Using Economic Theory to Guide Numerical Analysis: Solving for Equilibria in Models of Asymmetric First-Price Auctions

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Abstract

In models of first-price auctions, when bidders are *ex ante* heterogenous, deriving explicit equilibrium bid functions is typically impossible, so numerical methods are often employed to find approximate solutions. Recent theoretical research concerning asymmetric auctions has determined conditions under which equilibrium bid functions must cross. Plotting the relative expected pay-offs of bidders is a quick, informative way to decide whether the approximate solutions are consistent with theory. While some researchers have argued low-order polynomials provide sufficiently accurate approximations, using examples we illustrate that polynomials must be of high degree to obtain solutions that are even qualitatively correct. We simulate auctions from the approximated solutions and find that low-degree polynomial approximations are poor and can lead to incorrect policy recommendations concerning auction design, suggesting researchers need to take care to obtain quality solutions.

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JEL classification: C20, D44, L1.

1. Introduction

While first-price auctions have been used extensively, perhaps for centuries, their properties are not yet fully understood by economists. In the most studied model of equilibrium behavior at first-price auctions, the private valuations of potential buyers of the object for sale are assumed to be independent and identically-distributed draws from a common distribution—the symmetric independent private-values paradigm (IPVP). In many applications, however, it is natural to assume that bidders are *ex ante* heterogeneous, their valuation draws coming from different distributions, perhaps independently. Under such an assumption, however, formal analysis becomes complicated, even within the otherwise simple IPVP. The main problem is technical in nature:

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a Lipschitz condition at the lower bound is not satisfied by the system of differential equations that characterizes the equilibrium bid functions. This makes proofs of uniqueness difficult for theorists and precludes application of standard numerical methods for solving systems of differential equations. Typically it is impossible to derive equilibrium bid functions explicitly in asymmetric first-price auction models.[1] Therefore, beginning with Marshall et al. [25], several researchers have employed different techniques from the numerical analysis literature in an effort to investigate various properties of these auctions.

In parallel to these efforts, contributors to another literature have pursued theoretical analyses of the problem, basically investigating qualitative features of equilibrium bidder interaction. For example, Lebrun [21] as well as Maskin and Riley [26] have established that a weak bidder bids more aggressively than a strong one: weakness leads to aggression, as Krishna [20] put it. Typically, a bidder is referred to as *strong* if the cumulative distribution function from which his valuation is drawn dominates that of the weak bidder in a particular sense, *first-order stochastic dominance*—the cumulative distribution function of the strong bidder is everywhere to the right of that for the weak bidder. Recently, Kirkegaard [19] has derived results under much weaker assumptions concerning the primitives of the economic environment. For example, he has shown that when first-order stochastic dominance does not hold, so the cumulative distribution functions of bidders cross, the equilibrium bid functions must cross as well. Indeed, under certain conditions, Kirkegaard has determined the exact number of times equilibrium bid functions will cross.

Although the sharp predictions of the model with strong and weak bidders make it an attractive one, little reason exists to suggest that the model is necessarily an accurate description of real-world bidder asymmetries. For example, in an empirical analysis of winning-bid data from sequential, oral, ascending-price auctions of fish in Denmark, Brendstrup and Paarsch [3] estimated the cumulative distribution functions of valuations for major and minor bidders. They found that the two unconditional cumulative distribution functions crossed twice, while the cumulative distribution functions, conditional on being above the observed reserve price, crossed once. The analysis of Kirkegaard [19] allows one to make specific predictions concerning equilibrium behavior at first-price auctions in this case.

We demonstrate how the predictions of the theoretical literature can be useful in assessing the performance of numerical approximations to the equilibrium bid functions when bidders are heterogeneous. At the most general level, the number of times the cumulative distribution functions cross and certain other features of the approximate equilibrium-bid functions (however derived) can be compared to those theoretical predictions. Building on Marshall et al. [25], a number of researchers have proposed various strategies for approximating the equilibrium-bid functions. Specifically, Li and Riley [23] as well as Gayle and Richard [11] extended the (reverse) shooting method of Marshall et al. [25], while Fibich and Gavish [6] proved that shooting methods are inherently unstable and, instead, proposed a fixed-point iteration approach. Bajari [2] as well as Hubbard and Paarsch [14] proposed modeling the inverse-bid functions as polynomials, the coefficients of which are chosen to solve approximately the system of differential equations that characterize equilibrium behavior. These approaches are discussed extensively and compared by Hubbard and Paarsch [15].

Our analysis and the informative measures that the theoretical literature suggests hold, regardless of how the bidding strategies are approximated; however, in our work, we chose to in-

[1] For examples in which bid functions can be characterized, see Vickrey [32], Griesmer et al. [13], Plum [28], Cheng [5] as well as Kaplan and Zamir [18].

vestigate further the polynomial approximation approach as this allows us to illustrate the value theoretical insights provide in the cleanest way. Gayle and Richard [11] have provided the following evaluation of the polynomial approach as suggested and implemented by Bajari [2]:

> [it] will often produce accurate numerical approximations [...]. It is found to be fast, at least with good starting values. It remains, however, an approximation whose accuracy needs to be carefully assessed.

We agree with this initial assessment and can use this critique to illustrate why it makes for a good method to consider in our work. First, the polynomial approach avoids the inherent instability of backwards shooting algorithms as shown by Fibich and Gavish [6]. Second, it can be used in a wide variety of settings for which standard conditions on the bid functions do not hold; see, for example, the case of bid preferences studied by Hubbard and Paarsch [14]. Third, it is faster than shooting algorithms which is important in at least two settings: structural econometric estimation of auction models and simulation of dynamic auction games. Empirical researchers may need to find the equilibrium within the inner loop of an estimation routine that, in turn, involves repeated simulation; see, for example, the Bayesian econometric work in Bajari's doctoral dissertation, [1], as well as Paarsch and Hong [27] for a book-length treatment of the structural econometric analysis of auction data. Likewise, if researchers need to simulate dynamic games, which require computing the equilibrium inverse-bid functions in each period, then speed is crucial because this may require solving for the equilibrium inverse-bid functions thousands (perhaps millions) of times. For example, Saini [29] solved for a Markov Perfect Bayesian Equilibrium in a dynamic, infinite-horizon, procurement auction in which asymmetries obtain endogenously due to capacity constraints and utilization. Likewise, Saini [30] admitted entry and exit and allowed firms to invest in capacity in a dynamic oligopoly model in which firms compete at auction in each period to investigate the evolution of market structure as well as optimality and efficiency of first- and second-price auctions in dynamic settings. To do this, he needed to solve for the equilibrium inverse-bid functions for each firm, at each state (which was determined by each firm's capacity), at each iteration, when computing the Markov Perfect Bayesian Equilibrium.

Given the generality that approximation of functions by polynomials offers and its speed advantage, it is an attractive method in many practical settings. However, by adopting this approach we can also provide some insight into the warning Gayle and Richard [11] provided in their assessment quoted above concerning the accuracy of the polynomial approximation approach. After demonstrating how theoretical results can be used to gauge the accuracy of an approximation, we simulate auctions from the solutions we obtained using polynomials of various degrees. In doing so, we illustrate the importance of obtaining a good numerical solution to asymmetric first-price auction models: poor approximations can lead to policy recommendations that are completely wrong, even concerning which type of auction will yield the most expected revenue to the seller. Researchers have focused largely on environments in which one bidder's valuation distribution first-order stochastically dominates that of the other bidder. We provide insight into the revenue ranking, incidence of inefficient allocations, and preferred pricing rules in examples where the distributions of valuations cross, something we hope will motivate further theoretical research.

Another disadvantage of the polynomial approach is that success of the method often hinges on having a good initial guess. We cast the problem within the Mathematical Programming with Equilibrium Constraints (MPEC) approach advocated by Su and Judd [31] which we show allows us to obtain an initial guess that already satisfies key features of the true solution at essentially no cost.[2] In doing so, we discuss how our problem is related to collocation methods.

The primary shortcoming of all research concerning the solution of asymmetric first-price auctions is that no one has proven that any of the computational approaches ensure convergence to the true, unknown solution. While we have identified some qualitative predictions that equilibrium inverse-bid functions must satisfy, this is no substitute for a convergence result. Regardless, our predictions at least provide some minimal standards that should be met for a problem in which no such checks exist. Our predictions will allow both theoretical and empirical researchers to perform simple "tests" that will ensure the validity of the approximate equilibrium inverse-bid functions they have calculated.

Our paper is organized as follows: in the next section, we outline the polynomial method and introduce some modifications motivated by an MPEC approach, while in section 3, we describe how the theoretical results can be used to evaluate the validity of numerical approximations. We provide examples to illustrate the main points. In section 4, we provide some simulation results based on the solved examples of section 3. Finally, we summarize and conclude our research in section 5. To reduce clutter, we present the proof of a proposition in an appendix at the end of the paper.

2. Theoretical Model and Numerical Methods

Consider a set $\mathcal{N} = \{1, 2, \ldots, N\}$ containing N risk-neutral bidders who are vying to win an object sold at a first-price auction. A particular bidder, indexed by n, is assumed to draw a valuation V_n independently from a continuously differentiable cumulative distribution function F_n that has a finite and strictly positive probability density function f_n on the common support $[\underline{v}, \overline{v}]$ where $\overline{v} > \underline{v} \geq 0$, $n = 1, 2, \ldots, N$. Here, the realized valuation draw v_n represents bidder n's monetary value of the object for sale.

