expected revenue in stage one is

$$ER_1(\hat{v}) = \left\{ \begin{array}{l} \left. \begin{array}{l} \hat{v} - \frac{3}{2}\hat{v}^2 + \hat{v}^3 - \frac{1}{3}\hat{v}^4 \\ + \frac{k^2}{2}(1 - \hat{v} + \hat{v}^3 - \frac{1}{2}\hat{v}^4) \end{array} \right\} \quad \text{if } \hat{v} \geq k \\ \left. \begin{array}{l} \hat{v} - \frac{3}{2}\hat{v}^2 + \hat{v}^3 - \frac{1}{3}\hat{v}^4 \\ + \frac{k^2}{2}(1 - \hat{v} + \hat{v}^3 - \frac{1}{2}\hat{v}^4) \\ - \frac{1}{2k}(1 - \hat{v})(k^3 - \hat{v}^3 - 3\hat{v}^2(k - \hat{v})) \end{array} \right\} \quad \text{if } \hat{v} \leq k \end{array} \right.$$

Thus, for $\hat{v} \geq k$, we have

$$\begin{aligned} ER_1'(\hat{v}) &= 1 - 3\hat{v} + 3\hat{v}^2 - \hat{v}^3 + \frac{k^2}{2}(-1 + 3\hat{v}^2 - 2\hat{v}^3) \\ &= A_1(\hat{v}) + \frac{k^2}{2}A_2(\hat{v}) \\ &= -(1 - \hat{v})^2(1 + k^2)\left(\hat{v} - \frac{2 - k^2}{2(1 + k^2)}\right). \end{aligned}$$

Hence, $ER_1'(\hat{v}) = 0$ at $\hat{v} = \frac{2 - k^2}{2(1 + k^2)}$ (which is the only interior value of \hat{v} for which this is true). Now, $\frac{2 - k^2}{2(1 + k^2)}$ is decreasing in k , which implies that the optimal cut-off, \hat{v} , is decreasing in k . This requires that $\hat{v} \geq k$, and that it is a maximum, rather than a minimum. To see that it is indeed a (local) maximum, notice that $ER_1(\hat{v})$ is increasing to the left, and decreasing to the right of $\frac{2 - k^2}{2(1 + k^2)}$. For existence, we require $\hat{v} = \frac{2 - k^2}{2(1 + k^2)} \geq k$ which reduces to $0 \geq k^2 + 2k + 2k^3 - 2$. This, in turn, is satisfied for $k \in [0, \bar{k}]$, where $\bar{k} \simeq 0.60149$.⁹ Hence, for $k \leq \bar{k}$, we have a local maximum above k , but for $k > \bar{k}$, there is no such maximum.

For $\hat{v} \leq k$, we have

$$\begin{aligned} ER_1'(\hat{v}) &= A_1(\hat{v}) + \frac{k^2}{2}A_2(\hat{v}) + \frac{1}{2k}(k^3 - \hat{v}^3 - 3\hat{v}^2(k - \hat{v})) - \frac{1}{2k}(1 - \hat{v})(-6\hat{v}(k - \hat{v})) \\ &= \frac{1}{2k}(8 - 2k - 2k^3) \left[\hat{v}^3 - \frac{6 + 3k - 3k^3}{8 - 2k - 2k^3}\hat{v}^2 + \frac{2k}{8 - 2k - 2k^3} \right], \end{aligned}$$

and

$$ER_1''(\hat{v}) = \frac{1}{2k}(8 - 2k - 2k^3)3\hat{v} \left[\hat{v} - \underbrace{\frac{2 + k - k^3}{4 - k - k^3}}_{\in (0.5, 1)} \right]$$

⁹To be exact $\bar{k} = \frac{1}{6} \left(\sqrt[3]{125 + 6\sqrt{471}} - \frac{11}{\sqrt[3]{125 + 6\sqrt{471}}} - 1 \right)$.

Hence, $ER_1(\hat{v})$ is concave if $\hat{v} \leq \frac{2+k-k^3}{4-k-k^3}$. Clearly, $\frac{2+k-k^3}{4-k-k^3} > k$, and it follows that $ER_1(\hat{v})$ is concave in \hat{v} , for all $\hat{v} \leq k$. Recalling that $ER_1(\hat{v})$ is differentiable, we conclude that if $ER_1(\hat{v})$ has a local maximum above k , this is the only maximum (and if there is no local maximum above k , there must be exactly one below).

