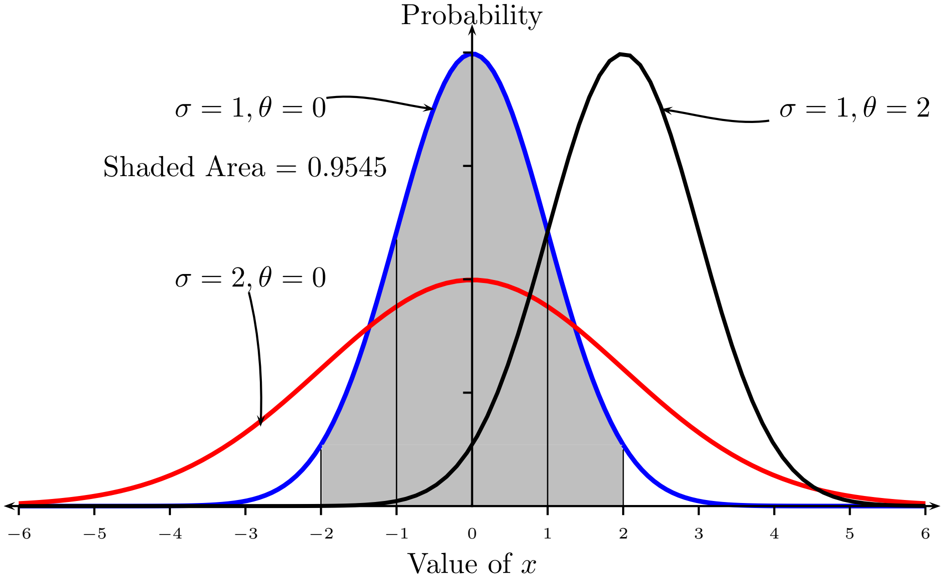
Univariate Normal Probability Density Function

A random variable, x, is normally distributed if, and only if, its probability density function has the following form:

$$\text{Prob}(x \mid \theta, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x-\theta)^2\right], \quad -\infty < x < \infty. \tag{1}$$

This probability density function has two parameters: a location parameter θ, with $-\infty < \theta < \infty$, and a scale parameter σ, with restrictions that $0 < \sigma < \infty$. It is often denoted by $\mathcal{N}(\theta, \sigma^2)$. Moreover, the function has a single mode or maximum frequency at $x = \theta$ and such point $\text{Prob}(x = \theta)$ is $\frac{1}{\sigma\sqrt{2\pi}}$. If σ is 1 (or the standard normal), then probability is about 0.4.



We will first prove that above function described in equation (1) is a proper normalized probability density function; that is $\int_{-\infty}^{\infty} \text{Prob}(x \mid \theta, \sigma) \partial x = 1$.

- First note that $\text{Prob}(x \mid \theta, \sigma) > 0$ for all x such that $-\infty < x < \infty$.
- Make the change of variable $z = \frac{x-\theta}{\sigma}$ to obtain following:

$$\text{Prob}(z) = \frac{1}{\sqrt{2\pi}} \exp -\frac{z^2}{2}, \quad -\infty < z < \infty. \tag{2}$$

- Let $u = \frac{z^2}{2}$, we have $0 < u < \infty$ and $\partial u = z \partial z$, or $\partial z = \frac{\partial u}{\sqrt{2u}}$.
- Using these substitutions, note that[1]

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{z^2}{2}\right] \partial z \quad = \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2u}} \exp(-u) \partial u$$

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[1] The factor 2 appear in the numerator to take account of the area under both positive and negative values of z.

$$
\begin{aligned}
&= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}} \exp(-u)\partial u && (3)\\
&= \frac{1}{\sqrt{\pi}} \int_0^\infty u^{-\frac{1}{2}} \exp(-u)\partial u
\end{aligned}
$$

- Note that the right hand side in above expression is the gamma function with argument $\frac{1}{2}$; that is, the gamma function denoted by $\Gamma(q)$ is defined as

  $$\Gamma(q) = \int_0^\infty u^{q-1} \exp(-u)\partial u, \quad 0 < q < \infty. \tag{4}$$

  This would result in above integral to be equal to $(1/\sqrt{\pi})\Gamma(\frac{1}{2})$. Moreover, since it is shown in calculus that $\Gamma(\frac{1}{2})$ is equal to $\sqrt{\pi}$.

- We conclude that the right hand side is equal to one.

• Moment Generating Function for Normal Distribution

The moment generating function is defined as the expected value of $\exp(t\tilde{x})$; that is,

$$M(z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(tz) \exp\left[-\frac{z^2}{2}\right] \partial z \tag{5}$$

Above integral may be evaluated by completing the square in the exponent and observing that the integrand becomes the normal density with mean t and variance of 1. That is,

$$
\begin{aligned}
M(z,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{z^2 - 2zt}{2}\right] \partial z\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\frac{z^2 - 2zt + t^2 - t^2}{2}\right] \partial z\\
&= \frac{1}{\sqrt{2\pi}} \exp(t^2/2) \int_{-\infty}^{\infty} \exp\left[-\frac{(z-t)^2}{2}\right] \partial z\\
&= \exp(t^2/2) && (6)
\end{aligned}
$$

This expression simplifies because integral is equal to 1. That is, integral is for a random variable z with its mean of t and variance of 1. Moreover, to convert moment generating function with respect to x, note that $M(z+\theta,t) = \exp(\theta t)M(z,t)$ and $M(z\sigma^2,t) = M(z,t\sigma^2)$. Using these two results, it possible to write the general expression for moment-generating function for the normal distribution as

$$M(x,t \mid \theta,\sigma^2) = \exp[\theta t + (\sigma^2 t^2/2)] \tag{7}$$

Equation (7) can be used to determine rth moment by differentiating a moment generating function rth time and then evaluating resulting expression at $t = 0$. That is,

$$m_1 = \left[\frac{\partial M(x,t)}{\partial t}\right]_{t=0} = (\theta + \sigma^2 t)(\exp[\theta t + (\sigma^2 t^2)/2]) = \theta$$

$$\begin{aligned}
m_2 = \left[\frac{\partial^2 M(x,t)}{\partial t^2}\right]_{t=0} &= [(\theta + \sigma^2 t)^2 + \sigma^2)](\exp[\theta t + (\sigma^2 t^2)/2]) = \theta^2 + \sigma^2 \\
m_3 = \left[\frac{\partial^3 M(x,t)}{\partial t^3}\right]_{t=0} &= [(\theta + \sigma^2 t)^3 + 3\sigma^2(\theta + \sigma^2 t)](\exp[\theta t + (\sigma^2 t^2)/2]) \\
&= \theta^3 + 3\sigma^2\theta \\
m_4 = \left[\frac{\partial^4 M(x,t)}{\partial t^4}\right]_{t=0} &= [(\theta + \sigma^2 \theta)^4 + 6\sigma^2(\theta + \sigma^2 t)^2 + 3\sigma^4](\exp[\theta t + (\sigma^2 t^2)/2]) \\
&= \theta^4 + 6\sigma^2\theta^2 + 3\sigma^4 \qquad (8)
\end{aligned}$$

Note that above moments can be used to obtain the mean, variance, skewness and kurtosis using following relationships.

$$\begin{aligned}
\text{mean} \equiv \mu_1 &= m_1 \\
\text{variance} \equiv \mu_2 &= m_2 - m_1^2 \\
\text{skewness} \equiv \mu_3 &= m_3 - 3m_1 m_2 + 2m_1^3 \\
\text{kurtosis} \equiv \mu_4 &= m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4 \qquad (9)
\end{aligned}$$

