An Empirical Analysis of Catastrophe-linked Security Markets: Evidence from PCS Call Spread Options Traded at CBOT[1]

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Abstract

Catastrophe-linked securities whose payoffs are tied to the occurrence of natural disasters allow insurers to better diversify their risks through capital markets while at the same time offering an attractive new asset class to investors. Up to now, this class of assets is still under development and many works have been done in criticizing the existing asset designs. However, most of the results are based on simulated data and hypothetical experiments, and few empirical applications have been carefully analyzed. In this paper, true transaction data -- PCS options traded at the Chicago Board of Trade (CBOT) between 1996 and 1999, are examined carefully. Two issues have been explored:

(a) Theoretical prices are derived by assuming a complete and arbitrage free market. Comparing market prices to theoretical prices, three puzzles are observed.

(b) Comparing PCS call spread option contracts to traditional excess-of-loss catastrophe reinsurance contracts; I find that pricing patterns of these two types of assets are similar to each other, and other factors, not limited capacity, are capable of explaining these special pricing patterns.

JEL Category: **C22; G12**

Keywords: Traditional excess-of-loss catastrophe reinsurance; PCS call spread options; PCS index; Unit price

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[1] PCS is named after the Property Claim Services, who is a branch of the Insurance Service Office (ISO) and recognized industry authority for catastrophe loss estimates.

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I. Introduction

Catastrophe losses caused by natural disasters, such as earthquakes, hurricanes, windstorms, and so on, are traditionally insured by the property/casualty insurance industry. Even though mega-catastrophe events are rare, the occurrence probability is not trivial. Once such an event occurs, the insured damages can be huge. Thus, covering catastrophic risks requires a large amount of liquid capital reserves. However, in practice, no insurers are willing to put aside such large amount of capital for low-probability events.[3] Instead, primary insurers usually buy reinsurance protections from reinsurers. With better-capitalization and better-diversified portfolios, the reinsurers are expected to absorb more risks with more efficiency. However, many experts, as well as the U.S. federal government, believe that this traditional catastrophe reinsurance system is no longer capable of coping with the potential mega-catastrophes. Hurricane Andrew, for example, landed in Florida in August 1992 and incurred $15.5 billion insured losses (in 1996 currency), which left at least 11 primary insurers insolvent and many losses were uninsured. In the wake of this disaster, reinsurers tried to unload their risks. As a result, reinsurance prices rose significantly and primary insurers found themselves could not buy enough reinsurance protections at cheaper prices. Many researchers (see Froot and O'Connell (1999), and references therein) believe that it is the limited capital capacity that leads to the market failure in the catastrophe reinsurance industry and argue that the catastrophic risks can be better diversified by utilizing plentiful capitals in the financial market.

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[3] Many authors (Jaffee and Russell (1997); Kleffner, Anne and Doherty (1996); and references therein) argue that double taxation and agency problems are the main reasons that primary insurers do not hold more collateral for self-insurance against catastrophe events.

Responding to high demands for catastrophe reinsurance protection, the catastrophe-linked securities had been innovated to improve the catastrophe reinsurance market's capacity since 1992. There have been dozens of bonds, swaps, and exchange-based futures and options on the futures designed to transfer catastrophe risks into the capital market. Most of cat bonds and exchange-based options have similar payoff formulas as the excess-of-loss reinsurance contracts. These new instruments are expected to allow insurers to better diversify their risks through capital markets, while at the same time offering an attractive new asset class to the capital market investors (Litzenberger, Beaglehole, and Reynolds (1996); Cummins, Lewis, and Phillips (1998); and references therein).

In this chapter, I examine the market for one such instrument, the catastrophic-linked options traded at the Chicago Board of Trade. In particular, I develop a theory for pricing these options based on the assumptions of complete market and no-arbitrage, and I compare the theoretical prices to market prices. My results show that, like in the traditional reinsurance market, the market prices of the catastrophic-linked options are significantly higher than the theoretical prices, and that the over-pricing pattern in the catastrophic-linked option market is very similar to that found in the traditional reinsurance market. Since the option market is less likely subject to the moral hazard, agency, and limited capacity problems that may exist in the traditional reinsurance market, my results suggest that over-pricing of catastrophic risk may be due to some other reasons, such as investors' ambiguity about occurrence probability and loss severity of catastrophic events. In addition, a unit price (ratio of market prices to theoretical prices) puzzle identified in the catastrophe-linked option market is similar to that found in the traditional reinsurance market in general. Other factors, not limited capacity, are capable of explaining the puzzle.

The chapter is organized as follows. Section II provides background information about the catastrophic-loss-claim index (PCS index) and the call spread options based on this index. Section III analyzes the pricing of these options based on the assumptions of complete market and no-arbitrage. Since the closed-form solution of the price equation is not available, the conditional Monte Carlo (CMC, in short) simulations are applied and the simulation procedures are explained in detail in Section III. In Section IV, theoretical prices are compared to the observed market prices for the 3rd quarter E contracts, and robustness analyses are conducted for other regional contracts with different loss periods. In Section V, I compare the over-pricing (relative to the theoretical prices) patterns in the catastrophe-linked option market and the traditional excess-of-loss catastrophe reinsurance market examined by Froot (2001). Finally, Section VI concludes.