2.1. Characterizing Equilibrium Bid Functions

Let $\sigma_n(v)$ denote bidder n's equilibrium bid function. Thus, when bidder n has value v_n, he tenders s_n which equals $\sigma_n(v_n)$. Denote by $\varphi_n(s)$ bidder n's equilibrium inverse-bid function, so $\varphi_n(\cdot)$ is $\sigma_n^{-1}(\cdot)$. Under relatively weak assumptions, Lebrun [22] has demonstrated that there exists a unique equilibrium in strictly increasing strategies.

When bidder n has value v_n, his decision problem is to choose a bid s to maximize expected profit, or

$$\max_{<s>} (v_n - s) \prod_{m \neq n} F_m [\varphi_m(s)] .$$

In equilibrium, bidder n tenders s if his value v_n equals $\varphi_n(s)$. Substituting into the first-order condition and re-arranging yields the following:

$$1 - \sum_{m \neq n} \frac{[\varphi_n(s) - s] \, f_m [\varphi_m(s)] \, \varphi_m'(s)}{F_m [\varphi_m(s)]} = 0 \quad n = 1, 2, \ldots, N. \tag{1}$$

Because the supports of the distributions of valuations are the same for all bidders, standard theoretical arguments can be used to show that all bidders submit bids in the same interval $[\underline{v}, \overline{s}]$ where $\overline{s} \in (\underline{v}, \overline{v})$. To wit, the following conditions must be satisfied:

[2] For more on the MPEC approach, see Luo et al. [24].

1a. $\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$;

1b. $\varphi_n(\overline{s}) = \overline{v}$ for all $n = 1, 2, \ldots, N$.

Conditions 1a and 1b mean that $\sigma_n(\underline{v})$ equals $\underline{v}$ and $\sigma_n(\overline{v})$ equals $\overline{s}$, the highest equilibrium bid, for all $n = 1, 2, \ldots, N$. Thus, the equilibrium inverse-bid functions are the solution to a system of differential equations, characterized by (1), with the boundary conditions mentioned above. Although it is typically impossible to solve this system of differential equations in closed-form, some properties can be deduced by studying the system at the endpoints, as s approaches $\underline{v}$ or $\overline{s}$. For example, Fibich et al. [7] proved the following properties hold:

2a. $\sum_{m \neq n} (\overline{v} - \overline{s}) f_m(\overline{v}) \varphi'_m(\overline{s}) = 1$ for all $n = 1, 2, \ldots, N$.

2b. If $f_n(\underline{v}) \in \mathbb{R}_{++}$ and $\varphi_n(s)$ is differentiable at $s = \underline{v}$ for all $n = 1, 2, \ldots, N$, then $\varphi'_n(\underline{v}) = [N/(N-1)]$.

While Lebrun [22] proved that $\varphi_n(s)$ is differentiable on $(\underline{v}, \overline{s}]$, it is still unknown whether this property holds at $\underline{v}$, as assumed in Property 2b; see Lebrun [22], footnote 8, for a discussion of this point which we will return to later. The assumption that $f_n(\underline{v}) \in \mathbb{R}_{++}$ is also important since Property 2b can fail otherwise; see, for example, Cheng [5] who considered a model in which two bidders draw valuations from different power distributions with $\underline{v}$ equal zero (but where bidders' supports have different upper end-points). In his case, the two bidders use different linear bidding strategies and thus cannot satisfy the tangency condition of Property 2b. We, however, have assumed the densities are bounded away from zero everywhere.

2.2. *Approximating Bid Functions by Polynomials*

Since it is typically impossible to solve the system of differential equations characterized by (1) in closed-form, Bajari [2] proposed approximating $\varphi_n(s)$ by an ordinary polynomial. That is, he represented $\varphi_n(s)$ by

$$\hat{\varphi}_n(s) = \overline{s} + \sum_{k=0}^{K} \alpha_{n,k} (s - \overline{s})^k, \ s \in [\underline{v}, \overline{s}], \ n = 1, 2, \ldots, N. \tag{2}$$

and recast the problem as one in which $\overline{s}$ as well as the $\alpha_{n,k}$s must be estimated for all $n = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, K \geq 3$. To accomplish this, Bajari proposed selecting a large number T of grid points uniformly from the interval $[\underline{v}, \overline{s}]$. Under the functional form in (2), the left-hand side of (1) can then be calculated at any grid point. Bajari proposed choosing the parameters that minimize

$$\begin{aligned} H(\overline{s}, \boldsymbol{\alpha}) = & \sum_{n=1}^{N} \sum_{t=1}^{T} [\text{left-hand side of (1) for bidder } n \text{ at grid point } t]^2 + \\ & \sum_{n=1}^{N} [\underline{v} - \hat{\varphi}_n(\underline{v})]^2 + \sum_{n=1}^{N} [\overline{v} - \hat{\varphi}_n(\overline{s})]^2 \end{aligned} \tag{3}$$

where $\boldsymbol{\alpha}$ denotes a vector that collects the $N \times (K+1)$ coefficients of the polynomials. If all the first-order conditions as well as the boundary conditions are satisfied, then $H(\overline{s}, \boldsymbol{\alpha})$ will equal zero.

Note that, under Bajari's approach, Conditions 1a and 1b (viz., that $\varphi_n(\underline{v})$ equal $\underline{v}$ and $\varphi_n(\overline{s})$ equal $\overline{v}$) are not imposed, although he chose to weight these terms by T when it came to implementation. Hubbard and Paarsch [14] modified Bajari's algorithm in three ways in their application: first, instead of regular polynomials, they employed Chebyshev polynomials, which are orthogonal polynomials and, thus, more stable numerically;[3] second, they cast the problem within the MPEC approach advocated by Su and Judd [31] to discipline the approximated solution. Specifically, the Chebyshev coefficients in the approximations are chosen so that the first-order conditions defining the equilibrium inverse-bid functions are approximately satisfied, subject to constraints that the boundary conditions defining the equilibrium strategies are satisfied. Finally, they imposed monotonicity on candidate solutions.

Under their assumptions, the approximate equilibrium inverse-bid function for bidder n can be expressed as

$$\hat{\varphi}_n(s; \boldsymbol{\alpha}_n, \overline{s}) = \sum_{k=0}^{K} \alpha_{n,k} \mathbb{T}_k[x(s; \overline{s})] \quad n = 1, 2, \ldots, N \tag{4}$$

where $x(\cdot)$ lies in the interval $[-1, 1]$ and where, for completeness, we have explicitly defined it as a transformation of the bid s under consideration. Here, $\mathbb{T}_k(\cdot)$ denotes the k^{th} Chebyshev polynomial of the first kind and the vector $\boldsymbol{\alpha}_n$ collects the polynomial coefficients for bidder n. Thus,

$$\begin{aligned} \mathbb{T}_0(x) &= 1 \\ \mathbb{T}_1(x) &= x \\ \mathbb{T}_{k+1}(x) &= 2x\mathbb{T}_k(x) - \mathbb{T}_{k-1}(x) \quad k = 1, 2, \ldots, K-1. \end{aligned}$$

Hubbard and Paarsch used the Chebyshev nodes on the interval $[-1, 1]$, which are

$$x_t = \cos\left(\frac{2t-1}{2T}\pi\right), \quad t = 1, \ldots, T.$$

The points $\{s_t\}_{t=1}^{T}$ are found using the following transformation:

$$x_t \equiv x(s_t; \overline{s}) = \frac{2s_t - \underline{v} - \overline{s}}{\overline{s} - \underline{v}}$$

or

$$s_t = \frac{\overline{s} + \underline{v} + (\overline{s} - \underline{v})x_t}{2}$$

which maps the s_ts from the Chebyshev nodes. The Chebyshev nodes have the property of minimizing the maximum interpolation error when approximating a function. As such, they are often considered the best choice for the approximating grid.

In summary, Hubbard and Paarsch approximated the equilibrium inverse-bid functions by Chebyshev polynomials and chose the parameters $\boldsymbol{\alpha}$ and $\overline{s}$ to minimize

$$H_I(\overline{s}, \boldsymbol{\alpha}) = \sum_{n=1}^{N} \sum_{t=1}^{T} [\text{left-hand side of (1) for bidder } n \text{ at grid point } t]^2 \tag{5}$$

[3] Judd [17] has advocated using Chebyshev polynomials, which are orthogonal with respect to the L_2 norm, the basis of the nonlinear least squares objective.

subject to constraints that Conditions 1a and 1b are satisfied and that the approximated solutions are all monotonic for each bidder.

Imposing Conditions 1a and 1b, which concern the boundaries, is important because these theoretical constraints are not captured by the first-order conditions (1). Note, too, that by representing the inverse-bid functions as polynomials, the assumption concerning differentiability of the inverse-bid functions in Property 2b which was made implicitly by Fibich et al. [7] will be satisfied. In going forward, we maintain the spirit of the approach taken by Hubbard and Paarsch, but incorporate Properties 2a and 2b as well. In doing so, for each bidder, four constraints are imposed on the equilibrium inverse-bid functions, which are approximated by Chebyshev polynomials of degree K. The unknown parameters are chosen to minimize (5) subject to the constraints described by Conditions 1a and 1b as well as Properties 2a and 2b and monotonicity conditions. We employ this approach in deriving the observations that follow.