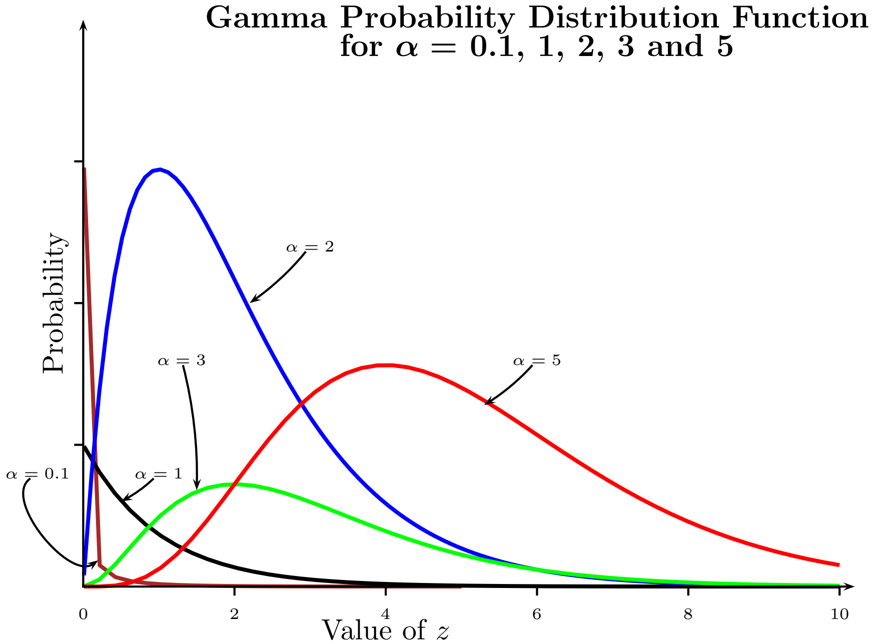
Using equation (9), note that the mean is equal to θ and variance is equal to σ^2. After some algebraic manipulation, we conclude that skewness is equal to zero. Finally, after considerable algebraic manipulation, we may write kurtosis is equal to $3\sigma^4$. Measure of skewness, that is, of departures from symmetry is measured by the coefficient of skewness (γ_1) and it is ratio of μ_3 to $\mu_2^{3/2}$ or $\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}$. To characterize, peakedness as well as long tail of distribution, coefficient of kurtosis (γ_2) is employed which is equal to ratio of μ_4 to square of μ_2 minus 3 or $\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3$. Note that for $\gamma_2 > 0$, the observed distribution are leptokurtic (more sharply peaked), and those for which $\gamma_2 < 0$, platykurtic (more flat-topped) than the normal curve. Both γ_1 and γ_2 for normal distribution are zero.

Gamma Probability Density Function

The Gamma probability density function is closely related to the gamma function. A random variable, x, is distributed according to the gamma distribution if, and only if, its probability density function is given by

$$\text{Prob}(x \mid \alpha, \gamma) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\gamma^\alpha} \exp\left[-\frac{x}{\gamma}\right], \quad 0 < x < \infty, \qquad (10)$$

and α and γ are strictly positive parameters; that is, $\alpha, \gamma > 0$. This probability density function is often denoted by $G(\alpha, \gamma)$. From equation (10) note that γ is a scale parameter. When α is less than or equal to 1, the probability density function has a single mode at $x = \gamma(\alpha - 1)$. For small values of α the probability density function has a long tail to the right. For larger values of α and for any given value of γ, the probability density function becomes more symmetric and approaches a normal form.



Gamma Probability Distribution Function for α = 0.1, 1, 2, 3 and 5

The gamma probability density function can be brought into a standardized form by the change of variable $z = \frac{x}{\gamma}$, which results in $x^{\alpha-1} = z^{\alpha-1}\gamma^{\alpha-1}$ and $\partial z = \frac{\partial x}{\gamma}$. This means that $x^{\alpha-1}\partial x = z^{\alpha-1}\gamma^{\alpha}\partial z$. Consequently, the standardized form of gamma probability density function may be written as

$$\text{Prob}(z \mid \alpha) = \frac{1}{\Gamma(\alpha)} z^{\alpha-1} \exp(-z), \qquad 0 < z < \infty. \tag{11}$$

Application of equation (4) leads to conclusion that probability density function in equation (11) is a proper probability density function.

- **Moment Generating Function for Gamma**

  To obtain moment generating function, following integral must be evaluated;

$$M(z,t) = \int_0^\infty \exp(tz) \frac{1}{\Gamma(\alpha)} z^{\alpha-1} \exp(-z)\partial z, \qquad 0 < z < \infty. \tag{12}$$

With following substitution, $z - tz = u$, note that $\partial z(1-t) = \partial u$ and $z = \frac{u}{1-t}$. Substituting these in equation (12) would result in

$$\begin{aligned}
M(z,t) &= \int_0^\infty \frac{\left[\frac{u}{1-t}\right]^{\alpha-1}}{\Gamma(\alpha)} \frac{\exp(-u)\partial u}{1-t} \\
&= \int_0^\infty \frac{u^{\alpha-1}}{(1-t)^{\alpha-1}\Gamma(\alpha)} \frac{\exp(-u)\partial u}{1-t} \\
&= \frac{1}{(1-t)^{\alpha}} \int_0^\infty \frac{1}{\Gamma(\alpha)} u^{\alpha-1} \exp(-u)\partial u \\
&= \frac{1}{(1-t)^{\alpha}} = (1-t)^{-\alpha} \qquad (13)
\end{aligned}$$

Repeated differentiation with respect to t, and setting $t = 0$ results in following four moments.

$$\begin{aligned}
\frac{\partial M(z,t)}{\partial t} &= \alpha(1-t)^{-(\alpha+1)} \\
\frac{\partial^2 M(z,t)}{\partial t^2} &= \alpha(\alpha+1)(1-t)^{-(\alpha+2)} \\
\frac{\partial^3 M(z,t)}{\partial t^3} &= \alpha(\alpha+1)(\alpha+2)(1-t)^{-(\alpha+3)} \\
\frac{\partial^4 M(z,t)}{\partial t^4} &= \alpha(\alpha+1)(\alpha+2)(\alpha+3)(1-t)^{-(\alpha+4)} \qquad (14) \\
m_1 &= \alpha \\
m_2 &= \alpha(\alpha+1) \\
m_3 &= \alpha(\alpha+1)(\alpha+2) \\
m_4 &= \alpha(\alpha+1)(\alpha+2)(\alpha+3)
\end{aligned}$$

This would result in the mean and variance of gamma distribution both to be α and skewness and kurtosis is equal to 2α and $3\alpha(\alpha+2)$ respectively. Note that coefficient of skewness is equal to $\frac{2}{\sqrt{\alpha}}$ and coefficient of kurtosis is $\frac{6}{\alpha}$. For $\alpha > 1$, both of these coefficients approach zero as α becomes larger. Note also that the maximum value of distribution function occurs, when $z = \alpha - 1$. This is different than normal distribution where mean of distribution also is the maximum of distribution function or mode of distribution function.

If gamma distribution contains parameter γ as given in equation (10), then mean and variance is equal to $\alpha\gamma$ and $\alpha\gamma^2$ respectively. Similarly skewness and kurtosis is equal to $2\alpha\gamma^3$ and $3\alpha(\alpha+2)\gamma^4$ respectively. These results can be obtained by noting that $M(z,t) = (1-t\gamma)^{-\alpha}$ and then following above steps. Note that changing scale of distribution has no impact on the coefficient of skewness and kurtosis.

Some authors prefer to write gamma distribution as follows:

$$\text{Prob}(x \mid \alpha\gamma) = \frac{\gamma^\alpha x^{\alpha-1}}{\Gamma(\alpha)} \exp(-x\gamma), \quad 0 < x < \infty, \qquad (15)$$

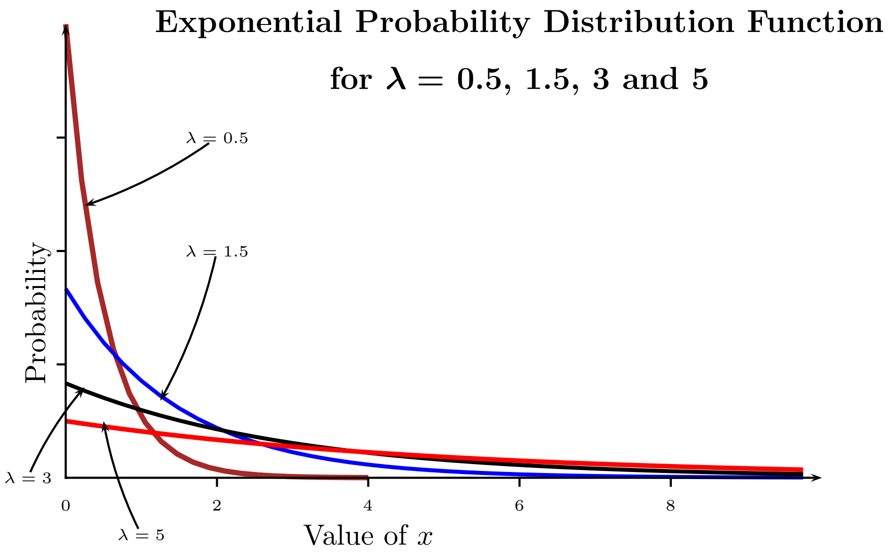
and α and γ are strictly positive parameters; that is, $\alpha, \gamma > 0$. Note that moment generating function for this form of gamma distribution is $M(z,t) = (1-\frac{t}{\gamma})^{-\alpha}$. As result mean and variance would be equal to $\frac{\alpha}{\gamma}$ and $\frac{\alpha}{\gamma^2}$ respectively.