II. PCS Index and PCS options

The CBOT began to trade cat futures and options on the futures in 1992. These cat futures and cat options based on ISO's loss ratio index failed and were substituted by PCS options in September 1995, which have stopped trading since the summer of 1999 due to inadequate open interests. During this period, cat bonds are more popular than exchange-based products;[4] however, the inactive secondary market of the cat bonds provides little information about the pricing patterns of the catastrophe-linked securities. For PCS options traded at the CBOT, even though less than 600 transactions in total completed over the three-and-half-year

[4] Cat bonds have successfully transferred around $4.1 billion of catastrophic risks till the end of 1999, while only more than $100 million of risks have been transferred via exchange-based securities (Sedgwick Lane Financial LLC, 1998).

experience,[5] the open out-cry transaction mechanism provides the latest bid/ask prices (even no deal made) to all participants. So final transaction prices might be close to the equilibrium price. In addition, PCS call spread option contracts have similar payoff functions to those of traditional excess-of-loss catastrophe reinsurance contracts, and the similarities between the two apparently make them good comparisons. Froot and O'Connell (chap. 5 in Froot (1999b)) explain that the limited capacity is the main factor that drives the reinsurance cycle problem. Exchange-based PCS options traded at the CBOT have no credit risks, so limited capacity would not be an issue. If the pricing cycle is not identified with PCS options, it would be a side support of their theory; if the cycle persists, then other reasons need to be explored. The following subsection briefly introduces some background knowledge of PCS options and the PCS index, and provides a summary of basic statistics on those transactions.

- **PCS Index.** The loss claim data provided by the PCS cover the period from 08/1949 to 12/1999, which includes the dates, location (i.e., states), and types of each catastrophic event, and the estimated total loss claims in then current dollars for the whole property/casualty insurance industry. PCS assigns a serial number to an event whose losses are over a predetermined threshold value and which affects a large number of policyholders and insurers simultaneously. The threshold value was $1 million prior to 1983, $5 million after 1983 and $25 million dollars since January 1, 1997. Nine PCS indices are provided to the CBOT daily: a National index; five regional indices covering

[5] All catastrophe-linked securities trade on some particular catastrophe loss index, and basis risks can be significant for small and medium sized primary insurers (Harrington and Niehaus (1999), and Major (chap. 10 in Froot, 1999b)). PCS options also suffer these basis risks. These risks could discourage those insurers from buying the options.

losses in Eastern, Northeastern, Southeastern, Midwestern, and Western exposures; and three state indices covering losses in Florida, Texas, and California. Each PCS index value represents the then current PCS estimates for insured catastrophic losses divided by $100 million and rounded to the nearest first decimal point. For instance, if estimated losses total $4,635,460,000, the index value would be 46.4. It usually takes three to five days for the PCS to release the first estimate after a catastrophe, and re-estimate is released within 60 days if its losses exceed $250 million. The PCS will continue to update its loss estimate until it is believed that the accurate estimate is obtained. Usually, the severer a catastrophe event, the longer it takes to obtain its accurate estimate.

- **PCS call spread options.** PCS options are standardized, exchange-based contracts that track the PCS catastrophe loss indices. The option is European style, i.e. it can not be exercised before the expiration date, but can be closed by selling or buying exactly the same option contract. Each contract is characterized with four factors: regional coverage, lower and upper strike prices, loss period, and development period.

1. **Region covered.** Corresponding to the nine regional PCS indices, nine regional contracts are available: Eastern contracts, Midwestern contracts, National contracts, Northeastern contracts, Southeastern contracts, Western contracts, California contracts, Florida contracts, and Texas contracts.
2. **Strike prices.** Define k_1, k_2 to be the lower and upper strike prices, respectively, which are equivalent to the *retention level* and *upper limit* of a traditional excess-of-loss catastrophe reinsurance contract. Selling a call-spread contract k_1 / k_2 $(k_1 < k_2)$ is equivalent to selling a call option with a strike price k_1 and automatically buying a call

option with a strike price k_2. By selling a call spread option contract, an investor's losses are capped by $k_2 - k_1$ and his risk is reduced relative to a pure call option. His final payoff at maturity will be $x_{T_1} = \max[0, \min(L(T) - k_1, k_2 - k_1)]$, where $L(T)$ is the aggregate PCS index values.

3. **Loss period.** Define $[0, T]$ to be the loss period (measured in months). If $T = 12$, it is an annual contract, which covers the aggregate insured losses occurring within a whole calendar year. If $T = 3$, it is a quarterly contract. There are four quarterly contracts, such as March/June/ September/December: March contracts cover losses occurring in the first quarter; June contracts cover losses in the second quarter; September contracts cover losses in the third quarter; December contracts covers losses in the fourth quarter. Only annual contracts are available for the Western and California regions. Both annual and quarterly contracts are provided for National contracts.

4. **Development period.** Define $(T, T_1]$ to be the development period, which follows the loss period. There are two choices: six months or a year. During the development period, the PCS continues to update its nine regional index values for catastrophe events occurring during each contract' loss period. The final payoff of a regional contract will be determined by the final, updated PCS index values on its expiration date.