Thus, for $\hat{v} \leq k$,

$$ER_1'(\hat{v}) = \frac{1}{2k}(8 - 2k - 2k^3) \left[\hat{v}^3 - \frac{6 + 3k - 3k^3}{8 - 2k - 2k^3} \hat{v}^2 + \frac{2k}{8 - 2k - 2k^3} \right]$$

which implies that any interior maximum must satisfy

$$A_3(v^*) \equiv (v^*)^3 - \frac{6 + 3k - 3k^3}{8 - 2k - 2k^3} (v^*)^2 + \frac{2k}{8 - 2k - 2k^3} = 0$$

Importantly, $A_3(v^*)$ is increasing in k . Implicit differentiation then implies that

$$\frac{dv^*}{dk} = -\frac{\partial A_3(v^*)}{\partial k} / \frac{\partial A_3(v^*)}{\partial v^*} > 0$$

where the inequality follows from the fact that $ER_1(\hat{v})$ is concave, and that $A_3(v^*)$ is increasing in k . Hence, the optimal cut-off is increasing in k .

To summarize, for the uniform distribution with 3 bidders, the optimal cut-off is “U-shaped”. For small values of k , the optimal value of \hat{v} is above k , and is decreasing until k reaches $\bar{k} \simeq 0.60149$. Indeed, for $k = 0$, the optimal value of \hat{v} is one, implying that the buy-out is never exercised. The reason is that when $k = 0$, the sequence of (straight) auctions is optimal, and any changes to the design would make the first seller worse off. After $k > \bar{k}$, the optimal value of \hat{v} starts to increase. For $k = 1$, the optimal cut-off is 1, implying again that the buy-out price is never accepted. Again, the reason is that with $k = 1$, the sequence of (straight) auctions is optimal. The closer k is to zero or one, the closer the sequence of auctions is to being optimal (overall and for seller 1), and the less incentive there is to manipulate the auction format. That explains why v^* is close to one when k is close to either zero or one. Consequently, neither the optimal cut-off nor the probability that the buy-out price is accepted are monotonic in k .¹⁰

Fig. 14 illustrates the optimal optimal cut-off, v^* , as a function of k when $n = 3$ (fat).

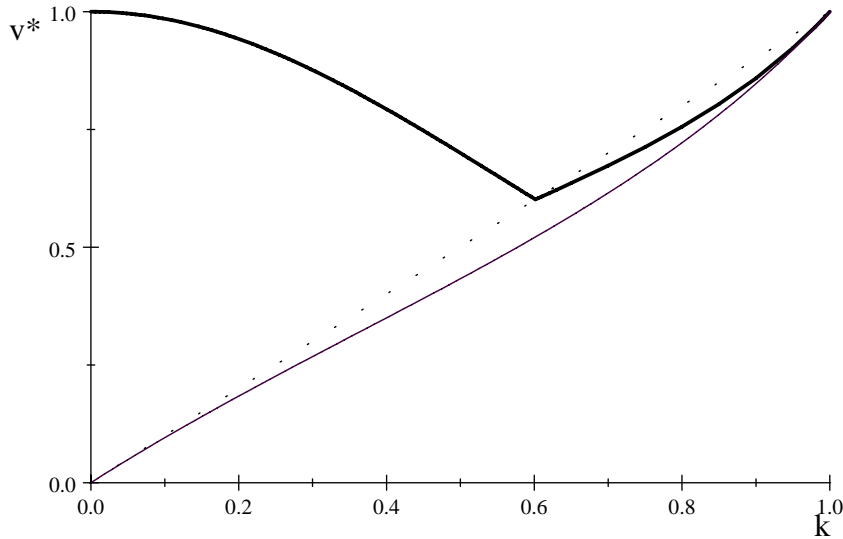


Fig. 14: Optimal cut-offs

¹⁰The “U-shape” of the optimal cut-off for $n > 2$ generalizes to any distribution of valuations, $F(v)$, for which $ER_1(\hat{v})$ is *single-peaked* in \hat{v} for all k .

For comparison, the figure also repeats the optimal cut-off when $n = 2$ (*thin*) and $\hat{v} = k$ (*dots*). When $n = 2$, we recall that the cut-off is everywhere increasing in k and below $\hat{v} = k$. We note that the addition of another bidder causes the optimal cut-off to increase. However, since there are more bidders, this does not necessarily mean that it becomes less likely that the buy-out is accepted.