The polynomial approach involves finding $N(K + 1) + 1$ parameters (the parameters in $\boldsymbol{\alpha}$ plus $\overline{s}$) that solve a constrained nonlinear minimization problem. Not surprisingly, it is important to have a good initial guess for this to method to have any success. As mentioned above, four conditions apply to each bidder. Thus, there are $4N$ conditions or constraints in total and TN points that enter the objective function. For the number of conditions (boundary and first-order together) to equal the number of unknowns

$$4N + TN = N(K + 1) + 1 \tag{6}$$

or

$$(T + 4) = (K + 1) + \frac{1}{N}. \tag{7}$$

Since at auctions, N weakly exceeds two, and T and K are integers, this equality cannot hold for any (T, K) choice.

We believe our approach is related to the spectral methods used to solve partial differential equations; for more on these methods, see Gottlieb and Orszag [12]. Related to the spectral family of methods is a family referred to as *collocation* methods. Under collocation methods, it is assumed that the solution can be represented by a candidate approximation, typically a polynomial; a solution is selected that solves the system exactly at a set of points over the interval of interest; such points are referred to as the *collocation points*. Because equality (7) cannot hold, collocation is infeasible in this case, but the MPEC-based approach can be thought of as an hybrid between collocation and least-squares as some constraints are explicitly imposed, leading to a constrained nonlinear optimization problem. If we ignore the T interior points, then, comparing the $N(K + 1) + 1$ parameters with the $4N$ conditions, note that, if K equals three and all the conditions are satisfied, then only one degree of freedom remains. Thus, a researcher can solve a simple degree-three approximation problem to get an initial guess at very little cost—the solution will satisfy Conditions 1a and 1b as well as Properties 2a and 2b and the free parameter $\overline{s}$ is chosen to minimize the nonlinear objective, providing the researcher with an initial guess for an approximation with $K > 3$.

2.3. Properties of Approximations

In order to compare the equilibrium inverse-bid functions of bidders m and n, define

$$D_{n,m}(s) = \hat{\varphi}_n(s) - \hat{\varphi}_m(s)$$

and

$$D^r_{n,m}(s) = \frac{\hat{\varphi}_n(s) - \hat{\varphi}_m(s)}{(s - \underline{v})^2}.$$

Note that $D^r_{n,m}$ is a measure of the difference between equilibrium inverse-bid functions in relation to the size of the bids.

Example 1: Assume that N is two, that $[\underline{v}, \overline{v}]$ are $[0, 1]$ and that $f_1(\overline{v})$ exceeds $f_2(\overline{v})$. The ordering of the probability density functions at $\overline{v}$ implies that bidder 1 is "strong at the top" when compared to bidder 2. That is, F_1 dominates F_2 near $\overline{v}$ in the sense of both first-order stochastic dominance and reversed hazard-rate dominance. Assume that a third-order polynomial is used to approximate each equilibrium inverse-bid function, so K is three. Solving the $4N$ or eight conditions for the eight coefficients in $\boldsymbol{\alpha}$, while allowing $\overline{s}$ to vary, yields the following conclusion:

$$D_{1,2}(s) = \hat{\varphi}_1(s) - \hat{\varphi}_2(s) = \left[\frac{f_1(1) - f_2(1)}{f_1(1)f_2(1)}\right]\left[\frac{s^2(\overline{s} - s)}{\overline{s}^2(1 - \overline{s})}\right].$$

Recall $\overline{v}$ exceeds $\overline{s}$, so regardless of the exact value of $\overline{s}$, $[\hat{\varphi}_1(s) - \hat{\varphi}_2(s)]$ is proportional to $s^2(\overline{s} - s)$ which, in turn, is strictly positive for all $s \in (0, \overline{s})$. Thus, $\hat{\varphi}_1(s)$ must exceed $\hat{\varphi}_2(s)$ for all $s \in (0, \overline{s})$, or $\sigma_1(v)$ is strictly less than $\sigma_2(v)$ for all $v \in (0, 1)$. This conclusion rests *only* on the assumption that $f_1(1)$ exceeds $f_2(1)$. In this example,

$$D^r_{1,2}(s) = \left[\frac{f_1(1) - f_2(1)}{f_1(1)f_2(1)}\right]\left[\frac{\overline{s} - s}{\overline{s}^2(1 - \overline{s})}\right].$$

Kirkegaard [19] showed that if $F_1(v)$ crosses $F_2(v)$, then theory predicts that the equilibrium bid functions must cross as well. Yet, when K is three, such a crossing cannot obtain for these approximate equilibrium-bid functions. □

When K is four or more, then, for a given $\overline{s}$, there are more parameters than we have conditions, so it is impossible to get "closed-form" expressions for these absolute and relative differences, respectively: too many degrees of freedom exist. While the analytic differences cannot be derived explicitly for higher order polynomials, Example 1 illustrates a general phenomenon, described in the following Proposition which describes some qualitative features of $D_{n,m}(s)$ for higher K.

Proposition 1. *Assume (i) $f_n(\underline{v}) \in \mathbb{R}_{++}$ and (ii) $\varphi_n(s)$ is a polynomial of order K, $K \geq 3$, with real coefficients that satisfy Conditions 1a and 1b as well as Properties 2a and 2b, for all $n = 1, 2, \ldots, N$. If $f_n(\overline{v}) \neq f_m(\overline{v})$, then $\varphi_n(s)$ and $\varphi_m(s)$ cross at most $(K - 3)$ times on $(\underline{v}, \overline{s})$, $m, n = 1, 2, \ldots, N$.*

Proof. See the Appendix. ■

Proposition 1 implies that a limited number of ways exist in which equilibrium inverse-bid functions approximated by polynomials can interact under Conditions 1a and 1b as well as Properties 2a and 2b. Proposition 2, below, contains a complementary result.

Proposition 2. *Under Condition 1a and Property 2b, $D^r_{n,m}$ is a polynomial of order $(K - 2)$.*[4]

[4] If only Condition 1a is imposed, then the function $[D_{n,m}(s)/(s - \underline{v})]$ is a polynomial of order $(K - 1)$.

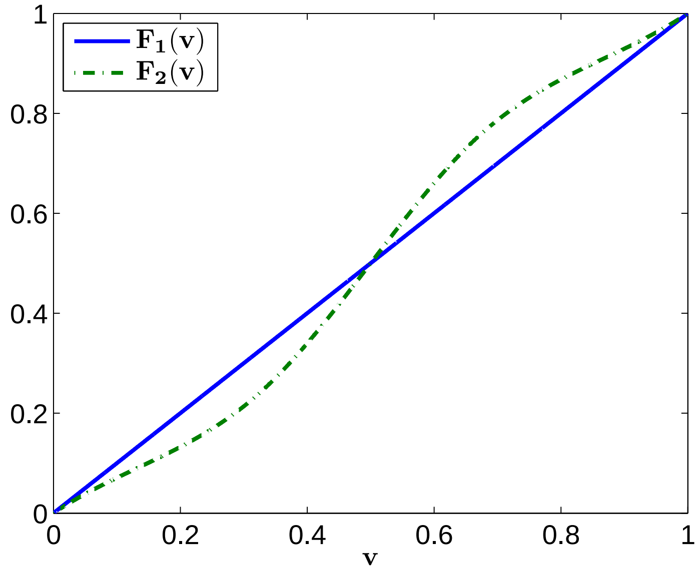


Figure 1: $F_1(v)$ and $F_2(v)$ for Example 2

Proof. From the proof of Proposition 1, Condition 1a and Property 2b imply that $D_{n,m}$ can be written

$$D_{n,m}(s) = \sum_{k=2}^{K} (\gamma_{n,k} - \gamma_{m,k})(s - \underline{v})^k,$$

which implies Proposition 2. ■

When K is three, then Proposition 2 implies that $D^r_{n,m}$ is linear. If K is four, then $D^r_{n,m}$ is a quadratic. Adding Condition 1b implies that $D^r_{n,m}(\overline{s})$ equals zero, while adding Property 2a determines $\mathrm{d}D^r_{n,m}(\overline{s})/\mathrm{d}s$. Thus, when K is small, if Conditions 1a and 1b as well as Properties 2a and 2b are imposed, then the relationships among the approximate equilibrium inverse-bid functions are almost predetermined. Above, in Example 1, $D^r_{n,m}$ is completely determined by $\overline{s}$ when K is three.

We now introduce a specific pair of cumulative distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ which are plotted in figure 1, and pursue a more quantitative analysis than in Example 1.