- **Exponential Distribution**

When $\alpha = 1$ and $\gamma = \lambda$, the resulting distribution is exponential distribution and it is written as

$$\text{Prob}(x \mid \lambda) = \frac{1}{\lambda} \exp\left[-\frac{x}{\lambda}\right], \quad 0 < x < \infty, \qquad (16)$$

where $\lambda > 0$. This distribution has closed form cumulative distribution function which does not exist for normal as well as gamma distribution. It is given by $F(x \mid \lambda) =$



$1 - \exp\left(-\frac{x}{\lambda}\right)$. We can also infer all four moments about mean for the exponential distribution. Note that the mean is equal to λ, variance of λ^2, skewness of $2\lambda^3$ and kurtosis as $9\lambda^4$. Consequently, the coefficient of skewness and kurtosis is equal to 2 and 6, which is unaffected by parameter of distribution. The moment generating function for exponential distribution is $M(x,t) = (1 - t\lambda)^{-1}$. Note that as λ increases, probability density function approaches uniform distribution. Finally note that the maximum for exponential distribution occurs at $x = 0$.

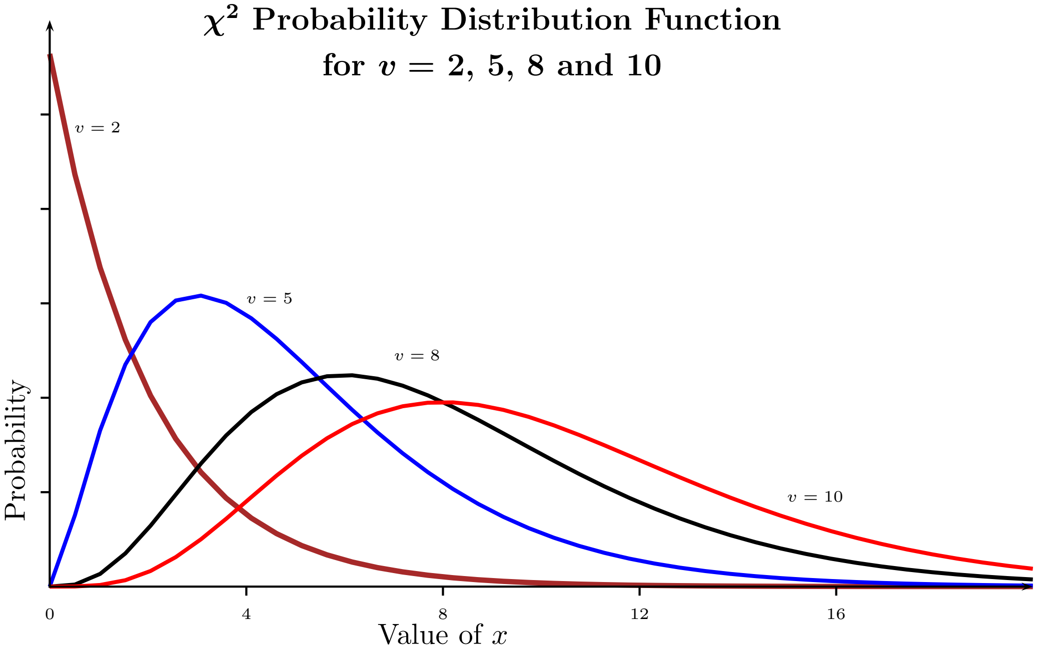
• χ^2 Distribution Function

The χ^2 probability density function is a special of the G probability density function given in equation (10) in which $\alpha = v/2$ and $\gamma = 2$; that is, the χ^2 probability density function has the following form:

$$\text{Prob}(x \mid v) = \frac{x^{v/2-1}\exp(-x/2)}{2^{v/2}\Gamma(v/2)}, \quad 0 < x < \infty, \tag{17}$$

where $v > 0$. The parameter v is usually referred to as the "number of degrees of freedom". Since equation (17) is a special case of equation (11), the standardized form moments are given in equation (14) with α replaced with $v/2$. Moments associated with unstandardized form are as follows:

$$\begin{aligned}
\text{Mean} &= \frac{v}{2} \times 2 = v \\
\text{variance} &= \frac{v}{2} \times 2^2 = 2v \\
\text{skewness} &= 2\frac{v}{2} \times 2^3 = 8v \\
\text{kurtosis} &= 3\frac{v}{2}(\frac{v}{2} + 2) \times 2^4 = 24v\left[\frac{v}{2} + 2\right].
\end{aligned}$$



χ^2 Probability Distribution Function for v = 2, 5, 8 and 10

Using above results, the coefficient of skewness is equal to $\sqrt{\dfrac{8}{v}}$ and coefficient of kurtosis to be $\dfrac{12}{v}$. As figure below indicates, χ^2 distribution approaches normal distribution as v increases. For smaller values of v, however χ^2 distribution is with a long tail on the right, and somewhat flatter in the middle. When $v = 1$ or 2, shape of distribution is similar to exponential distribution while all other integer values, the maximum for the distribution occurs at $x = v - 2$.

A very important property of the χ^2 probability density function is that any sum of squared independent, standardized normal random variables has a probability density function in the χ^2 form; that is, if $y = z_1^2 + z_2^2 + \cdots + z_n^2$, where the z_i's are independent, standardized normal random variables, then y has a probability density function in the form of equation (17) with $v = n$. Consider a case of $v = 1$. Let $z \sim \mathcal{N}(0, 1)$ and $y = z^2$. We can show that $y \sim G(\frac{1}{2}, \frac{1}{2})$ or χ^2 with $v = 1$. Note that

$$f_y(y) = f_z(h^{-1}(y)) \left| \frac{\partial h^{-1}(y)}{\partial y} \right|$$

for $a < y < b$ and $|\ |$ stands for the absolute value and a and b refer to smallest and the largest value z can take respectively. To apply this transformation lemma, we must have $\frac{\partial h(z)}{\partial z}$ is either increasing or decreasing but not both. Note that $\frac{\partial h(z)}{\partial z} = 2z$ which is increasing function for $z > 0$ and decreasing function for $z < 0$. Consider, however, region $(-\infty, 0]$ for z function y is decreasing. Thus we can apply transformation lemma in two segments. That is, $z = \pm\sqrt{y}$. This would mean that

$$\frac{\partial h^{-1}(y)}{\partial y} = \pm\frac{1}{2\sqrt{y}}.$$

Thus, we may write

$$\begin{aligned} f_y(y) &= f_z(\sqrt{y})\left[\frac{1}{2\sqrt{y}}\right] + f_z(-\sqrt{y})\left[\frac{1}{2\sqrt{y}}\right] \quad \text{for } y > 0 \\ &= \left[\frac{1}{2\sqrt{y}}\right]\left[\frac{1}{\sqrt{2\pi}}\exp(-\frac{y}{2}) + \frac{1}{\sqrt{2\pi}}\exp(-\frac{y}{2})\right] \\ &= \frac{y^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})\frac{1}{2}^{\frac{1}{2}}}\exp(-\frac{y}{2}) \quad \text{for } y > 0 \end{aligned} \tag{18}$$

This is $G(\frac{1}{2}, \frac{1}{2})$ or χ^2 with $v = 1$.