- **Basic Statistics.** Data under examination contains 474 call spread option contracts and 20 pure put option contracts (all quarterly contracts), and 36 pure call options with strike price 150 (all annual contracts). The ratio between annual contracts and quarterly contracts was around 1 to 2.5. Grouped by regional coverage and loss period, there are 29 different types of contracts traded and the distributions of these transactions are reported in Table 1. Among them, most of the E (east), NE (northeast), SE (southeast) and FL (Florida) contracts cover the 3rd and 4th quarter losses, when the hurricane activities are more intensive. The 2nd quarter losses are the greatest concerns in MW (Midwest) and TX (Texas). In addition, 54% of quarterly contracts were traded prior to the first catastrophe event and around 40% for the annual contracts. Only few contracts were traded during the development period.

Table 1. Distributions of contracts with different loss period and regional coverage

In the sixth column, 3rd +4th refers that the loss period includes both the 3rd and 4th quarters; and I put the number of contracts with loss period including the 2nd, 3rd and 4th quarters in the parenthesis.

Quarterly contracts						
Region \ Quarter	1st	2nd	3rd	4th	3rd +4th (+2nd)	Total
E	18	1	60	17	22	118
NE	2	1	30	15	9	57
SE	2	0	51	5	0	58
FL	3	0	5	2	6	16
MW	0	53	0	0	0	53
TX	0	14	5	0	1(+2)	22
NA	19	18	4	1	9(+2)	53
Total	44	87	155	40	51	377
Annual contracts						Total
NA	86	W	47	CA	20	153

In the following sections, I will concentrate on the analysis of contracts covering the 3rd quarter losses in eastern region (or 3rd quarter E contracts, in short). There are mainly three reasons for me to pick this type of contracts: First, different natural perils may follow different distributions. For example, hurricanes, earthquakes, and windstorms behave differently and their stochastic properties will be different also. So a type of contracts only covering one particular type of natural perils will provide more accurate distribution estimation. Losses due to hurricane activities dominate the 3rd quarter losses in the eastern coast of U.S. Second, within a short period, quarterly contracts are more liquid than annual contracts and the 3rd quarter E contracts are the most popular quarterly contracts (see Table 1). Therefore, the market prices for this type of contracts might be closer to equilibrium prices relative to other type of contracts. Third, the most important reason is that I have information about watches (weekly or monthly forecasted) and short-term warnings (released three or four days earlier) of hurricane activities, which were released by the NOAA.[6] When an investor writes a call spread option contract, the investor will treat a potential hurricane forecasted and warned by NOAA as another catastrophe event and updates his price accordingly, even though the hurricane has not occurred at the trading date. So such information can be treated as investors' prior information in our computation of theoretical prices, see case 2 at Section 3.2 in details. The availability of forecasts of hurricane activities makes the 3rd quarter E contracts superior to other type of contracts. Therefore, I decide to analyze the 3rd quarter E contracts in details and the results from other types of contracts are considered for the sake of robustness in the end. The loss period of a 3rd quarter contract is between July 1 and September 30. The twelve-month development period extends from October 1 to September 30 in the next year, and the contract will be settled on September 30, one year after the loss period.

[6] NOAA refers to National Oceanic and Atmospheric Administration, who will release long-term severe weather forecasts and short-tern warnings (two or three days prior to a hurricane).

III. Pricing PCS call spread option contract

As a benchmark, I price the PCS options based on the assumption of complete markets and no-arbitrage.[7] Under these assumptions, the price at time t of any contingent claim with a payoff function F at time $T_1 > t$ is $E_t\left[M_{t,T_1} F\right]$, where M_{t,T_1} is the so-called stochastic discount factor (See, for example, Harrison and Kreps (1979)). As defined in Section II, let $L(T)$ be the aggregate loss index at maturity T_1, then the payoff of a call-spread option with call spread k_1 / k_2 is

$$F(L(T)) = \max\{0, \min[L(T) - k_1, k_2 - k_1]\} \tag{1}$$

and the price of the option at time t is

$$P_{c,t} = E_t\left[M_{t,T_1} F(L(T))\right] = E_t\left(M_{t,T_1}\right) E_t\left[F(L(T))\right] + Cov_t\left\{M_{t,T_1}, F(L(T))\right\}. \tag{2}$$

Note that $E_t\left[M_{t,T_1}\right]$ is the price of a risk free zero-coupon bond that pays one dollar at maturity T_1. So we have

$$E_t\left[M_{t,T_1}\right] = e^{-r_f (T_1 - t)} \tag{3}$$

and

$$P_{c,t} = e^{-r_f (T_1 - t)} E_t\left[F(L(T))\right] + Cov_t\left\{M_{t,T_1}, F(L(T))\right\}, \tag{4}$$

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[7] Cummins and Geman (1993 and 1995), and Geman and Yor (1997) price PCS options as Asian options under the assumption of an arbitrage free and complete insurance market. This method requires continuous loss claim data from individual underwriters. Embrechts and Meister (1997); Balas, Longarela, and Lucia (1999) price options in an incomplete catastrophe insurance market. The former paper is to identify the unique equivalent martingale measure by applying utility maximization pricing theory to the insurance derivatives, especially CAT future. The latter one examined several annual national contracts traded on relatively liquid dates. Cummis, Lewis, and Phillips (1998) propose a hypothetical excess-of-loss reinsurance contract and calculate its price directly from the estimated loss severity distribution and probability distribution of event occurrence.