Example 2: Assume that N is two and that $[\underline{v}, \overline{v}]$ are $[0, 1]$. Here, $F_1(\cdot)$ is a standard uniform distribution, while $F_2(\cdot)$ is a cumulative distribution fuction constructed using piecewise polynomials so that the function is monotonic, has $F_2(0)$ equal zero, and has $F_2(1)$ equal one. Furthermore, $\mathrm{d}^2F_2(v)/\mathrm{d}v^2$ is continuous, which means that $\mathrm{d}f_2(v)/\mathrm{d}v$ exists. This latter property is unnecessary under this approach; we require only that the probability density functions be continuous, strictly positive, and finite everywhere. Finally, F_2 has been constructed so that F_1 is a mean-preserving spread over F_2; the common mean is 0.5. In figure 2, we plot the absolute differences in the approximate equilibrium inverse-bid functions $\hat{D}_{1,2}(s)$ when the order of the approximating

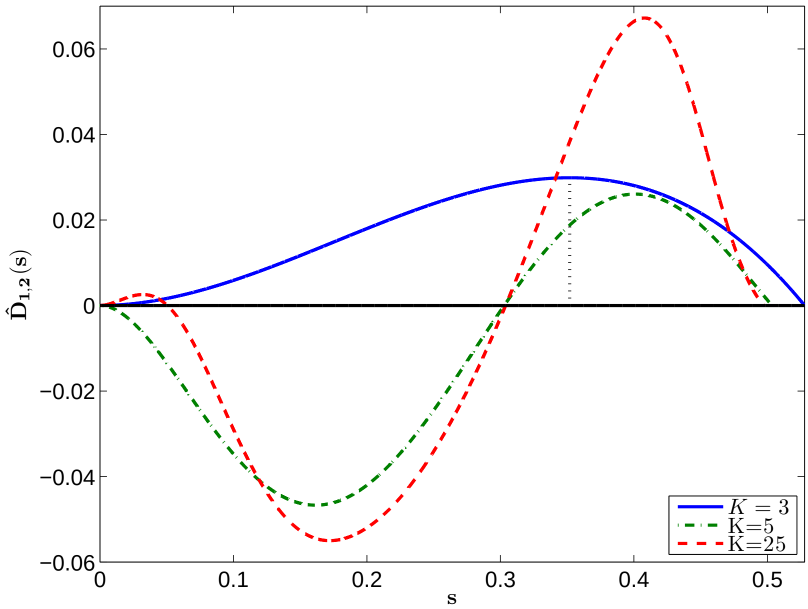


Figure 2: $\hat{D}_{1,2}(s)$ for $K = 3, 5, 25$ for Example 2

polynomials K increases from three to five to twenty-five.[5]

We denote by $\bar{s}_K$ the approximated high bid in the case when an order K polynomial is used to approximate the equilibrium inverse-bid functions. When K equals three, the absolute difference in the approximate equilibrium inverse-bid functions and the theoretical absolute difference coincide exactly. We do not plot the theoretical difference, but the two are indistinguishable. Note that, when K equals three, $D_{1,2}(s)$ is maximized at $\frac{2}{3}\bar{s}$, which we identify in figure 2. Note, too, that the difference does not cross zero in the interior of $(0, \bar{s}_3)$, which implies that the approximate equilibrium-bid functions do not cross. As the order of the approximating polynomials increases to five, a crossing obtains: the approximate equilibrium-bid functions cross in the in-

[5]We used the modelling language AMPL (which is described in detail in Fourer et al. [9]) to calculate the coefficients of our approximating polynomials. For most of our work, we used the solver `SNOPT 7.2-8`, but we also checked to see whether the results were sensitive to the solver used; for example, we used the solver `KNITRO 6.0.0` as well. While there are always some differences in solutions across solvers, we did not find these to be important. We will provide the code at

`http://www.myweb.ttu.edu/timhubba/code/hkpcode.zip`

along with a user's guide that will also detail the exact specification of the distributions used in our examples and describe how they were constructed. To get an initial guess for the high-bid parameter $\bar{s}_K$, we used the third-order approximation first because, in that case, $\bar{s}_3$ is the only free parameter. For the parameters of subsequent, higher-order polynomials, we used the previous convergent estimates as starting values, and set the highest-order coefficient to zero; this is reasonable because the Chebyshev polynomials are orthogonal polynomials. The approximations for a given degree K take less than one second to solve and the objective (5) is decreasing in K. In practice, a researcher can select a K which satisfies the theoretical predictions we describe in the next section and meets some error tolerance criterion—for example, the objective (6) for our order twenty-five approximation is below 10^{-6}. While all of our work was done in the L_2 norm, one could carry out the work in the L_1 norm as well. This is suggested by Hubbard and Paarsch [15] and considered further by Hubbard et al. [16].

terior of $(0, \overline{s}_5)$ which is shown by the absolute difference crossing the zero line once. Finally, in the twenty-fifth order case, the absolute difference in the approximate equilibrium inverse-bid functions equals zero twice in the interior of $(0, \overline{s}_{25})$, meaning that the approximate equilibrium-bid functions cross one another twice. Note, too, that the value of the high bid $\overline{s}_K$ changes as the order of the approximations change. For low values of K, $\overline{s}_K$ exceeds the common mean of the two distributions (which is 0.5). However, the opposite holds for large values of K. Using a different approximation technique, Fibich and Gavish [6] provided an example in which it is also the case that $\overline{s}$ falls below the common mean. This observation is also consistent with Gavious and Minchuk [10] who noted that if both bidders' distributions are "almost" uniform, small asymmetries between them will tend to suppress $\overline{s}$. In comparison, if the two bidders are symmetric, then it is well known that $\overline{s}$ will equal the mean of the distribution. □

Note that the difference $D_{1,2}(s)$, which we were able to characterize explicitly in the case of K equal three, is most definitely *not* the true difference between the equilibrium bid functions. It is derived from the constraints and requires knowledge only of $f_1(\overline{v})$ and $f_2(\overline{v})$, so it cannot possibly describe the true differences in the equilibrium bid functions. This is precisely our point: with too few degrees of freedom, the approximations are essentially predetermined and the difference in the approximations when K equals three will look like the analytic expression we derived for this example. As we discussed, we employ this result to inform our initial guess.

While our results thus far illustrate why polynomials of too low an order will result in poor approximations, we have yet to show how to evaluate whether an approximation is sufficiently "good"—i.e., consistent with theory. In the next section, we discuss the theoretical predictions that can be made when the cumulative distribution functions of valuations cross.[6] Armed with such predictions, it is possible to assess better the accuracy of approximate equilibrium-bid functions.

3. Theoretical Predictions and a Test

To begin, let

$$P_{n,m}(v) = \frac{F_m(v)}{F_n(v)}, \; v \in (\underline{v}, \overline{v}]$$

measure bidder n's strength (power) relative to bidder m at a given value v. The larger is $P_{n,m}$, the stronger bidder n when compared to bidder m at that value. For example, if $P_{n,m}(v)$ exceeds one, then $F_n(v)$ is less than $F_m(v)$. Similarly, define $U_n(v)$ as bidder n's equilibrium expected pay-off (profit) at an auction if his value is v, and let

$$R_{n,m}(v) = \frac{U_n(v)}{U_m(v)}, \; v \in [\underline{v}, \overline{v}]$$

denote bidder n's equilibrium pay-off relative to bidder m's equilibrium pay-off at a given value. Note that $P_{n,m}(v)$ is exogenous, while $R_{n,m}(v)$ is endogenous.

Kirkegaard [19] demonstrated that the two ratios can be used to make predictions concerning the properties of $\sigma_n(v)$ and $\sigma_m(v)$ or, equivalently, $\varphi_n(s)$ and $\varphi_m(s)$. At v equal $\overline{v}$, the two bids

[6] If they do not cross, then one distribution dominates the other in the first-order stochastic sense, and the results described in the introduction may provide some guidance.

coincide and so too do the two ratios, or $\sigma_n(\overline{v})$ equals $\sigma_m(\overline{v})$ and $R_{n,m}(\overline{v})$ equals $P_{n,m}(\overline{v})$, which is one. In fact, comparing the two ratios at any $v \in (\underline{v}, \overline{v}]$ is equivalent to comparing the equilibrium bids at v, or

$$R_{n,m}(v) \gtreqless P_{n,m}(v) \Longleftrightarrow \sigma_n(v) \gtreqless \sigma_m(v), \text{ for } v \in (\underline{v}, \overline{v}]. \tag{8}$$

Moreover, it turns out that the motion of $R_{n,m}$, which is endogenous, is determined by how it compares to $P_{n,m}$, which is exogenous. Specifically,

$$R'_{n,m}(v) \gtreqless 0 \Longleftrightarrow R_{n,m}(v) \gtreqless P_{n,m}(v), \text{ for } v \in (\underline{v}, \overline{v}]. \tag{9}$$

Coupled with the condition that

$$R_{n,m}(\overline{v}) = P_{n,m}(\overline{v}) = 1, \tag{10}$$

(9) allows one to make a number of predictions.

In figures 3.a and 3.b, we depict (8) and (9). In this example, F_n and F_m cross twice in the interior, so $P_{n,m}$ equals one twice in the interior. The boundary condition in (10) and the relationship in (9) imply that $R_{n,m}$ and $P_{n,m}$ must cross exactly twice in the interior. Therefore, by (8), the equilibrium bid functions must cross *exactly* twice in the interior. Note, too, that if the assumptions in Property 2b are satisfied, then by l'Hôpital's rule

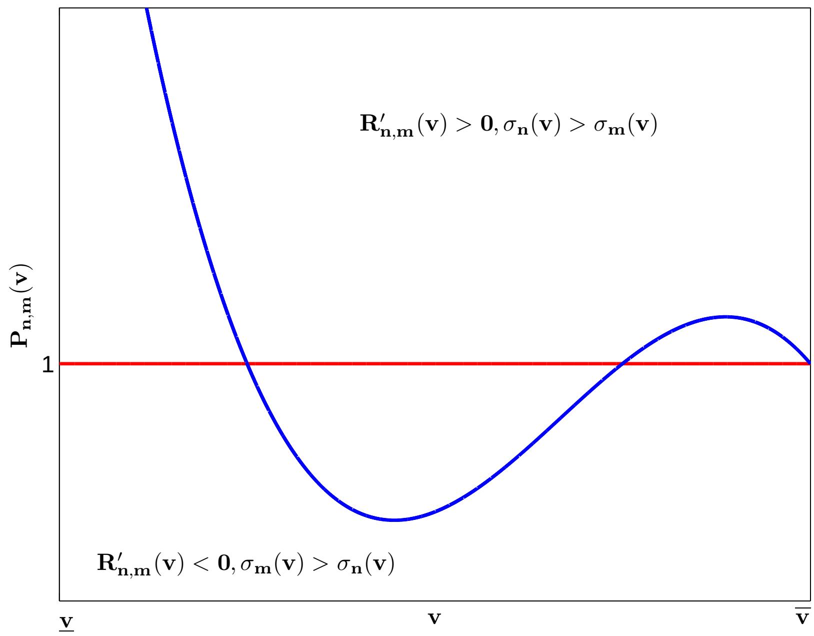
$$\lim_{v \to \underline{v}} R_{n,m}(v) = \frac{f_m(\underline{v})}{f_n(\underline{v})} = \lim_{v \to \underline{v}} P_{n,m}(v). \tag{11}$$