The Inverse Gamma (IG) Probability Density Function

If x^{-1} has gamma distribution with paramters α and γ as given in equation (10), then x has the inverse-gamma distribution. That is, inverse-gamma distribution can be obtained by substituting $y = x^{-1}$ in equation (10). To accomplish this transformation, note that $x = y^{-1}$ and $\left|\frac{\partial x}{\partial y}\right| = y^{-2}$. Substituting these in equation (10), following may be obtained

$$\begin{aligned} \text{Prob}(y \mid \alpha, \gamma) &= \frac{y^{-(\alpha-1)}y^{-2}}{\Gamma(\alpha)\gamma^{\alpha}}\exp\left[-\frac{1}{y\gamma}\right], \quad 0 < y < \infty, \\ &= \frac{y^{-(\alpha+1)}}{\Gamma(\alpha)\gamma^{\alpha}}\exp\left[-\frac{1}{y\gamma}\right], \quad 0 < y < \infty. \end{aligned} \tag{19}$$

Note that according to gamma function,

$$\int_0^{\infty} y^{-(\alpha+1)}\exp\left[-\frac{1}{y\gamma}\right]\partial y = \Gamma(\alpha)\gamma^{\alpha},$$

which leads to conclusion that equation in (19) is proper probability density function. Note also that the maximum[2] for such function occurs at $y^{\star} = \frac{1}{\gamma(\alpha+1)}$. Following graph usefully describes inverse gamma distribution.

- **Moments of Inverse Gamma Distribution**

To obtain moments of this distribution, it is not easy to derive the moment generating function for probability density function based on equation (19). To derive first four

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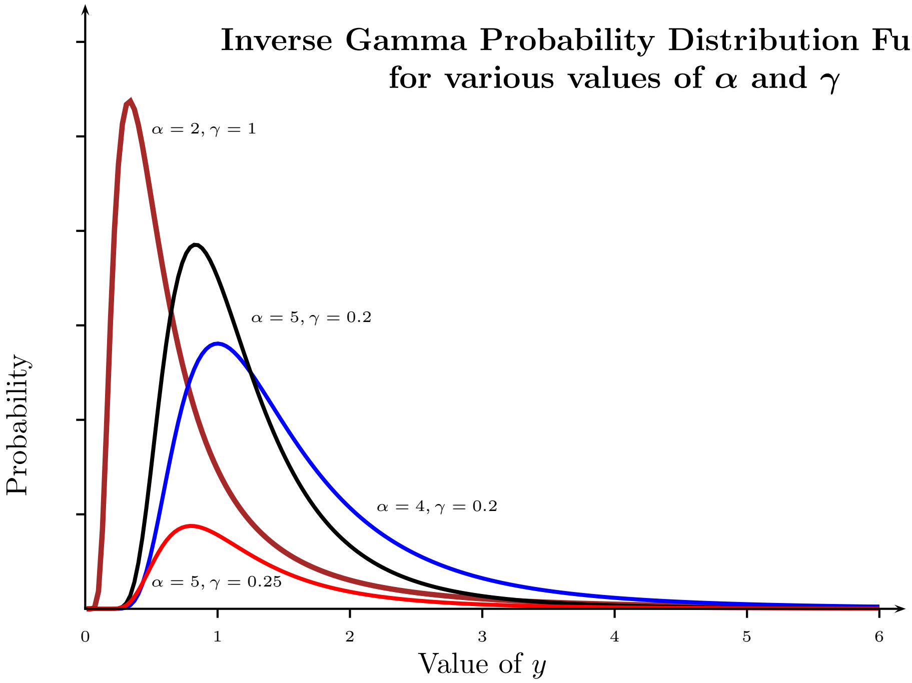
[2] The maximum is unique because $\frac{\partial^2\text{Prob}(y)}{\partial y^2} = -y^{-(\alpha+3)}\exp\left[-\frac{1}{y\gamma}\right](\alpha+1)$ is always negative at the maximum value.



moment, we use the relation between the expected value of y and the moment generating function of x given by Cressie et al. (1981)[3]. This relationship is given by

$$\mathcal{E}(y^r) = \int_0^\infty \frac{t^{r-1} M(x,-t)\partial t}{\Gamma(r)}$$

where $r > 0$. Since $M(x,t)$ for gamma distribution is $(1-\gamma t)^{-\alpha}$, we may write $M(x,-t) = (1+\gamma t)^{-\alpha}$. Substituting expression for $M(x,-t)$, we may write,

$$\mathcal{E}(y^r) = \int_0^\infty \frac{t^{r-1}(1+\gamma t)^{-\alpha}\partial t}{\Gamma(r)}.$$

To solve this integral, first we need to convert above integral in beta functional format, which then can be evaluted. To accomplish the first task, substutite $u = \dfrac{\gamma t}{1+\gamma t}$. Note that $\partial u = \dfrac{\gamma \partial t}{(1+\gamma t)^2}$, $t = \dfrac{u}{\gamma(1-u)}$ and $1 + \gamma t = \dfrac{1}{1-u}$. Note also that substitution also affects limits of integration from 0 to 1. With these substitution, we may write

$$\begin{aligned}\mathcal{E}(y^r) &= \int_0^1 \left[\frac{u}{\gamma(1-u)}\right]^{r-1} \left[\frac{1}{1-u}\right]^{-\alpha} \frac{\partial u}{\Gamma(r)\gamma(1-u)^2} \\ &= \frac{1}{\Gamma(r)\gamma^r} \int_0^1 u^{r-1}(1-u)^{\alpha-r-1}\partial u\end{aligned}$$

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[3] Cressie, Noel, Anne S. Davis, J. Leroy Folks and George E. Policello II (1981) "The Moment-Generating Function and Negative Integer Moments", *The American Statistician*, vol. 35, pp. 148-50.

$$= \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)\gamma^r}. \tag{20}$$

The last step could be written because definition of beta function is

$$\begin{aligned} B(r, \gamma - r) &= \int_0^1 u^{r-1}(1-u)^{\alpha-r-1} \partial u \\ &= \frac{\Gamma(r)\Gamma(\alpha - r)}{\Gamma(\alpha)}. \end{aligned}$$

There is an alternative and simpler approach to obtain moments of inverse gamma random variable. Suppose we want the expected value of r^{th} power, that is, $\mathcal{E}(y^r)$ which is given by

$$\begin{aligned} \mathcal{E}(y^r) &= \int_0^\infty y^r \text{Prob}(y \mid \alpha, \gamma) \partial y \\ &= \int_0^\infty y^r \frac{y^{-(\alpha+1)}}{\Gamma(\alpha)\gamma^\alpha} \exp\left[-\frac{1}{y\gamma}\right] \partial y \\ &= \frac{1}{\Gamma(\alpha)\gamma^\alpha} \int_0^\infty y^{-(\alpha-r+1)} \exp\left[-\frac{1}{y\gamma}\right] \partial y \end{aligned}$$

Note that it follows from the definition of $\Gamma()$ function that

$$\int_0^\infty x^{-(p+1)} \exp\left[-\frac{1}{ax}\right] \partial x = a^p \Gamma(p).$$

In our case $p = \alpha - r$ and $a = \gamma$. Hence we may write

$$\begin{aligned} \mathcal{E}(y^r) &= \frac{\gamma^{\alpha-r}\Gamma(\alpha - r)}{\Gamma(\alpha)\gamma^\alpha} \\ &= \frac{\Gamma(\alpha - r)}{\Gamma(\alpha)\gamma^r} \end{aligned}$$

which is same as (20).
Using $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ and equation (20), we may write following moments of inverse gamma distribution.

$$\begin{aligned} \mathcal{E}(y) &= \frac{\Gamma(\alpha - 1)}{\gamma\Gamma(\alpha)} = \frac{1}{\gamma(\alpha - 1)} \qquad \alpha > 1 \\ \mathcal{E}(y^2) &= \frac{\Gamma(\alpha - 2)}{\gamma^2\Gamma(\alpha)} = \frac{1}{\gamma^2(\alpha - 1)(\alpha - 2)} \qquad \alpha > 2 \\ \mathcal{E}(y^3) &= \frac{\Gamma(\alpha - 3)}{\gamma^3\Gamma(\alpha)} = \frac{1}{\gamma^3 \prod_{i=1}^3 (\alpha - i)} \qquad \alpha > 3 \\ \mathcal{E}(y^4) &= \frac{\Gamma(\alpha - 4)}{\gamma^4\Gamma(\alpha)} = \frac{1}{\gamma^4 \prod_{i=1}^4 (\alpha - i)} \qquad \alpha > 4 \end{aligned}$$