where r_f is the risk-free interest rate on the zero-coupon bond. Since we will look at quarterly contract, we use 90-day Treasury bill rate for r_f. We can simplify this pricing formula by imposing the following two additional assumptions: a) catastrophic risk represented by $L(T)$ is zero-beta or diversifiable risk; and b) the capitalization of the catastrophe-linked security markets amount for approximately zero percentage of capitalization of the whole capital markets (See numbers in Froot, 1995). We examine empirically the validity of these assumptions in the Appendix I. Under these two assumptions, the second term in the above pricing equation is zero. Therefore, we have

$$P_{c,t} = e^{-r_f(T_1-t)}E_t\left[F\left(L(T)\right)\right] = e^{-r_f(T_1-t)}E_t\left[\max\left\{0, \min\left[L(T)-k_1, k_2-k_1\right]\right\}\right] \qquad (5)$$

Equation (5) means that $P_{c,t}$ is equal to the expected discounted value of contract payoff, or it is the actuarial fair price of the contract. To calculate the expected discounted value, I adopt the historical simulation method that is also used by Cummins, Lewis, and Phillips (1998). This method requires us to estimate frequency and loss severity distributions of the loss claims data for each type of contracts.

3.1. Modeling event losses and event occurrences

In order to calculate theoretical price, I assume a simple $Crame'r - Lundberg$ kind of model. The stationary count process and independencies assumed below will simplify the computation of prices.

(1) *The number of natural perils occurring up to time t is denoted by* $N(t)$. Assume that $N(0)=0$, $N(t)\le N(t+h), h>0$, $N(t_1)-N(t_2)$ and $N(t_3)-N(t_4)$ are independent for any $t_1>t_2\ge t_3>t_4$, i.e. the process has stationary and independent increments. Poisson and negative binomial distributions are the two most popular discrete processes used to explain the count data.

(2) *The size of the* i^{th} *natural perils is* U_i. The sequence $\{U_i, i=1,2,\cdots\}$ of consecutive loss claims is assumed to be independent and identically distributed. Finally, I assume that the sequences $\{U_i\}$ and $\{N(t)\}$ are independent of each other.

(3) *The aggregate loss amount up to time t is* $L(t)=\sum_{i=1}^{N(t)} U_i$, while $L(t)=0$ if $N(t)=0$. Thus, the aggregate loss is a random sum of random variables.

Therefore, as long as we know the distributions of $\{U_i\}$ and $\{N(T)\}$, assuming above model, we can compute $E_t\{\max[0,\min(L(T)-k_1,k_2-k_1)]\}$ by simulation if no closed form solution is available: first, generate $N(T)$, the total number of events during the loss period from its distribution, then generate $\{U_i\}_{i=0}^{N(T)}$ and define $L(T)=\sum_{i=0}^{N(T)} U_i I(U_i>d_i)$, d is the threshold value defined in Section II; $L(T)=0$ if $N(T)=0$; repeat this procedure for J times, and by the law of large numbers, we know that $\frac{1}{J}\sum_{j=1}^{J}\max[0,\min(L_j(T)-k_1,k_2-k_1)]$ will converge to $E_t\{\max[0,\min(L(T)-k_1,k_2-k_1)]\}$ if J is sufficiently large.

Since the underlying distributions are unknown, we have to estimate them from data first. Data under consideration are the historical records from the PCS as mentioned in Section II. A natural peril will be reported as a catastrophe event only when the insured property losses exceed a threshold value d and a large number of policyholders and insurers are affected simultaneously. Thus, the loss claim data have already been truncated, not drawn from the whole distribution. That is, we observe $N(T)$ such that $\{U_i \geq d\}_{i=0}^{N(T)}$. Therefore, the above independence between observed $N(T)$ and $\{U_i \geq d\}_{i=0}^{N(T)}$ will not hold with the historical data. Fortunately, Theorem 2 and Theorem 3 in Appendix II show that if the occurrence number of natural perils follows Poisson or negative binomial process, then $N(T)$ still follow Poisson or negative binomial process for truncated data, though with different parameters. Hence, in order to compute $P_{c,t}$, now we have to generate $N(T)$ from its conditional distribution instead, then generate $\{U_i \geq d\}_{i=0}^{N(T)}$ from its truncated distribution. Calculate aggregate loss $L(T) = \sum_{i=0}^{N(T)} U_i$; and repeat this procedure for J times; then compute the average value $\frac{1}{J}\sum_{j=1}^{J} \max\left[0, \min\left(L_j(T) - k_1, k_2 - k_1\right)\right]$. This is the basic computation method used in this chapter. However, further adjustments of the raw historical data are necessary before we actually price PCS call spread options.