Thus, simply by plotting $P_{n,m}$, it is often possible to predict qualitative behavior with great accuracy. The example in figures 3.a and 3.b satisfies the *diminishing wave property*. What does this mean? Well, first, the extrema of $P_{n,m}(v)$—including $\lim_{v \to \underline{v}} P_{n,m}(v)$—alternate between being above and below one; second, the extrema that are above one get closer to one as v increases; and, third, the extrema that are below one also get closer one as v increases. In this case, the exact number of times equilibrium bid functions cross can be identified, as in figure 3.a. The number of crossings between $R_{n,m}$ and $P_{n,m}$ in the interior is identical to the number of interior stationary points of $P_{n,m}$. When the diminishing wave property is not satisfied, the number of interior stationary points, instead, provides an upper bound on the number of times equilibrium bid functions can cross in the interior.

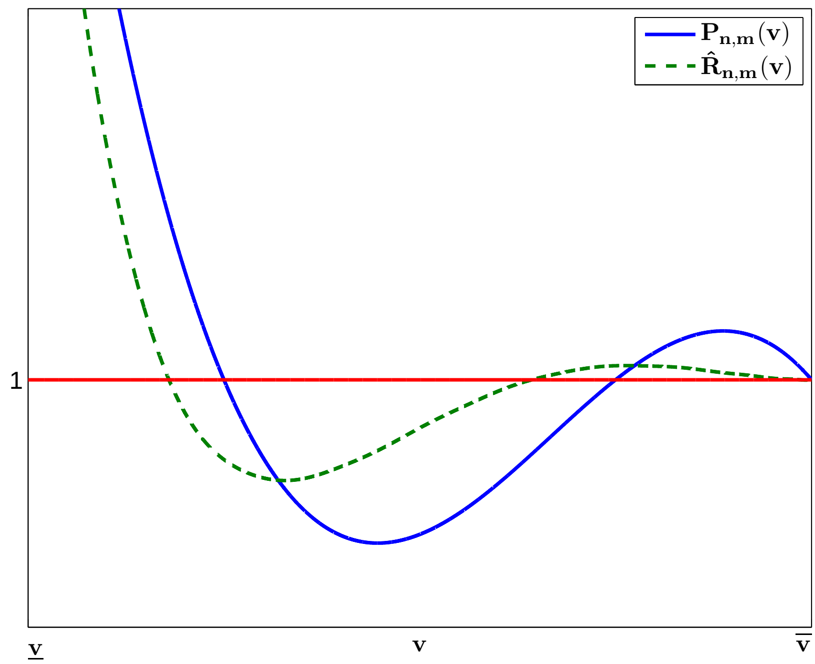
If approximate equilibrium-bid functions do not cross the theoretically-predicted number of times, then these approximations are clearly suspect.[7] However, even when the approximate equilibrium-bid functions are consistent with theoretical predictions, the theoretical relationship between $R_{n,m}$ and $P_{n,m}$ suggests a visual test of the accuracy of approximate equilibrium-bid functions. Based on the approximate equilibrium-bid functions, denoted $\hat{\sigma}_n$, the ratio of expected pay-offs can be computed. Denote the estimated ratio by $\hat{R}_{n,m}$. If $P_{n,m}$ and $\hat{R}_{n,m}$ are plotted in the same figure, then they should interact in a manner consistent with (9), as illustrated in figure 3.b. The steepness of $\hat{R}_{n,m}$ at a point of intersection with $P_{n,m}$ and the location of the intersections can be used to eliminate inaccurate solutions.

1. **Slope**: At any point where $P_{n,m}$ and $\hat{R}_{n,m}$ intersect (i.e., where $\hat{\sigma}_n$ equals $\hat{\sigma}_m$), the latter should be *flat*, have a derivative that equals zero. If $\hat{R}_{n,m}$ is steep at such a point, then this is

[7]Note that these observations hold for approximations derived using shooting methods, the iterative approach of Fibich and Gavish [6], or any other method of approximating the equilibrium in a model of an asymmetric first-price auction.



3.a: Comparing $R_{n,m}(v)$ and $P_{n,m}(v)$



3.b: A possible path for $R_{n,m}(v)$

Figure 3: Comparing $R_{n,m}(v)$ and $P_{n,m}(v)$ and a path consistent with (8)–(11).

an indication that the approximate equilibrium-bid function is inaccurate as the first-order conditions are not close to being satisfied. Note, too, that this is true any time bids coincide (for any $v > \underline{v}$, including $\overline{v}$).

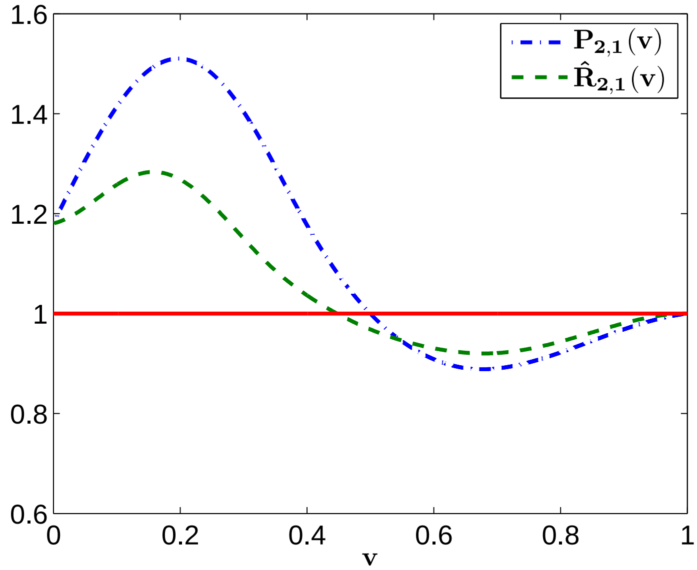
2. **Location**: The location of the intersections of $P_{n,m}$ and $\hat{R}_{n,m}$ must also be consistent with theory. In particular, $P_{n,m}$ and $R_{n,m}$ can cross *at most* once between any two peaks of $P_{n,m}$; with diminishing waves, they *must* cross between any two peaks (not counting v equals $\overline{v}$). In figure 3.b, for example, $P_{n,m}$ and $R_{n,m}$ must cross once to the left of the point where $P_{n,m}$ is minimized, and once between the two interior stationary points.

Example 2 (continued): Above, we demonstrated that the third-order polynomial approximations were poor by appealing to the absolute difference in the approximate equilibrium-bid functions. Consider now plotting the exogenous ratio $P_{2,1}$ and the approximate endogenous ratio $\hat{R}_{2,1}$ for these distributions. In figure 4.a, we depict the two ratios when polynomials of order five are used to approximate the equilibrium inverse-bid functions. First, note that this example does not satisfy the diminishing wave property as both $\lim_{v \to \underline{v}} P_{2,1}(v)$ and the first interior stationary point of $P_{2,1}$ are greater than one. Because the diminishing wave property is not satisfied, we can only use theory to bound the number of times the equilibrium bid functions should cross: if $P_{2,1}$ does not get closer to one as v increases, then the equilibrium bid functions can cross at most twice. (Remember that they have to cross at least once because the cumulative distributions cross once.) The fifth-order approximations are an improvement over the third-order approximations in the sense that the approximate equilibrium-bid functions at least cross; the relationship in (8) implies this since $\hat{R}_{2,1}$ equals $P_{2,1}$ once over $v \in (\underline{v}, \overline{v})$. However, we can use the slope property formalized above to demonstrate why this approximation is insufficient. Specifically, note that $\hat{R}_{2,1}$ is not at a stationary point when it intersects $P_{2,1}$. This observation is a corollary of the more general property that $\hat{R}_{2,1}$ should be decreasing (increasing) when it is below (above) $P_{2,1}$. It is clear that this is violated for the fifth-order approximations as $\hat{R}_{2,1}$ peaks around v equal to 0.2 in figure 4.a.