These could be used to derive variance, skewness and kurtosis. After various algebraic manipulations, note that

$$\begin{aligned}\text{variance} &= \frac{1}{\gamma^2(\alpha-1)^2(\alpha-2)}\\ \text{skewness} &= \frac{4}{\gamma^3(\alpha-1)^3(\alpha-2)(\alpha-3)}\\ \text{kurtosis} &= \frac{3(a+5)}{\gamma^4(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}\end{aligned}$$

The coefficient of skewness for inverse gamma then is equal to $\gamma_1 = \dfrac{4\sqrt{(\alpha-2)}}{(\alpha-3)}$ and coefficient of kurtosis is given by $\gamma_2 = \dfrac{6(5\alpha-11)}{(\alpha-3)(\alpha-4)}$.
The inverse gamma probability density function appear with many special forms, which may be derived from the one given above in equation (19). First consider, a scaled version which is obtained by substituting $\gamma = \dfrac{1}{\beta}$. This would result in,

$$\text{Prob}(y \mid \alpha, \beta) = \frac{y^{-(\alpha+1)}\beta^{\alpha}}{\Gamma(\alpha)} \exp\left[-\frac{\beta}{y}\right], \quad 0 < y < \infty. \tag{21}$$

A summary for this and other forms of inverse gamma distribution are provided on following table. As shown above that χ^2 distribution is special case of gamma probability density function, the inverse χ^2 and scaled inverse χ^2 probability density function are also special cases of IG distribution. To obtain, inverse χ^2 distribution, substitute $\alpha = \dfrac{v}{2}$ and $\gamma = 2$. This would result in

$$\text{Prob}(y \mid v) = \frac{y^{-(v/2+1)}}{\Gamma(v/2)2^{v/2}} \exp\left[-\frac{1}{2y}\right], \quad 0 < y < \infty. \tag{22}$$

Finally scaled inverse χ^2 probability density function is obtained by substituting $\alpha = \dfrac{v}{2}$ and $\gamma = \dfrac{2}{vs^2}$. This would result in

$$\text{Prob}(y \mid v, s^2) = \frac{y^{-(v/2+1)}(v/2)^{v/2}s^{v/2}}{\Gamma(v/2)} \exp\left[-\frac{vs^2}{2y}\right], \quad 0 < y < \infty. \tag{23}$$

Zellner (1971)[4] obtained the IG probability density function from the gamma probability density function in equation (10) by letting y equal the positive square root of $\frac{1}{x}$; that is, $y = \left|\sqrt{\frac{1}{x}}\right|$ and thus $y^2 = \frac{1}{x}$. With this change of variable the IG probability density function is

$$\text{Prob}(y \mid \gamma, \alpha) = \frac{2}{\Gamma(\alpha)\gamma^{\alpha}y^{2\alpha+1}} \exp\left[-\frac{1}{\gamma y^2}\right], \quad 0 < y < \infty, \tag{24}$$

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[4] Zellner, Arnold (1971) *An Introduction to Bayesian Inference in Econometrics*, Wiley:New York, pp. 371-373.

A Summary of Various forms of Inverse Gamma Distributions

	Original as equation (19)	Scaled IG equation (21)	Inverse χ^2 equation (22)	Scaled inverse χ^2 equation (23)
Substitution	—	$\gamma = \frac{1}{\beta}$	$\alpha = \frac{v}{2}, \gamma = 2$	$\alpha = \frac{v}{2}, \gamma = \frac{2}{vs^2}$
Mean	$\frac{1}{\gamma(\alpha-1)}$	$\frac{\gamma}{(\alpha-1)}$	$\frac{1}{v-2}$	$\frac{vs^2}{v-2}$
Variance	$\frac{1}{\gamma(\alpha-1)^2(\alpha-2)}$	$\frac{\gamma}{(\alpha-1)^2(\alpha-2)}$	$\frac{2}{(v-2)^2(v-4)}$	$\frac{2v^2s^4}{(v-2)^2(v-4)}$
Mode	$\frac{1}{\gamma(\alpha+1)}$	$\frac{\gamma}{(\alpha+1)}$	$\frac{1}{v+2}$	$\frac{vs^2}{v+2}$

where γ and $\alpha > 0$. Since this probability density function is encountered frequently in connection with prior and posterior probability density function for a standard deviation, in equation (24) let $\sigma = y$ (note that σ here is a random variable), $\alpha = v/2$ and $\gamma = 2/vs^2$ to obtain

$$\text{Prob}(\sigma \mid v, s) = \frac{2}{\Gamma(v/2)} \left[\frac{vs^2}{2}\right]^{v/2} \frac{1}{\sigma^{v+1}} \exp\left[-\frac{vs^2}{2\sigma^2}\right], \quad 0 < \sigma < \infty, \tag{25}$$

where $v, s > 0$. The probability density function in equation (25) has an unique maximum or mode at

$$\sigma^\star = s\sqrt{\frac{v}{v+1}},$$

and $\sigma^\star \to s$ as v tends to infinity. Note that $v = 10$, σ is equal to $0.9535s$, while at $v = 50$, σ is equal to $0.9901s$ which indicates that even for small v, σ is very close to s. To obtain moments of this form of inverse gamma distribution, we need to evaluate

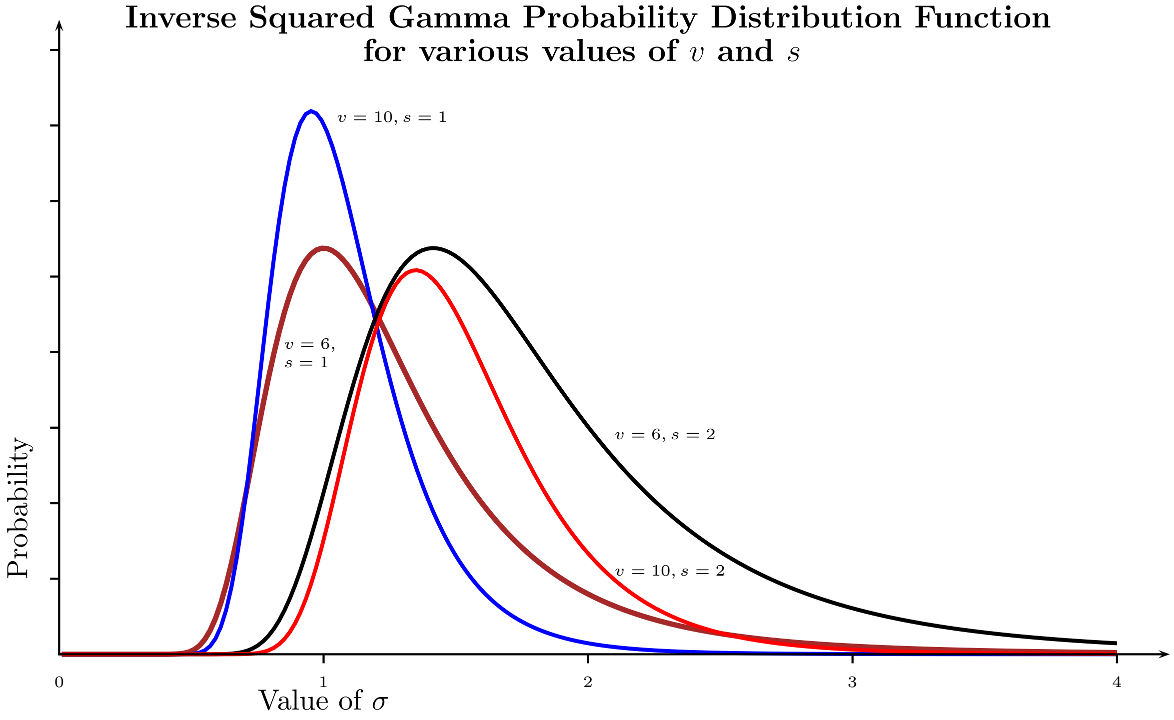
$$\begin{aligned}
\mathcal{E}(\sigma^r) &= \int_0^\infty \sigma^r \text{Prob}(\sigma \mid v, s) \partial\sigma \\
&= \int_0^\infty \sigma^r \frac{2}{\Gamma(v/2)} \left[\frac{vs^2}{2}\right]^{v/2} \frac{1}{\sigma^{v+1}} \exp\left[-\frac{vs^2}{2\sigma^2}\right] \partial\sigma \\
&= \frac{2}{\Gamma(v/2)} \left[\frac{vs^2}{2}\right]^{v/2} \int_0^\infty \sigma^{-(v-r+1)} \exp\left[-\frac{vs^2}{2\sigma^2}\right] \partial\sigma
\end{aligned}$$