We have to be cautious in using historical raw data since systemic changes may have occurred such that prior catastrophes may not be recognized as single events in the future, then the conditional distributions of the number of events observed and observed event losses may change across time. For example, during the sample period from 1949 to 1999, systemic changes, such as climatological changes, price changes, and population growth, may have effects on both the frequency and severity of various catastrophic perils. Also threshold values have changed three times since 1949 mentioned in Section II, which adds jumps to a stationary count process. In order to derive distributions independent of time, I make the following adjustments except for the climatological changes, which I did not find an appropriate variable to proxy hurricane cycle, if the cycle does exist. For price effects and population growth effects, I use the monthly construction cost index provided by the Engineering News Record to measure price movements and denote it as C_t;[8] and state population series $\{POP_t\}$ are from the U.S. Bureau of Census. State losses are adjusted to their own population levels. Therefore, the adjustment factor is defined as $\tilde{C}_t = C_t \cdot POP_t$. For the 3rd quarter E contracts, first we have observations $\left\{\{U_{i,s}\}_{i=0}^{N_s(T)}, N_s(T), d_s\right\}_{s=1950}^{1999}$, where $N_s(T)$ is the number of reported catastrophe events occurring during the 3rd quarter of year s, and U_i^s is the corresponding loss claims of the i^{th} event, and d_s is the threshold value in year s. Second, I define $\tilde{U}_{i,s} = U_{i,s} / \tilde{C}_s$, and $\tilde{d}_s = d_s / \tilde{C}_s$, which can be treated as the real losses and real threshold value per person in year s.[9] Third, define $\tilde{d}_{\max,s}$ to be the largest real threshold value during the sample period and treat it as the benchmark value. Let $\tilde{N}_s(T)$ such that $\tilde{U}_{i,s} > \tilde{d}_{\max,s}, i = 0, \cdots, \tilde{N}_s(T)$. Finally, $\left\{\{\tilde{U}_{i,s}\}_i, \tilde{N}_s(T), \tilde{d}_s, \tilde{d}_{\max,s}\right\}_{s=1950}^{1999}$ will be used to estimate the underlying distributions.

Figure 1 plots the raw and adjusted event losses for the 3rd quarter losses in the eastern region. After adjusted to eastern population and construction cost index, the past events occurred during 1950's and 1960's are severer than the corresponding nominal values. For

[8] The Construction cost index (1950:1-1999:12) are provided by the Engineering News Record and annual data are used between 1950 and 1963 and monthly data are used after 1963.

[9] If a 3rd quarter E contract was traded at 19950625, i.e. June 5, 1995, then $s = 1995$. Both C_s and POP_s are observations in June, 1995.

instance, the real values of two events occurring in 1954 and 1965 are \$1.67 billion and \$3.48 billion (in 1996 currency); and the corresponding nominal values are \$136 million and \$515 million, respectively. Two more random variables are introduced here: C_s and POP_s. The expected adjustment factor at maturity, $\hat{C}_t = E_t\left(\tilde{C}_s\right)$, is estimated for each type of contracts (in Appendix II).

Frequency of cat event occurrence

The Poisson and negative binomial distributions are the two most popular examples of the claim number distributions. Recall that

- Poisson(λ): $p_k = e^{-\lambda} \frac{\lambda^k}{k!}$ for $k \in \mathrm{N}, \lambda > 0$, where $E(N) = \lambda, Var(N) = \lambda$.
- $NB(\alpha, p)$: $p_k = \dfrac{\Gamma(\alpha + k)}{\Gamma(\alpha)\Gamma(k+1)}(1-p)^{\alpha} p^k, k \in \mathrm{N}, p \in (0,1), \alpha > 0$ where

$$EN = \alpha p/(1-p), Var(N) = \alpha p/(1-p)^2.$$

By applying the maximum likelihood method, I fit the data with $Poisson(\lambda)$ and $NB(\alpha, p)$ distributions. The estimators are reported in Table 6 through Table 12.

Poisson or Negative binomial -- which one provides better fit? A simple LR (likelihood ratio) test can be applied, since a negative binomial distribution becomes a Poisson distribution as $p = \dfrac{\lambda}{\lambda + \alpha}$ and $\alpha \to \infty$. The limiting distribution of the LR test statistic is chi-squared with one degree of freedom under the null hypothesis that Poisson distribution is correct; and its 95% critical value is 0.6703. For the 3rd quarter E contracts, based on the negative log-likelihood values calculated at Table 8, the LR statistic is equal to 4.6424, which is far greater than the 95% critical value. Therefore, Poisson distribution is rejected at the significance level of 5%, and negative binomial distribution is selected. Even though, negative binomial distribution is better than Poisson distribution, it does not imply that negative binomial distribution will fit the data. So two test statistics are used to test whether the data follows the negative binomial distribution: one is called the Kolmogorov-Smirnov test; the other is the Pearson's *goodness-of-fit* statistic with the null hypothesis H_0 : $F(x) = F_0(x)$:

(1) $D_n = \max_x |F_n(x) - F_0(x)|$, $F_n(x)$ is the empirical c*df* and $F_0(x)$ is the c*df* under the null hypothesis, or negative binominal distribution here. Under H_0, D_n has a distribution that is not known.

(2) $W_n = \sum_{i=2}^{k} \frac{(f_i - np_i)^2}{np_i}$. Divide the real line into *k* classes $(c_1, c_2], (c_2, c_3], \cdots, (c_{k-1}, c_k]$ and f_i is the frequency that observations fall into the i^{th} class and $p_i = F_0(c_i) - F_0(c_{i-1})$.

$W_n \xrightarrow{d} \chi^2(k-1)$ for a large sample.

Since there are only 46 observations, the critical values are calculated through bootstrap method with $J = 100{,}000$ replications. Test procedures are as follows:

(i) Calculate D_n and W_n based on fitted negative binomial distribution.