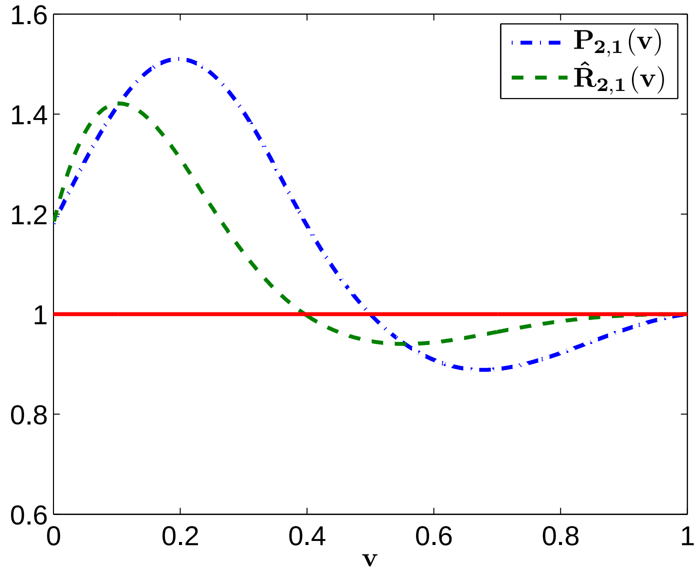
In contrast, we depict in figure 4.b the exogenous and endogenous ratios for the twenty-fifth order approximations, which are consistent with all of the theoretical properties. $\hat{R}_{2,1}$ intersects $P_{2,1}$ twice in the interior and, at each crossing, the slope of the endogenous ratio is zero. Furthermore, the locations of the crossings are appropriate: $\hat{R}_{2,1}$ intersects once between $\underline{v}$ and the first interior stationary point and once between the two interior extrema. Equation (10) specifies that both ratios take a value of one at $\overline{v}$. Thus, because $\hat{R}_{2,1}$ must be increasing when it exceeds $P_{2,1}$, as v approaches the high valuation $\overline{v}$, $\hat{R}_{2,1}$ should be bounded between $P_{2,1}$ and one, as it is in figure 4.b. In short, $\hat{R}_{2,1}$ is horizontal at $\overline{v}$. While this convergence obtains in figure 4.a as well, $\hat{R}_{2,1}$ approaches at a much steeper angle and we know from our slope property that $\hat{R}_{2,1}$ should be flat when it intersects $P_{n,m}$, as in figure 4.b. □

While Example 2 has allowed us to investigate approximations of the equilibrium inverse-bid functions when the diminishing wave property is not satisfied, theory provides precise predictions concerning the number of crossings when the diminishing wave property is satisfied. In the next example, we consider a situation in which the diminishing wave property is satisfied and the cumulative distribution functions cross twice.

Example 3: Assume that N is two and that $[\underline{v}, \overline{v}]$ are $[0, 1]$. Here, $F_1(\cdot)$ is again a standard uniform distribution, while $F_2(\cdot)$ is a cumulative distribution function (different from that used



4.a: $K = 5$



4.b: $K = 25$

Figure 4: $P_{2,1}(v)$ and $\hat{R}_{2,1}(v)$ for Example 2

in EXAMPLE 2) constructed using piecewise polynomials such that the function is monotonic, has $F_2(0)$ equal zero, and has $F_2(1)$ equal one. Furthermore, $\mathrm{d}^2 F_2(v)/\mathrm{d}v^2$ is continuous, which means that $\mathrm{d}f_2(v)/\mathrm{d}v$ exists. As mentioned above, this is not required for our approach—we require only that the probability density functions be continuous, strictly positive, and finite everywhere, which they are. We plot the cumulative distribution functions in figure 5.a. In figure 5.b, we depict $P_{2,1}$, the ratio of the cumulative distribution functions; note that the diminishing wave property is satisfied in this example.

When the equilibrium inverse-bid functions are approximated by third-order or fourth-order polynomials, the approximate equilibrium-bid functions do not cross, suggesting that serious inadequacies exist with the approximations: for these cases, $\hat{R}_{2,1}$ is everywhere below $P_{2,1}$, but not monotonic. However, the fifth-order approximations cross twice, the exact number of times predicted by theory. In figure 6.a, we depict the exogenous and endogenous ratios in the case when the equilibrium inverse-bid functions are approximated by polynomials of order five. We restrict the figure to the interval $v \in [0.2, 1]$, so the crossings can be seen clearly.[8] While the number of crossings is correct, the slope property described above is again violated because $\hat{R}_{2,1}$ is not flat when it intersects $P_{2,1}$; the location property is nearly violated as well. Note, too, that both intersections are close to lying between the interior stationary points of $P_{2,1}$. In figure 6.b, we depict the ratios for the case when the equilibrium inverse-bid functions are approximated by polynomials of order twenty-five. These approximations are consistent with all of our theoretical checks. By comparing figure 6.a with figure 6.b, we note that the location of the crossings are clearly different—the twenty-fifth order approximations cross at lower values of v relative to the fifth-order approximations. □

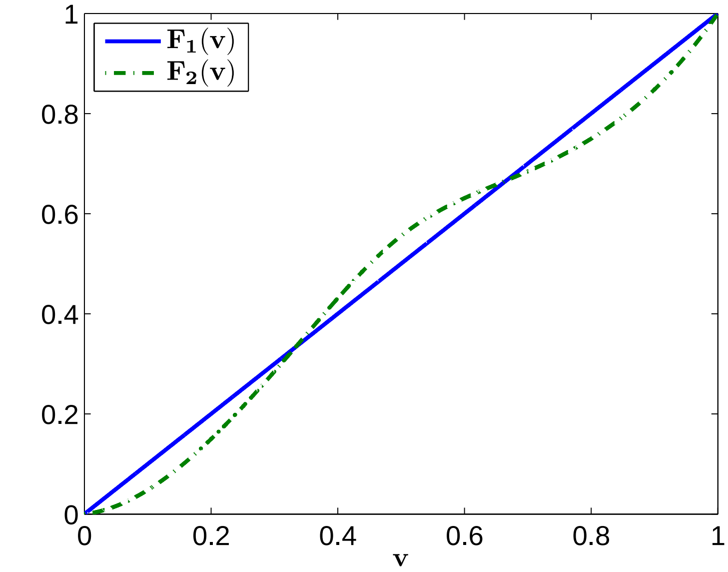
It is also interesting to note that if the diminishing wave property were satisfied, then the number of times $P_{n,m}$ and $R_{n,m}$ (or σ_m and σ_n) cross would be *independent* of either the number of rival bidders they face or which cumulative distribution functions characterize these opponents, provided their supports are the same. That is, theory tells us that only pairwise comparisons between bidders are required. Thus, if bidders were to draw valuations from three (or more) different distributions (so F_1, F_2, and F_3), then our insights could be used to compare pairs of bidders—class 1 versus class 2, class 1 versus class 3, and class 2 versus class 3. To wit, all that matters is the relationship between pairs of distribution functions.

4. Some Simulation Results

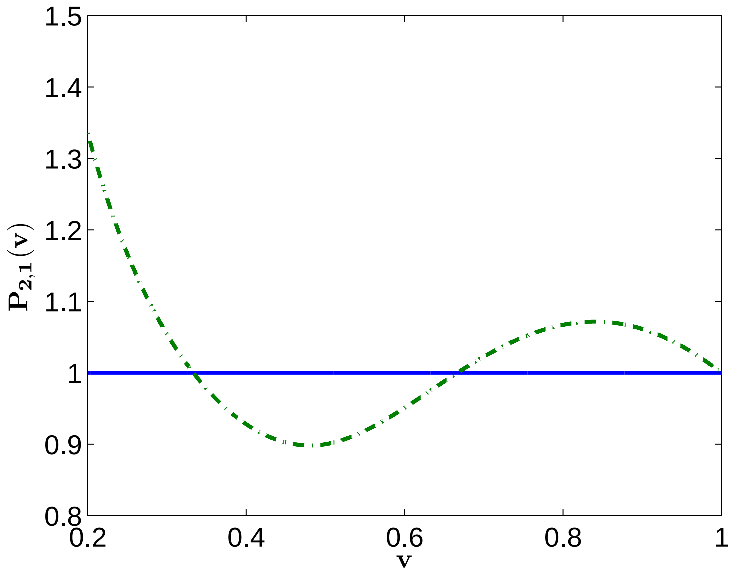
We also undertook a simulation study to investigate the quantitative importance of using good approximations. While Hubbard and Paarsch [15] considered a formal error analysis comparing the various approximation methods, we illustrate in this simulation study the importance of obtaining solutions that are consistent with theory. Given that our work is heavily based on theoretical results, we hope to give back by suggesting directions of future theoretical research.

In our simulation experiments, we approximated the equilibrium inverse-bid functions using polynomials of various degrees. We approximated each equilibrium only once for a given degree

[8]Over the interval $v \in [0, 0.2]$ both $\hat{R}_{2,1}$ and $P_{2,1}$ increase as v decreases and $\hat{R}_{2,1}$ converges to $P_{2,1}$ as $v \to \underline{v}$ which is consistent with (11). In this example, $\hat{R}_{2,1}$ converges to $P_{2,1}$ from below as the first interior stationary point obtains below one, whereas in EXAMPLE 2, $\hat{R}_{2,1}$ converged to $P_{2,1}$ from above as the first interior stationary point obtained above one.