Note that

$$\int_0^\infty x^{-(p+1)} \exp\left[-\frac{a}{x^2}\right] \partial x = \frac{1}{2} a^{-\frac{p}{2}} \Gamma(\frac{p}{2}).$$

In our case, $p = v - r$ and $a = \frac{vs^2}{2}$. Hence we may write,

$$\begin{aligned}
\mathcal{E}(\sigma^r) &= \frac{2}{\Gamma(\frac{v}{2})} \left[\frac{vs^2}{2}\right]^{\frac{v}{2}} \frac{1}{2} \left[\frac{vs^2}{2}\right]^{-\frac{(v-r)}{2}} \Gamma(\frac{v-r}{2}) \\
&= \left[\frac{vs^2}{2}\right]^{r/2} \frac{\Gamma(v-r)/2}{\Gamma(v/2)}, \quad v > r, \qquad (26)
\end{aligned}$$



Inverse Squared Gamma Probability Distribution Function for various values of v and s

a compact expression for the moments about zero. This expression only helps understand various moments, when r is even. That is,

$$
\begin{aligned}
m_1 \equiv \mathcal{E}(\sigma) &= \frac{\Gamma\left(\frac{v-1}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\sqrt{\frac{v}{2}}s, \quad v > 1 \\
m_2 \equiv \mathcal{E}(\sigma^2) &= \frac{\Gamma\left(\frac{v-2}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\frac{v}{2}s^2 = \frac{vs^2}{v-2}, \quad v > 2 \\
m_3 \equiv \mathcal{E}(\sigma^3) &= \frac{\Gamma\left(\frac{v-3}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\left[\frac{v}{2}\right]^{3/2} s^3, \quad v > 3 \\
m_4 \equiv \mathcal{E}(\sigma^4) &= \frac{\Gamma\left(\frac{v-4}{2}\right)}{\Gamma\left(\frac{v}{2}\right)}\left[\frac{v}{2}\right]^{2} s^4 = \frac{v^2 s^4}{(v-2)(v-4)}, \quad v > 4
\end{aligned}
$$

To get better feel for these moments, following Table provides computed moments about zero and about the mean as well as coefficient of skewness (γ_1) and kurtosis (γ_2). We could conclude that as v becomes larger, measures associated with skewness and kurtosis approach that of normal distribution. Note also that for v of 100 distribution implied by equation (25) is skewed to the right and slightly flat in the middle.

The Beta (B) Probability Density Function

A random variable, x is said to be distributed according to beta distribution if, and only

Computed First four moments of Inverse Gamma when $s = 1$

v	Mean m_1	$\mathcal{E}(\sigma^2)$ m_2	$\mathcal{E}(\sigma^3)$ m_3	$\mathcal{E}(\sigma^4)$ m_4	variance	Skewness	Kurtosis	γ_1	γ_2
5	1.1894	1.6667	2.9735	8.3333	0.2520	0.3918	2.3291	3.0981	33.6895
10	1.0837	1.2500	1.5482	2.0833	0.0755	0.0298	0.0425	1.4342	4.4548
15	1.0537	1.1538	1.3172	1.5734	0.0435	0.0097	0.0101	1.0647	2.3256
20	1.0396	1.1111	1.2230	1.3889	0.0304	0.0047	0.0042	0.8837	1.5696
25	1.0313	1.0870	1.1719	1.2940	0.0234	0.0028	0.0023	0.7716	1.1837
50	1.0153	1.0417	1.0801	1.1322	0.0108	0.0006	0.0004	0.5215	0.5304
100	1.0076	1.0204	1.0387	1.0629	0.0052	0.0001	0.0001	0.3609	0.2520

Computed First four moments of Inverse Gamma when $s = 2$

5	2.3788	6.6667	23.7883	133.3333	1.0078	3.1346	37.2658	3.0981	33.6895
10	2.1674	5.0000	12.3854	33.3333	0.3022	0.2382	0.6807	1.4342	4.4548
15	2.1075	4.6154	10.5373	25.1748	0.1740	0.0773	0.1612	1.0647	2.3256
20	2.0791	4.4444	9.7841	22.2222	0.1217	0.0375	0.0677	0.8837	1.5696
25	2.0626	4.3478	9.3755	20.7039	0.0935	0.0221	0.0366	0.7716	1.1837
50	2.0306	4.1667	8.6410	18.1159	0.0432	0.0047	0.0066	0.5215	0.5304
100	2.0152	4.0816	8.3099	17.0068	0.0208	0.0011	0.0014	0.3609	0.2520

if, its probability density function has the following form:

$$\text{Prob}(x \mid \alpha, \beta, c) = \frac{1}{cB(\alpha,\beta)} \left[\frac{x}{c}\right]^{\alpha-1} \left[1 - \frac{x}{c}\right]^{\beta-1}, \quad 0 \leq x \leq c, \tag{27}$$

where α, β and $c > 0$ are parameters of distribution and $B(\alpha, \beta)$ denotes the beta function which is equal to

$$\begin{aligned} B(\alpha,\beta) &= \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\partial z, \quad 0 \leq z \leq 1, 0 < \alpha, \beta < \infty \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \end{aligned}$$

which converges for all values of α and β greater than 0. Note that probability density function given in equation (27) ranges from 0 to c. By a change of variable, $z = \frac{x}{c}$, we can obtain the standardized beta probability density function,

$$\text{Prob}(z \mid \alpha, \beta) = \frac{1}{B(\alpha,\beta)} z^{\alpha-1}(1-z)^{(\beta-1)}, \quad 0 \leq z \leq 1, \tag{28}$$

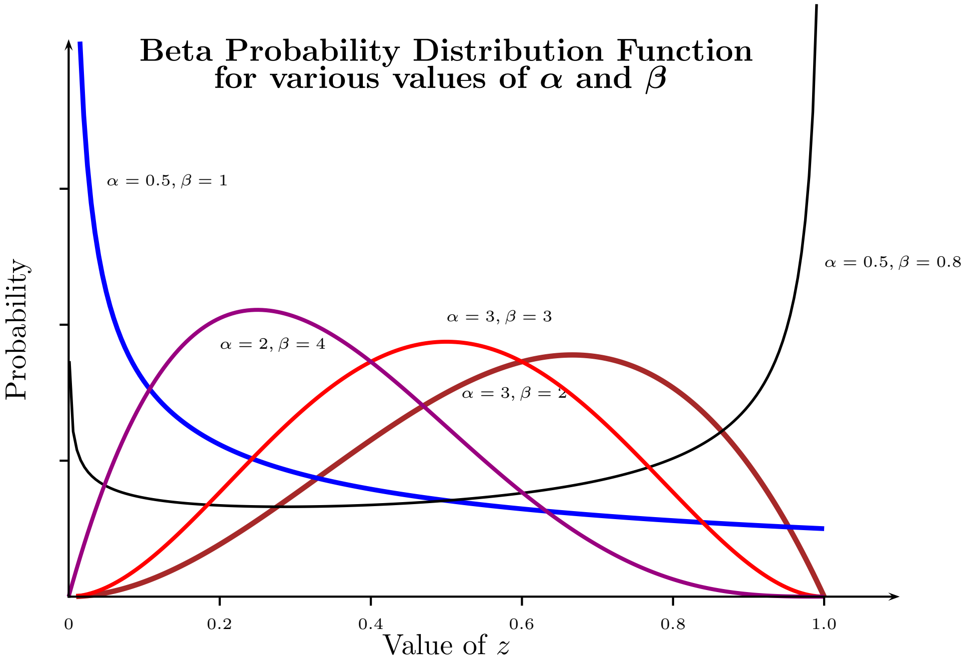
which has a range of zero to one. From definition of Beta function, we conclude that equation (28) is proper probability density function. Beta distribution takes variety of shapes depending upon parameter values of α and β. For α and $\beta > 1$, there is unique maximum[5] or mode for probability density function given in equation (28) at

$$z^\star = \frac{\alpha - 1}{\alpha + \beta - 2},$$

and for $0 < \alpha, \beta < 1$, the probability density function approaches ∞ as z approaches 0 or 1.