(ii) Resample observed data, then calculate $\hat{D}$ and $\hat{W}$ based on the bootstrap resample.

(iii) Repeat (ii) for J times; and sort $\{\hat{D}_i\}_{i=1}^{J}$ and $\{\hat{W}_i\}_{i=1}^{J}$ in increasing order and rename them as $\{\tilde{D}_i\}_{i=1}^{J}$ and $\{\tilde{W}_i\}_{i=1}^{J}$, respectively; then the 95% bootstrap critical values for the two tests are $\tilde{D}_{95,000}$ and $\tilde{W}_{95,000}$.

(iv) If D_n or W_n is greater than its 95% critical value, then the null hypothesis will be rejected at the significance level of 5%.

Table 2 implies that the negative binomial distribution is not rejected by either of the two tests at the significance level of 5%. Therefore, the negative binomial distribution is chosen for the 3rd quarter E contracts. Similar analyses are applied to other regional contracts with different loss periods.

Table 2. Hypothesis Tests based on data during 1950-1995

Test Statistic	H_0	*Negative Binomial*
Goodness-of-fit *With* $k=8$	W_n	3.6533
	$\tilde{W}_{95,000}$	27.288
Kolmogorov-mirnov	D_n	0.0386
	$\tilde{D}_{95,000}$	0.1691

In addition, if a contract is traded *prior to the development period.* The future population growth and construction cost index at the end of loss period are not known. Then the investor has to forecast his expected adjusted factor $\hat{C}_s$, which will affect the event frequency. Hence, at the transaction time *t*, if the expected threshold value $\hat{d}_s = d/\hat{C}_s$ is larger than $\tilde{d}_{\max,s}$, then $\Pr(\hat{U} > \hat{d}_s) < \Pr(\hat{U} > \tilde{d}_{\max,s})$. In light of Theorem 2 and Theorem 3 in Appendix II, the number of event occurrences still follows Negative binomial process but with different parameter values. The adjustment is made as follows:

$$NB(\hat{\alpha}, \hat{p}) \text{ is changed to } NB(\hat{\alpha}, \overline{p}), \text{ where } \overline{p} = \frac{\hat{p}q}{1-\hat{p}(1-q)},$$

and

$$q = \Pr\left(\hat{U} > \hat{d}_s\right) / \Pr\left(\hat{U} > \tilde{d}_{\max,s}\right).$$

Loss severity distribution

Recall the construction of the adjusted event loss claim data, the catastrophic loss claim data have already been truncated:

$$Y = \begin{cases} X, & X > d \\ \textit{not} \quad \textit{defined}, & \textit{otherwise} \end{cases}$$

Suppose the cumulative distribution function and density function of X are $F_X(x)$ and $f_X(x)$, then the *cdf* and *pdf* of the variable Y are

$$F_Y(x) = \begin{cases} 0, & x \le d \\ \Pr(X \le x \mid X > d) & \\ \dfrac{F_X(x) - F_X(d)}{1 - F_X(d)} & x > d \end{cases} \quad \text{and} \quad f_Y(x) = \begin{cases} 0, & x \le d \\ \dfrac{f_X(x)}{1 - F_X(d)} & x > d \end{cases}$$

Therefore, the maximum likelihood function is

$$l(\theta; \tilde{U}, \tilde{d}) = \sum_{i=1}^{n} \left\{ \ln(f(\tilde{U}_i; \theta)) - \ln(1 - F(\tilde{d}_i; \theta)) \right\} \tag{6}$$

where θ is the parameter vector to be estimated.

Since catastrophe events with extremely high losses are very rare, the nonparametric density estimate will be biased with limited historical records. Therefore, parametric models are used to fit the data by the maximum likelihood method. Heavy-tailed distributions are usually used to estimate loss severity distributions. Eight heavy-tailed distribution families are considered: Burr, Gamma, Generalized Pareto, Loggamma, Loglogistic, Lognormal, Pareto and Weibull distributions. Gamma, Loglogistic, Lognormal, and Weibull have all the moments, but Loglogistic have the heaviest tails among them. Burr, Generalized Pareto, Logamma, Pareto distributions are more rightward-skewed and some of the moments do not exist. First, for each kind of contracts, eight distribution models are fitted with the real loss claims data. The next step is to choose the best fit among them. Two or three costliest events have larger power in the nonparametric density estimation and more or less, formal test statistics require estimating density or cumulative distribution function non-parametrically, which may lead to biased choices due to very limited observations. Thus, two steps are performed in model selection: First, use the likelihood ratio test to choose between Pareto / Loglogisitic and Burr distributions, and between Pareto and generalized Pareto distributions; second, select models with the largest log-likelihood value. Since the estimated price level relies on the underlying distribution assumptions, an alternative distribution model is also reported for each type of contracts for robustness analysis. Table 13 - Table 20 contains all estimation results, and the best-fitted distribution models are in boldface.

Figure 5 and Figure 6 plot the *pdf* and *cdf* for losses over 0.1612 (or $16.12 million in 1996 dollars) between the years 1950 and 1995 (for 3rd quarter E contracts). Figure 6 indicates that the Loglogistic distribution has fatter tail than the Weibull distribution. The nonparametric density imposes heavy weights at three points, shown by the enlarged right tail of the density in Figure 7.