As in the preceding case with two bidders, we can also capture central features of the example in an alternative way when there are three bidders.

k	ER_1^{SSP}	v^*	$ER_1(v^*)$	G
0.20	0.26000	0.94231	0.26000	≈ 0
0.40	0.29000	0.79310	0.29018	0.06
0.60	0.34000	0.60294	0.34282	0.83
0.80	0.41000	0.75587	0.41252	0.61
0.90	0.45250	0.85878	0.45311	0.13

From the last column we note that the gain to the first seller from an optimally chosen buy-out price is small when $n = 3$ compared to the case where $n = 2$.

As in the previous subsection we conclude by commenting on the (potential) efficiency loss associated with buy-out in the first auction. Generally, with n buyers, welfare in a sequence of auctions *without* a buy-out price (that is, an efficient sequence) is

$$\begin{aligned}
W_n &= n \int_0^{\bar{v}} v F(v)^{n-1} f(v) dv + n \int_0^{\bar{v}} kv F(kv)^{n-1} f(v) dv \\
&\quad + n(n-1) \int_0^{k\bar{v}} v \left(F\left(\frac{v}{k}\right) - F(v) \right) F(v)^{n-2} f(v) dv \\
&\quad + n(n-1) \int_{k\bar{v}}^{\bar{v}} v(1 - F(v)) F(v)^{n-2} f(v) dv
\end{aligned}$$

The first two terms are expected social surplus from the bidder with the highest valuation, and the last two terms capture surplus from the bidder with the second-highest valuation. This just generalizes the expression above for $n = 2$.

Assuming that $n = 3$, and that the distribution is uniform on $[0, 1]$, this reduces to

$$W_3 = \frac{1}{4}k^3 + \frac{5}{4}$$

Now, turning to the efficiency loss associated with a buy-out price, we start by considering *high* buy-out prices. That is, the cut-off, \hat{v} , is assumed to be above $k\bar{v}$, in which case the only type of inefficiency that may arise is that a player other than buyer 1 or 2 wins the first object, when buyer 1 and buyer 2 should have shared the two objects. With 3 buyers, welfare changes by

$$L_3(\hat{v}) = \frac{1}{3} \left(\int_{\hat{v}}^{\bar{v}} 3v(1 - F(v))^2 f(v) dv - \int_{\hat{v}}^{\bar{v}} 6v(1 - F(v))(F(v) - F(\hat{v})) f(v) dv \right)$$

since welfare changes if *all* buyers have valuation above \hat{v} , *and* buyer 3 wins stage 1 (which occurs with probability $\frac{1}{3}$). In this case, the surplus is $v_1 + v_3$, rather than $v_1 + v_2$, so the change in welfare is $v_3 - v_2 \leq 0$. With the uniform distribution, this reduces to

$$L_3(\hat{v}) = \frac{1}{3} \left(\int_{\hat{v}}^1 3v(1 - v)^2 dv - \int_{\hat{v}}^1 6v(1 - v)(v - \hat{v}) dv \right) = \left(-\frac{1}{12} \right) (1 - \hat{v})^4$$

For given \hat{v} , notice that the welfare loss is independent of k , as long as $\hat{v} \geq k$. However, the optimal value of \hat{v} depends on k . Earlier, we found that for $k \leq \bar{k} \simeq 0.60149$, the optimal cut-off, $v^* = \frac{2-k^2}{2(1+k^2)}$, is indeed above k . We use this in $L_3(\hat{v})$ to write

$$L_3(v^*) = \left(-\frac{1}{12}\right) \left(1 - \frac{2-k^2}{2(1+k^2)}\right)^4 = \left(-\frac{1}{12}\right) \left(\frac{3k^2}{2(1+k^2)}\right)^4$$

to get an expression for the loss in welfare (valid for $k \leq \bar{k}$),

$$l_3 = \frac{L_3(v^*)}{W_3} = \frac{\left(-\frac{1}{12}\right) \left(\frac{3k^2}{2(1+k^2)}\right)^4}{\frac{1}{4}k^3 + \frac{5}{4}}$$

In Fig. 15, this is depicted as a function of k .