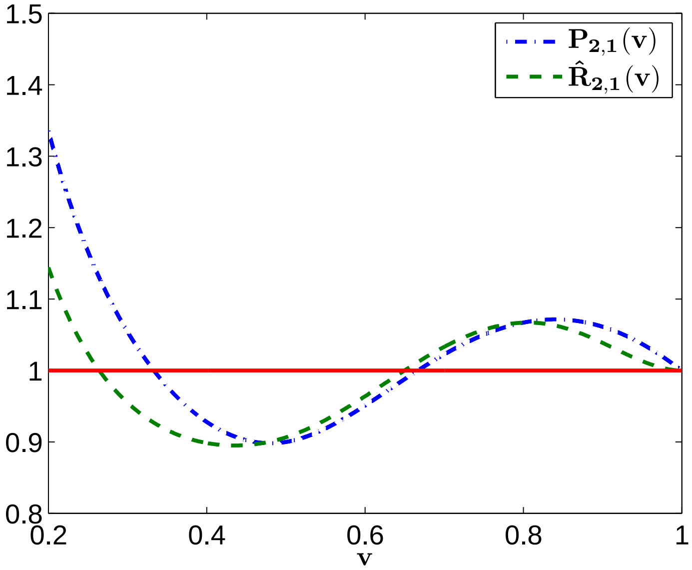


5.a: $F_1(v)$ and $F_2(v)$ for Example 3

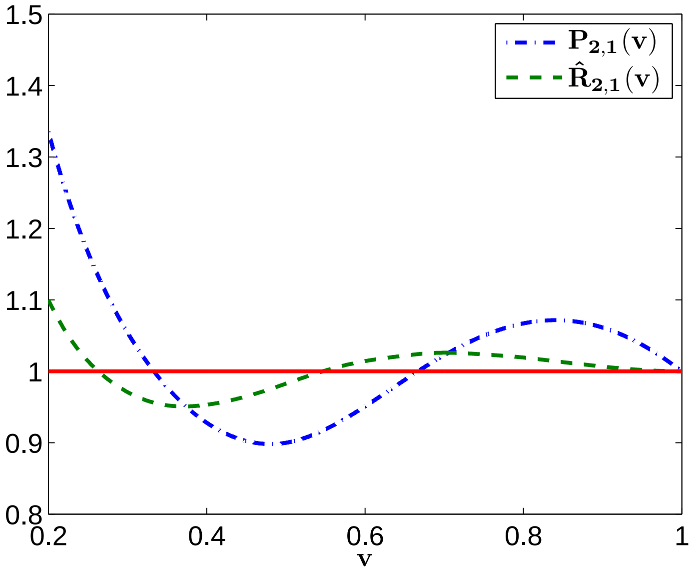


5.b: $P_{2,1}$ for Example 3

Figure 5: $F_1(v)$, $F_2(v)$, and $P_{2,1}$ for Example 3



6.a: $K = 5$



6.b: $K = 25$

Figure 6: $P_{2,1}(v)$ and $\hat{R}_{2,1}(v)$ for Example 3

Table 1: Results of Simulation Study with One Trillion Auctions

	Order of $\hat{\varphi}_n$	Expected Revenue	Proportion of Inefficiencies	Prop. Wins Player 1	Prop. Wins Player 2	$\mathbb{E}(U_1)$	$\mathbb{E}(U_2)$
Example 2	$K = 3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
	$K = 4$	0.3473	0.0223	0.5209	0.4791	0.1645	0.1434
	$K = 5$	0.3458	0.0227	0.5089	0.4911	0.1634	0.1459
	$K = 10$	0.3431	0.0337	0.5028	0.4972	0.1635	0.1481
	$K = 15$	0.3432	0.0338	0.5027	0.4973	0.1634	0.1481
	$K = 20$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 25$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 30$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	SPA	0.3445	0.0000	0.5000	0.5000	0.1555	0.1555
Example 3	$K = 3$	0.3475	0.0721	0.5491	0.4509	0.1568	0.1568
	$K = 4$	0.3452	0.0670	0.5287	0.4713	0.1584	0.1613
	$K = 5$	0.3364	0.0557	0.5041	0.4959	0.1605	0.1678
	$K = 10$	0.3334	0.0704	0.4964	0.5036	0.1632	0.1699
	$K = 15$	0.3330	0.0719	0.4947	0.5053	0.1634	0.1702
	$K = 20$	0.3328	0.0725	0.4944	0.5056	0.1635	0.1703
	$K = 25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	$K = 30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

polynomial, and used each approximation to generate information concerning one trillion auctions. Specifically, we generated pairs of independent uniform random draws which represented the cumulative distribution functions of the two players. We then inverted the respective cumulative distribution functions introduced above in Example 2 and Example 3 at these uniform draws to obtain the respective valuation draws. We conducted the auctions, first assuming a first-price rule, where we varied the order of the polynomials used to approximate the equilibrium inverse-bid functions, and then mapped the random valuations into random bids. Thus, the valuations are constant across our results and differences in the approximated inverse-bid functions generate differences in the simulated bids. For comparison, we also considered a second-price auction rule—within the IPVP each player has a dominant strategy to bid his valuation at a second-price auction.

In table 1, we present some statistics of interest which summarize the simulation study. Specifically, we computed expected revenues under the different pricing rules as well as with different orders of approximating polynomials, the number of auctions that involved inefficient allocations, the number of times each bidder won the auction, as well as a measure of the *ex ante* equilibrium expected pay-off for bidder n for a given approximation. We computed the *ex ante* equilibrium expected pay-off using the following formula:

$$\mathbb{E}[U_n^{\mathrm{FP}}(V)] = \int_{\underline{v}}^{\overline{v}} [v - \hat{\varphi}_n^{-1}(v)] F_m \left(\hat{\varphi}_m[\hat{\varphi}_n^{-1}(v)]\right) f_n(v) \, \mathrm{d}v$$

where $\hat{\varphi}_n(\cdot)$ corresponds to player n's approximated equilibrium inverse-bid function. The table shows that the approximations are in line when polynomials of sufficiently high degree are used as the measures presented converge to the same values.

We divide our discussion of the simulation results into three parts; expected revenues garnered by the seller, inefficient allocations among buyers, and the *ex ante* equilibrium expected

Table 2: Vickrey's Example with F_1 a Mean-Preserving Spread over F_2

Pricing Rule	Expected Revenue	Probability Player 1 Wins	Probability Player 2 Wins	$\mathbb{E}(U_1)$	$\mathbb{E}(U_2)$
First-Price	0.3833	0.5625	0.4375	0.1677	0.0625
Second-Price	0.3750	0.5000	0.5000	0.1250	0.1250

pay-offs of bidders.

Consider first expected revenues to the seller in Example 2. It is interesting to compare expected revenues across pricing rules. A second-price auction yields an expected revenue of 0.3445. Note, too, from the entries in table 1 that, when the polynomial is of a low degree, the first-price auction appears more profitable, on average. On the other hand, when polynomials of higher order are used, the results are exactly the opposite.[9] In summary, while expected revenues change only a bit quantitatively as K increases, in terms of expected-revenue ranking, the qualitative predictions are sensitive to changes in K—a poor approximation can lead to an improper policy suggestion concerning the preferable auction format for the seller.

Now, Vickrey [32] was able, analytically, to derive equilibrium strategies and expected revenues when F_1 is the uniform distribution and F_2 is degenerate. In this case, a first-price auction is more profitable, on average, than a second-price one, provided F_2 has all of its mass located at a point greater than 0.43. In particular, F_1 can be viewed as a mean-preserving spread over F_2 where the latter has all of its mass concentrated at 0.5. In table 2, we summarize some properties of this model (based on Vickrey's analysis, not simulations), as a counterpart to Table 1. Recall that, in our Example 2, F_1 is also uniform and a mean-preserving spread over F_2. However, in Example 2, for large K, we found that the second-price auction garners more revenue, on average, for the seller than the first-price auction. While most expected-revenue comparisons in the literature concern distributions that can be ordered according to first-order stochastic dominance, Example 2 illustrates that relaxing this assumption may prove interesting.

Fibich et al. [8] used perturbation analysis to examine auctions with "small" asymmetries and concluded that small departures from symmetry have the same first-order effect on revenue in both the first-price auction and the second-price auction. Gavious and Minchuk [10] then considered second-order effects. In their model, bidders' distributions were derived by slightly perturbing a uniform distribution. Although their examples all involve first-order stochastic dominance, the theory they developed does not require this. Consider, for example, an environment where

$$F_1(v) = v + \varepsilon v(1 - v)\left(\frac{1}{2} - v\right)$$

and

$$F_2(v) = v - \varepsilon v(1 - v)\left(\frac{1}{2} - v\right),$$

where the weakly-positive ε is "small." In this case, $F_1(v)$ is a mean-preserving spread over $F_2(v)$ whenever ε is strictly positive and bidders are symmetric if ε equals zero—in which case the Revenue Equivalence Proposition applies. Theorem 2 in Gavious and Minchuk [10], however, can be applied to demonstrate that a small increase in ε, starting at ε equal to zero, leads the

[9] Incidentally, the same pattern emerges in Example 3, where expected revenue is 0.3399 at second-price auctions.

second-price auction to become more profitable than the first-price auction. Thus, this theoretical example is consistent with the findings for Example 2 of our simulation study.

Next consider inefficiencies. In both Example 2 and Example 3, the larger is K the larger is the quantitative effect on the efficiency properties of the first-price auction. Starting from a K of four (so dismissing a K of three as pathological, based on the analysis presented in Example 1), the number of inefficiencies increases by fifty percent in Example 2 as K grows to thirty. Note, too, that the number of inefficiencies is non-monotonic in K in Example 3. Likewise, in Example 3, small K also leads to incorrect predictions regarding which bidder is more likely to win the first-price auction.