---

[5] Note that $\frac{\partial^2 \text{Prob}(z)}{\partial z^2} = -z^{(\alpha-1)}(1-z)^{(\beta-1)} \left[\frac{1}{\alpha-1} + \frac{1}{\beta-1}\right]$ is always negative for α and $\beta > 1$.



To obtain moments of Beta distribution, we need to evaluate

$$\begin{aligned}\mathcal{E}(z^r) &= \int_0^1 z^r \text{Prob}(z \mid \alpha, \beta) \partial z \\ &= \int_0^1 z^r \frac{1}{B(\alpha, \beta)} z^{\alpha-1}(1-z)^{(\beta-1)} \partial z \\ &= \frac{1}{B(\alpha, \beta)} \int_0^1 z^{\alpha+r-1}(1-z)^{(\beta-1)} \partial z\end{aligned}$$

Note that integral in the above expression is Beta function with parameters $(\alpha + r)$ and β. Consequently,

$$\begin{aligned}\mathcal{E}(z^r) &= \frac{B(\alpha + r, \beta)}{B(\alpha, \beta)} \\ &= \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+r)}{\Gamma(\alpha+\beta+r)\Gamma(\alpha)},\end{aligned} \tag{29}$$

which is a simple expression but not easy to compute. Using recursive relationship $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$, the first four moments about zero could be written as

$$\begin{aligned}m_1 \equiv \mathcal{E}(z) &= \frac{\alpha}{\alpha+\beta} \\ m_2 \equiv \mathcal{E}(z^2) &= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \\ m_3 \equiv \mathcal{E}(z^3) &= \frac{\alpha(\alpha+1)(\alpha+2)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)} \\ m_4 \equiv \mathcal{E}(z^4) &= \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}\end{aligned}$$

Since variance is equal to $m_2 - m_1^2$, after algebraic manipulations, we may write

$$\text{Variance} = \mu_2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

and skewness equal to $m_3 - 3m_2m_1 + 2m_1^3$. After considerable algebraic manipulations, we may write

$$\text{Skewness} = \mu_3 = \frac{2\alpha\beta(\beta-\alpha)}{(\alpha+\beta)^3(\alpha+\beta+1)(\alpha+\beta+2)}.$$

Since kurtosis is equal to $m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4$, after considerable algebraic manipulations, we may write

$$\text{Kurtosis} = \mu_4 = \frac{3\alpha\beta[\alpha^2(\beta+2) - 2\alpha\beta + \beta^2(\alpha+2)]}{(\alpha+\beta)^4(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)}.$$

As can be seen from above expressions, there is not much insights about distributional characteristics based on skewness and kurtosis. Note that when $\alpha = \beta$, skewness is zero and indicating that distribution is symmetric about the middle point in the distribution and the coefficient of kurtosis, $\frac{\mu_4}{\mu_2^2}$ is equal to $\frac{-6}{2\alpha+3}$. This would indicate that beta distribution is more flat topped or platykurtic compared to the normal curve. Based on the skewness expression, we may conclude that if β is greater than α, there is positive skewness and distribution is likely to be skewed to the right. On the other hand, β is less than α, there is negative skewness and distribution is likely to be skewed to the left. The standardized form of Beta probability density function is closely related to the standardized gamma distribution. Suppose z_1 and z_2 be two independent random gamma variables with parameters α and β respectively. Then the random variable $z = \frac{z_1}{z_1+z_2}$ has a standardized beta probability density function with parameters α and β. To prove this result, first note that the joint probability of two independent gamma variate is given by

$$\text{Prob}(z_1, z_2 \mid \alpha, \beta) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} z_1^{\alpha-1} z_2^{\beta-1} \exp\left[-(z_1+z_2)\right] \partial z_1 \partial z_2.$$

First, let us change variables as follows: $v = z_1 + z_2$ and $z = \frac{z_1}{z_1+z_2}$. Thus,

$$\begin{aligned} g(z_1, z_2) &= (z_1+z_2, \frac{z_1}{z_1+z_2}) \\ g^{-1}(v, z) &= (zv, v - zv), \end{aligned}$$

where $z_1 = zv$ and $z_2 = v - zv$. Note that Jacobian of this transformation is given by

$$J_g^{-1}(v, z) = \begin{vmatrix} z & 1-z \\ v & -v \end{vmatrix} = -v.$$

We may write the joint probability of our transformation as,

$$\begin{aligned}
\text{Prob}(v, z \mid \alpha, \beta) &= \text{Prob}(g^{-1}(v, z) \mid \alpha, \beta)|J_g^{-1}(v, z)| \\
&= \int_0^\infty \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \exp(-v)(zv)^{(\alpha-1)}(v - zv)^{(\beta-1)} v \partial z \partial v. \\
&= \int_0^\infty \int_0^1 \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \exp(-v) v^{(\alpha+\beta-1)} z^{(\alpha-1)} (1 - z)^{(\beta-1)} \partial z \partial v.
\end{aligned}$$

Note that integral with respect to v is gamma function with parameter $\alpha + \beta$ and if we integrate out v, we have required result.

- **Inverted Beta (IB) Distribution Function**

A closely related beta distribution is the beta prime or inverted beta probability density function. If we substitute $z = \frac{1}{1+u}$ in equation (28), then Jacobian of transformation is given by

$$\left| \frac{\partial z}{\partial u} \right| = \frac{1}{(1+u)^2}$$

with $0 \leq u < \infty$. Thus, making substitution in (28), we may write

$$\begin{aligned}
\text{Prob}(u \mid \alpha, \beta) &= \frac{1}{B(\alpha, \beta)} \left[\frac{1}{1+u} \right]^{\alpha-1} \left[1 - \frac{1}{1+u} \right]^{(\beta-1)} \frac{1}{(1+u)^2}, \\
&= \frac{1}{B(\alpha, \beta)} \frac{u^{(\beta-1)}}{(1+u)^{(\alpha+\beta)}}, \qquad 0 \leq u < \infty, \qquad (30)
\end{aligned}$$

with α and β greater than 0. Note that integral

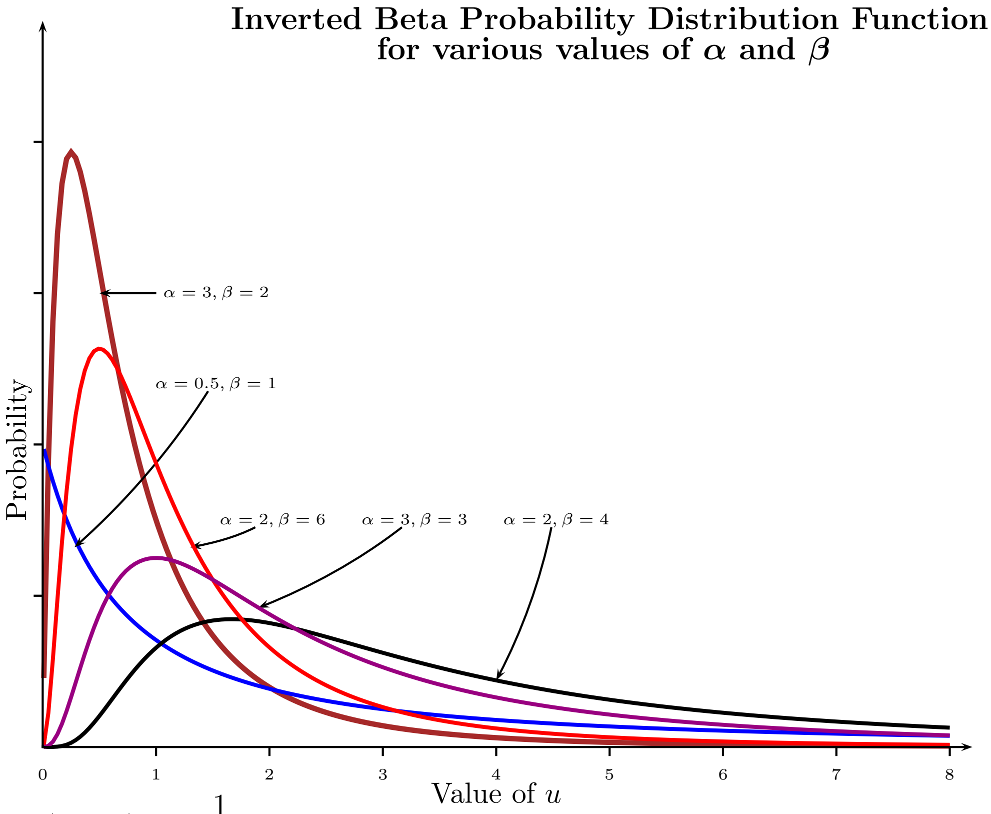
$$\int_0^\infty \cdots \int_0^\infty x_1^{p_1-1} \cdots x_n^{p_n-1} (1 + x_1 + x_2 + \cdots + x_n)^{-\sum_{s=1}^{n+1} p_s} \partial x_1 \cdots \partial x_n = \frac{\sum_{s=1}^{n+1} \Gamma(p_s)}{\Gamma(\sum_{s=1}^{n+1} p_s)}$$