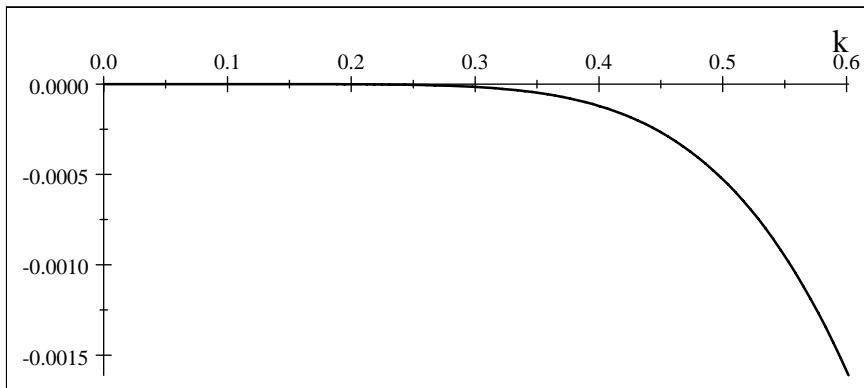


Fig. 14: Optimal buy-out and efficiency loss

For $k \simeq \bar{k}$, the loss corresponds to approximately 0.16%. While this loss is still small, it is significantly (an order of magnitude) larger than the maximum loss when $n = 2$. Intuitively, the reason is that there is an additional type of inefficiency when the extra bidder is added. When $n > 2$, any \hat{v} below \bar{v} gives rise to potential inefficiency. As illustrated by Fig. 14, the optimal cut-off is far below \bar{v} for $n = 3$ and intermediate values of k . On the other hand, for $n = 2$, inefficiency arises only if \hat{v} is below $k\bar{v}$. However, from Fig. 14 it is evident that v^* is only slightly below $k\bar{v}$ for $n = 2$. Finally, as the number of bidders increases without bound (intensive bidder competition), the efficiency loss clearly disappears. Thus, we conclude that the efficiency loss, when the buy-out price in the first auction is set optimally, is *non-monotonic* in the number of bidders.

2.2.4 Discussion

Comparing the result of the preceding two subsections, we conclude that for the uniform case, the competition between three bidders is essentially enough to wipe out the substantial gain to the first seller from buy-outs that exists when there are only two bidders. Of course, it never hurts the first seller to introduce a buy-out (*provided that it is suitably chosen*).

This is interesting from a practical perspective: We should expect the first seller to introduce a buy-out when he thinks that there will only be a few interested buyers. If the seller expects bidder competition to be just moderately strong, the potential gains from the introduction of a buy-out in an otherwise standard auction are insignificant compared to the costs that must be associated with setting the buy-out price

at the right level. For the latter, it is particularly important to note that the optimal buy-out price will depend on both the expected number of interested bidders (n) and details about the nature of individual demands (k). Getting the buy-out price just slightly wrong when there are more than two bidders will likely hurt the first seller.

By way of comparison, let us consider the effects of a reserve price in a *single* standard auction (in the uniform case). An optimal (revenue maximising) auction is implemented by a reserve price of $r = \frac{1}{2}$ independently of n (thus excluding bidders with negative marginal revenues). Expected revenues are given by

$$ER^{base} = \int_0^1 J(v)f_{1,n}(v)dv = \frac{n-1}{n+1}$$

in any baseline auction and by

$$ER^{opt} = \int_{\frac{1}{2}}^1 J(v)f_{1,n}(v)dv = \frac{n-1}{n+1} + \frac{1}{n+1}\left(\frac{1}{2}\right)^n$$

in an optimal auction. We define the gain as

$$G_n = \frac{ER^{opt} - ER^{base}}{ER^{base}} = \frac{\frac{n-1}{n+1} + \frac{1}{n+1}\left(\frac{1}{2}\right)^n - \frac{n-1}{n+1}}{\frac{n-1}{n+1}} = \frac{1}{n-1}\left(\frac{1}{2}\right)^n$$

and represent the most relevant information as follows

n	G_n	$\%G_n$
2	$\frac{1}{4}$	25.00
3	$\frac{1}{32}$	3.13
4	$\frac{1}{96}$	1.04
5	$\frac{1}{256}$	0.39

Thus, we note, again, that competition really does the trick: With two bidders, there is a sizable gain, while with three and certainly four the gain has (essentially) disappeared. Of course, with any number of bidders, the seller is not hurt by a reserve, but as in our case we would expect to see it mainly when the seller thinks there might be only a few potential buyers. In any case, the seller should only invest resources in finding the right value of the optimal reserve when there are few bidders. Note also that a reserve price introduces inefficiency, in the sense that the object is withheld with strictly positive probability. However, this efficiency loss is *monotonically decreasing* as the number of bidders increases.