3.2. Pricing by CMC

In Section 3.1., I have estimated the frequency distribution, loss severity distribution and estimated adjusted factors for each type of contracts. Now I am in a position to calculate theoretical price for each option traded. Except gamma distribution, there is no closed form solution to equation (5), so conditional Monte Carlo simulation method is applied here. Below, three cases are considered separately according to transaction times. In this section, I use A to stand for nominal value, $\tilde{A}$ for real value, and $\hat{A}$ for estimated value.

- **Case 1:** If an option is traded prior to the loss period, then no information about the future catastrophe events is available. That is $N(t)=0$ and $L(t)=0$. The algorithm is as follows:

(1) Generate $\tilde{N}_j \sim NB(\hat{\alpha}, \bar{p})$, where $\bar{p} = \dfrac{\hat{p}q}{1-\hat{p}(1-q)}$ and $q = \Pr\left(\tilde{U} > \hat{d}_s\right) / \Pr\left(\tilde{U} > \tilde{d}_{\max,s}\right)$ if $\hat{d}_s \neq \tilde{d}_{\max,s}$; generate $\tilde{N}_j \sim NB(\hat{\alpha}, \hat{p})$, otherwise.

(2) [10] Given $\tilde{N}_j$, generate $\tilde{U}_1^j = \max\left\{\tilde{U}_1^j, \cdots, \tilde{U}_{\tilde{N}_j}^j \,\middle|\, \tilde{U}_i^j > \hat{d}_s, i = 1, \cdots, \tilde{N}_j\right\}$ from the selected loss severity distribution and then generate $\left\{\tilde{U}_i^j\right\}_{i=2}^{\tilde{N}_j}$ with $\hat{d}_s \leq \tilde{U}_i^j < \tilde{U}_1^j, i = 2, \cdots, \tilde{N}_j$. Finally let $\tilde{L}_j = \sum_{i=1}^{\tilde{N}_j} \tilde{U}_i^j$; and $\tilde{L}_j = 0$ if $\tilde{N}_j = 0$.

(3) Let $Z_j = \max[0, \min(\hat{C}_s \tilde{L}_j - k_1, k_2 - k_1)]$, where $\hat{C}_s$ is the expected adjusted factor estimated at transaction time t and detailed estimation procedure could be found in

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[10] If $\{U\}_{i=1}^n$ is a sequence of i.i.d. random variables from a truncated distribution function $\frac{F(u)-F(d)}{1-F(d)}$, then $\Pr(U_{\max} < u) = \left[\Pr(U < u)\right]^n$. Then random variable $U_{\max}$ can be generated as follows: first, generate $r \sim U(0,1)$, and then $u_{\max} = F^{-1}\left[r^{1/n}(1-F(d)) + F(d)\right]$ will be one of observations of the random variable $U_{\max}$. Similarly, it is easy to generate a random variable from a bounded distribution $\Pr(u \mid d \leq u \leq u_{\max}) = \frac{F(u)-F(d)}{F(u_{\max})-F(d)}$. Random variable $U, d \leq U < u_{\max}$ can be generated by the similar method: first generate $r \sim U(0,1)$, then $u = F^{-1}\left[rF(u_{\max}) + (1-r)F(d)\right]$. In addition, generating a maximum random variable, then other variables in between would make the simulation procedure more efficient than the simple randomly drawing *n* times.

Appendix II. Here $\tilde{L}_j$ is the real loss index after removing the price and population growth effects, and $\tilde{L}_j \hat{C}_s$ is the nominal aggregate loss index. Since the contract is written on nominal values, the nominal aggregate losses are required. I multiply $\tilde{L}_j$ by $\hat{C}_s$ instead of sampling over C_s and POP_s here, since these two variables are less volatile within a short time period.

(4) Repeat (1) – (3) J times and define $\hat{V} = J^{-1}\sum_{j=1}^{J} Z_j$ to be the estimate of V. $J=10,000$ here.

(5) Repeat (1) – (4) 100 times and calculate the average and standard deviation of $\hat{V}$.

- **Case 2:** If the option is traded during the loss period. The investor knows $L(t)$ and $N(t)$. With Theorem 1 in Appendix II, I derive $\tilde{N}(T-t) \sim NB(\frac{T-t}{T}\hat{\alpha}, \bar{p})$. Some changes are made to the above algorithm.

(1)' Change $\hat{\alpha}$ to $\frac{T-t}{T}\hat{\alpha}$ and others are the same.

(2)' Two situations need to be considered separately:

(A) Similar to step (2) above. Define $m = m_1 + m_2$, where m_1 is the number of events observed, as yet without released estimate of loss claims, and m_2 is the number of events that will occur watched and warned by NOAA. Then total number of random variables to be sampled is $m + \tilde{N}_j$ instead of $\tilde{N}_j$ in (2), and $\tilde{L}_j^A = \frac{L(t)}{\tilde{C}_s} + \sum_{i=1}^{m+\tilde{N}_j} \tilde{U}_i^j$.