is known as the inverted Dirichlet integral. As consequence, we would conclude that equation (30) is proper integral.
To obtain moments of Inverted Beta distribution, we need to evaluate

$$\begin{aligned}
\mathcal{E}(u^r) &= \int_0^\infty u^r \text{Prob}(u \mid \alpha, \beta) \partial u \\
&= \int_0^\infty u^r \frac{1}{B(\alpha, \beta)} \frac{u^{(\beta-1)}}{(1+u)^{(\alpha+\beta)}} \partial u \\
&= \frac{1}{B(\alpha, \beta)} \int_0^\infty \frac{u^{(r+\beta-1)}}{(1+u)^{(\alpha+\beta)}} \partial u
\end{aligned}$$

To evaluate integral in above expression, substitute $u = \frac{x}{1-x}$. Then,

$$\left| \frac{\partial u}{\partial x} \right| = \frac{1}{(1-x)^2},$$



Inverted Beta Probability Distribution Function for various values of α and β

$(1+u) = \dfrac{1}{1-x}$, and limits of integration become 0 to 1. Then substituting these terms in above integral, we may write

$$\begin{aligned}
\int_0^\infty \frac{u^{(r+\beta-1)}}{(1+u)^{(\alpha+\beta)}}\partial u &= \int_0^1 \left[\frac{x}{1-x}\right]^{(r+\beta-1)} \left[\frac{1}{1-x}\right]^{(\alpha+\beta)} (1-x)^{-2}\partial x \\
&= \int_0^1 x^{(r+\beta-1)}(1-x)^{(\alpha-r-1)}\partial x \\
&= B(\beta+r, \alpha-r), \quad r < \alpha.
\end{aligned}$$

Consequently, we conclude that the moments of inverted beta distribution are given by

$$\mathcal{E}(u^r) = \frac{B(\beta+r, \alpha-r)}{B(\alpha, \beta)} \quad r < \alpha. \tag{31}$$

Using recursive relationship $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, the first four moments about zero could be written as

$$\begin{aligned}
m_1 \equiv \mathcal{E}(u) &= \frac{\beta}{\alpha-1} \quad \alpha > 1 \\
m_2 \equiv \mathcal{E}(u^2) &= \frac{\beta(\beta+1)}{(\alpha-1)(\alpha-2)} \quad \alpha > 2 \\
m_3 \equiv \mathcal{E}(u^3) &= \frac{\beta(\beta+1)(\beta+2)}{(\alpha-1)(\alpha-2)(\alpha-3)} \\
m_4 \equiv \mathcal{E}(u^4) &= \frac{\beta(\beta+1)(\beta+2)(\beta+3)}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}
\end{aligned}$$

Since variance is equal to $m_2 - m_1^2$, after algebraic manipulations, we may write

$$\text{Variance} = \mu_2 = \frac{\beta(\alpha+\beta-1)}{(\alpha-1)^2(\alpha-2)},$$

and skewness equal to $m_3 - 3m_2m_1 + 2m_1^3$. After considerable algebraic manipulations, we may write

$$\text{Skewness} = \mu_3 = \frac{2\beta[2\beta^2 + 3\beta(\alpha-1) + (\alpha-1)^2]}{(\alpha-1)^3(\alpha-2)(\alpha-3)} \qquad \alpha > 3.$$

Since skewness is positive, we would conclude that inverted Beta probability density function has usually a long tail to the right. Since kurtosis is equal to $m_4 - 4m_1m_3 + 6m_1^2m_2 - 3m_1^4$, after considerable algebraic manipulations, we may write

$$\begin{aligned}\text{Kurtosis} = \mu_4 \quad &= \quad \frac{3\beta[(\beta+1)(\beta+2)(\alpha-1)^2(\alpha+5\beta-\alpha\beta-1)]}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)} + \\ &\quad\ \ \frac{3\beta^3(\alpha-3)(\alpha-4)(\alpha\beta+2\alpha-2)]}{(\alpha-1)^4(\alpha-2)(\alpha-3)(\alpha-4)}.\end{aligned}$$

Unfortunately, such complex expression provides very little insight about nature of this distribution.

The inverted beta probability density function has an unique maximum or mode[6], if $\beta > 1$ at

$$u^\star = \frac{\beta-1}{\alpha+1}.$$

An alternative form of inverted beta probability density function can be obtained, if we substitute $u = \frac{y}{c}$ with $c > 0$ in equation (30), then we may obtain

$$\text{Prob}(y \mid \alpha, \beta, c) = \frac{1}{cB(\alpha,\beta)} \frac{\left(\frac{y}{c}\right)^{(\beta-1)}}{(1+\frac{y}{c})^{(\alpha+\beta)}}, \qquad 0 \le y < \infty, \tag{32}$$

and α, β and $c > 0$. We will show below that the univariate student's t probability density function as well as F distribution are special cases of equation (32). Since $y = cu$ and we already know moments of random variable u, moments y are equal to $c^r m_r(u)$.

Univariate t Probability Distribution Function

A random variable, x is distributed according to t probability density function if, and only if, it has the following functional form

$$\text{Prob}(x \mid \theta, h, v) = \frac{\Gamma(\frac{(v+1)}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{v}{2})} \sqrt{\frac{h}{v}} \left[1 + \frac{h}{v}(x-\theta)^2\right]^{-\frac{v+1}{2}}, \quad -\infty < x < \infty, \tag{33}$$

---

[6] Note that $\dfrac{\partial^2 \text{Prob}(u)}{\partial u^2} = -\dfrac{(\alpha+1)^3 u^{(\beta-1)}}{(b-1)(a+b)(1+u)^{(\alpha+\beta)}}$ is always negative for $\alpha > 0$ and $\beta > 1$.

where $-\infty < \theta < \infty$, $0 < h < \infty$, $v > 0$ and where Γ denotes the gamma function This probability density function has three parameters, θ, h and v. θ is associated with location, v is degrees of freedom and h is scale parameter. To prove that equation (33) is proper one, we need to convert this expression to normalized form by change of variable, $t = \sqrt{\frac{h}{v}}(x-\theta)$, with $\frac{\partial t}{\partial x} = \sqrt{\frac{h}{v}}$.

$$\text{Prob}(t \mid v) = \frac{\Gamma(\frac{(v+1)}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{v}{2})}\left[1+t^2\right]^{-\frac{v+1}{2}}, \quad -\infty < t < \infty. \tag{34}$$

Note that various terms involving Γ function is equal to

$$B\left(\frac{1}{2}, \frac{v}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{v}{2})}{\Gamma(\frac{v+1}{2})}.$$

To indicate that probability density function given in equation (34) is proper, substitute $z = \frac{1}{1+t^2}$. Note that $t^2 = \frac{1-z}{z}$ and $\left|\frac{\partial t}{\partial z}\right| = \frac{1}{2tz^2}$, which is equal to $\frac{1}{2}z^{-\frac{3}{2}}(1-z)^{-\frac{1}{2}}$. Note also that $\int_{-\infty}^{\infty} \text{Prob}(t \mid v)\partial t = 2\int_0^1 \text{Prob}(z \mid v)\partial z$, we have

$$\begin{aligned} \text{Prob}(z \mid v) &= \frac{1}{B\left(\frac{1}{2}, \frac{v}{2}\right)}\int_0^1 z^{(\frac{v+1}{2}-\frac{3}{2})}(1-z)^{-\frac{1}{2}}\partial z \\ &= \frac{1}{B\left(\frac{1}{2}, \frac{v}{2}\right)}\int_0^1 z^{(\frac{v}{2}-1)}(1-z)^{-\frac{1}{2}}\partial z = 1 \end{aligned}$$

We conclude from above that the probability density function described by (34) is proper.