Our examples also reveal some new insights into the *ex ante* equilibrium expected pay-offs of bidders and, hence, the preferences of bidders over the two pricing rules. Denote by μ_n the mean of F_n, $n = 1, 2$, and let $m \neq n$ denote bidder n's opponent. Bidder n's *ex ante* expected pay-off at a second-price auction is

$$\mathbb{E}[U_n^{\text{SP}}(V)] = \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} (v-u) f_m(u)\, \mathrm{d}u\, f_n(v)\, \mathrm{d}v$$

By Leibniz's rule,

$$\frac{\mathrm{d}}{\mathrm{d}v} \int_{\underline{v}}^{v} (v-u) f_m(u)\, \mathrm{d}u = (v-v) f_m(v) + \int_{\underline{v}}^{v} f_m(u)\, \mathrm{d}u = F_m(v).$$

Also, using integration-by-parts, we have

$$\begin{aligned}
\mathbb{E}[U_n^{\text{SP}}(V)] &= \int_{\underline{v}}^{\overline{v}} \left[\int_{\underline{v}}^{v} (v-u) f_m(u)\, \mathrm{d}u \right] f_n(v)\, \mathrm{d}v \\
&= \left[F_n(v) \int_{\underline{v}}^{v} (v-u) f_m(u)\, \mathrm{d}u \right]_{\underline{v}}^{\overline{v}} - \int_{\underline{v}}^{\overline{v}} F_n(v) F_m(v)\, \mathrm{d}v \\
&= F_n(\overline{v}) \left[\int_{\underline{v}}^{\overline{v}} (\overline{v}-u) f_m(u)\, \mathrm{d}u \right] - \int_{\underline{v}}^{\overline{v}} F_n(v) F_m(v)\, \mathrm{d}v \\
&= \overline{v} - \mu_m - \int_{\underline{v}}^{\overline{v}} F_n(v) F_m(v)\, \mathrm{d}v.
\end{aligned}$$

Hence,

$$\mathbb{E}(U_n^{\text{SP}}) - \mathbb{E}(U_m^{\text{SP}}) = \mu_n - \mu_m. \tag{12}$$

Thus, in Example 2, at a second-price auction, in expectation, both bidders are equally well-off. From table 1, it is interesting to note that $\mathbb{E}[U_1^{\text{FP}}]$ exceeds $\mathbb{E}[U_2^{\text{FP}}]$. Example 2 can easily be perturbed, by slightly increasing μ_2, to generate a new example where $\mathbb{E}[U_1^{\text{SP}}]$ is less than $\mathbb{E}[U_2^{\text{SP}}]$, but where $\mathbb{E}[U_1^{\text{FP}}]$ still exceeds $\mathbb{E}[U_2^{\text{FP}}]$. Of course, the same argument also applies to Vickrey's example of table 2. Nevertheless, we believe we are the first to point out that there are environments in which the two standard auctions favor different bidders, in the sense that different bidders win.[10]

[10] In Maskin and Riley [26], the weak bidder prefers the first-price auction to the second-price auction, while the strong bidder has the opposite preference. However, *ex ante*, the strong bidder is better off than the weak bidder under *both* pricing rules.

Continuing with expected utility, note that, in Example 2, $\mathbb{E}(U_1^{\text{FP}})+\mathbb{E}(U_2^{\text{FP}})$ exceeds $\mathbb{E}(U_1^{\text{SP}})+\mathbb{E}(U_2^{\text{SP}})$. That is, collectively, the bidders prefer the first-price auction, while the seller prefers the second-price auction. Since the latter pricing rule is efficient, the "total pie" is larger, but this benefits the seller, not the bidders. In contrast, in Vickrey's example of table 2, bidders collectively prefer the second-price auction to the first-price auction, while the opposite holds for the seller.

A final observation pertains to the probabilities of winning for each bidder. When F_1 is the uniform distribution, bidder 2's *ex ante* probability of winning a second price auction is

$$\int_0^1 F_1(v)f_2(v)\,\mathrm{d}v = \int_0^1 v f_2(v)\,\mathrm{d}v = \mu_2.$$

In Example 2, both μ_1 and μ_2 equal one-half, so the two bidders are *ex ante* equally likely to win the auction.[11] Changing the pricing rule barely changes the probabilities of winning. However, although bidder 1's probability of winning is only about one percent greater than bidder 2's winning probability, $\mathbb{E}(U_1^{\text{FP}})$ is about ten percent greater than $\mathbb{E}(U_2^{\text{FP}})$. Thus, the *ex ante* probabilities of winning may not be a particularly good indication of the distribution of pay-offs among bidders. Indeed, it is tempting to speculate that in some environments the bidder who wins more often may be worse off, at least *ex ante*.

5. Summary and Conclusions

In a model of a first-price auction, if the draws of potential bidders are independent, but from different distributions, then a Lipschitz condition at the lower bound $\underline{v}$ is not satisfied by the system of differential equations that characterizes the equilibrium bid functions. Analyzing asymmetric first-price auctions is challenging for theorists in deriving proofs of existence and uniqueness as well as for those who use numerical methods as conditions to employ standard approaches for solving systems of differential equations do not hold. Furthermore, because analytic solutions are scarce in this setting, numerical methods are often the only way to gain any insight into the properties of these auctions. Perhaps this is why numerical analysis can suggest directions for theoretical work; for example, Cantillon [4] "rationalized" many of the results discovered by Marshall et al. [25]. Theoretical work is also useful in informing numerical analysis; for example, our work uses many of the results of Kirkegaard [19]. Certainly, this interdependence will continue as researchers push the boundaries of understanding behavior at asymmetric first-price auctions forward in both strands of the literature.

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[11] The example in table 2 has the same property. Note, too, that in that example that $\mathbb{E}(U_1^{\text{SP}})$ equals $\mathbb{E}(U_2^{\text{SP}})$, which is consistent with equation (12).

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A. Appendix

Proof of Proposition 1. It is convenient to rewrite (2) as

$$\hat{\varphi}_n(s) = \sum_{k=0}^{K} \gamma_{n,k}(s - \underline{v})^k, \; s \in [\underline{v}, \overline{s}], \tag{13}$$

which can be obtained by expanding the terms under the summation in (2), collecting terms with like exponents, and defining the appropriate coefficients. Note that $\gamma_{n,k}$ implicitly depends on $\overline{s}$.

Now, compare the equilibrium bid functions of two bidders n and m, $n \neq m$,

$$D_{n,m}(s) \equiv \hat{\varphi}_n(s) - \hat{\varphi}_m(s) = \sum_{k=0}^{K}(\gamma_{n,k} - \gamma_{m,k})(s - \underline{v})^k. \tag{14}$$

Condition 1a implies $D_{n,m}(\underline{v})$ is zero, so $\gamma_{n,0}$ must equal $\gamma_{m,0}$. Likewise, Property 2b implies $D'_{n,m}(\underline{v})$ is also zero, so $\gamma_{n,1}$ must also equal $\gamma_{m,1}$.[12] Hence,

$$\begin{aligned} D'_{n,m}(s) &= \sum_{k=2}^{K}(\gamma_{n,k} - \gamma_{m,k})k(s - \underline{v})^{k-1} \\ &= (s - \underline{v}) \times \sum_{k=2}^{K}(\gamma_{n,k} - \gamma_{m,k})k(s - \underline{v})^{k-2}. \end{aligned}$$

The second term forms a polynomial of degree $(K - 2)$, which has $(K - 1)$ terms and whose coefficients can change signs at most $(K - 2)$ times. Therefore, by Descartes' rule of signs, the polynomial has at most $(K - 2)$ roots for (strictly) positive values of $(s - \underline{v})$. In other words, $D_{n,m}$ has at most $(K - 2)$ stationary points on $(\underline{v}, \overline{s}]$. By assumption (Property 2b) it also has a stationary point at $\underline{v}$. Recall, too, that $D_{n,m}(\underline{v})$ equals zero as does $D_{n,m}(\overline{s})$. Armed with these observations, the Proposition can now be proven.

The assumption that $f_m(\overline{v})$ does not equal $f_m(\overline{v})$ in combination with Property 2a imply that the two bidders use different strategies near the top. Thus, it can be ruled out that equilibrium bid functions coincide everywhere: $D_{n,m}(s)$ cannot equal zero everywhere. Therefore, by Descartes' rule of signs, $D_{n,m}$ has a finite number of roots; there can be no non-degenerate interval on which $D_{n,m}$ is zero. Let Q denote the total number of roots on $(\underline{v}, \overline{s})$, and let $s_{(q)}$ denote the q^{th} root, with $\underline{v} < s_{(1)} < s_{(2)} < \ldots < s_{(Q)} < \overline{s}$ if Q exceeds zero. Since $D_{n,m}(\underline{v})$ equals zero as does $D'_{n,m}(\underline{v})$, $D_{n,m}(s)$ cannot equal zero on the interval $(\underline{v}, \underline{v} + \varepsilon)$. In order for $D_{n,m}(s_{(1)})$ to equal zero, $D'_{n,m}(s)$ equals zero for some $s \in (\underline{v}, s_{(1)})$. That is, one of the at most $(K - 2)$ interior stationary points is between $\underline{v}$ and s_1. Likewise, there must be at least one stationary point between $s_{(1)}$ and $s_{(2)}$,

[12] Thus, the formulation in (13) lends itself to easy tests of approximated equilibrium-bid functions as well.

and so on. Since there are at most $(K-2)$ interior stationary points, there can be at most $(K-2)$ roots on $(\overline{v}, \overline{s}]$. However, since there is a root at $\overline{s}$, because $D_{n,m}(\overline{s})$ equals zero, there can be at most $(K-3)$ roots on $(\overline{v}, \overline{s})$. ■

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