(B) If $i_t \geq 1$ event(s) have occurred and corresponding loss claims have been released partially at time *t*: $\tilde{u}_i > \tilde{d}_s, i = 1, \cdots i_t$, then for an event with a released loss index over 2.5 (i.e. $250 million insured industry losses), generate $\{\tilde{U}_i\}_1^{i_t}$ such that $\tilde{u}_i \leq \tilde{U}_i \leq c\tilde{u}_i$, where *c* is some constant and $c > 1$. Then $\tilde{L}_j^B = \sum_{i_t \geq 1} \tilde{U}_i^j$.

Combining (A) and (B), the final aggregate losses would be $\tilde{L}_j = \tilde{L}_j^A + \tilde{L}_j^B$.

- **Case 3:** The option is traded after the loss period. $N(T)$ is known completely. However, $L(T)$ may be known only partially due to the delayed insured loss reports. Thus, (2)' (B) is proceeded; and $\tilde{L}_j = \frac{L(t)}{\tilde{C}_s} + \sum_{i=1}^{m} \tilde{U}_i^j, \tilde{u}_i < \tilde{U}_i^j < c\tilde{u}_i$ where $m \leq N(T)$ is the number of events with losses over $250 million. Of course, $L(T) = 0$ if $N(T) = 0$.

I set $c = 2$ in the simulations. After Hurricane George occurred on September 28, 1998, the initial loss estimate of PCS was $750 million on October 7, 1998 and the revised estimate released in January 1999 was $1.2 billion—***1.6*** times the initial estimate.[11] For a smaller event, the accuracy of the first loss estimate may not be worse than that in this case. Thus, it is reasonable to assume that the revised estimate will be at most twice the amount of the initial estimate. Similarly, I calculate $\Pr(L \geq k_1)$ and $\Pr(L \geq k_2)$ by defining $Z_j = I(\hat{C}_s \tilde{L}_j \geq k_1)$ and

[11] The numbers are taken from "The effect of wind duration on structures in the AIR tropical cyclone model", special report of Applied Insurance Research, Inc. , March 1999.

$Z_j = I\left(\hat{C}_s \tilde{L}_j \geq k_2\right)$ in (3), respectively. $I(\cdot)$ is the indication function, and equals one when losses exceeds k_i, $i = 1,2$.

IV. Market prices versus theoretical prices

In this subsection, ***three puzzles*** are found by comparing market prices to theoretical prices for 3rd quarter E contracts. Other regional contracts with different loss periods are also examined for robustness analysis. The three puzzles are consistently observed across different types of option contracts. No theoretical models are set up to explain these puzzles in this paper, however, we do suggest that the uncertainty aversion theory (see Epstein and Wang, 1994) might help.

- **Theoretical prices**

Theoretical prices are calculated according to equation (5), that is,

$$P_t = e^{-r_f(T_1-t)} E_t\left[F(L(T))\right] = e^{-r_f(T_1-t)} E_t\left\{\max\left[0, \min\left[L(T)-k_1, k_2-k_1\right]\right]\right\}.$$

This formula indicates that option prices are determined by five factors: risk free rate r_f (-), time to maturity $(T_1 - t)$ (+), the aggregate loss index $L(T)$ (+), lower strike price, k_1 (-), and higher strike price k_2 (+). Here I use plus/minus sign to stand for the positive/negative relationship between each factor and theoretical price. Among these factors, risk free rate was relatively stable within a short time period, and is assumed to be constant. I will provide a brief explanation for other factors:

(a) **Time effects:** time enters into the above pricing formula in two ways. One is through the discount factor $e^{-r_f(T_1-t)}$; the other is through occurrence probability. Since price changes through $e^{-r_f(T_1-t)}$ are negligible for short period of time, wherever time effects are referred to in this chapter, I mean time affects price through the updating of event occurrence probability. The occurrence probability distribution updates with time, such as $\tilde{N}(T-t) \sim NB(\frac{T-t}{T}\hat{\alpha}, \overline{p})$, and $E\left[\tilde{N}(T-t)\right] = \frac{T-t}{T}\hat{\alpha}\overline{p}/(1-\overline{p})$, and $Var\left[\tilde{N}(T-t)\right] = \frac{T-t}{T}\hat{\alpha}\overline{p}/(1-\overline{p})^2$. Both expected number of events and its variance decrease as transaction time closes to the end of loss period, so the possibility that a catastrophe event occurs is smaller as $t \to T$. Therefore, during loss period, theoretical prices decline as time reaches to the end of loss period, if other factors being equal. For contracts traded prior to and posterior to the loss period, time is valued through discount factor, so theoretical prices are flat across time.

(b) **Event effects:** At time t, $L(T) = L(t) + L(T-t)$, where $L(t)$ is the known aggregate losses incurring up to time t, and $L(T-t)$ is the future aggregate losses. The larger $L(t)$, the larger probability that $L(T) = L(t) + L(T-t) > k_1$. I define effects of past catastrophe losses on option prices to be event effects. It is obvious that event effects are positive.

(c) For unknown future aggregate losses $L(T-t)$, *the heavier tailed loss severity distribution function is, the larger chances* $L(T-t)$ are to be drawn as a large number; therefore, *the higher theoretical prices are.*

(d) The higher the retention level k_1 is, other things being equal, the lower is the option price. After all mega-catastrophe event occurs with very low probability and the larger k_1 is, the smaller the chance that the PCS index exceeds k_1, then the less valuable the option is with other factors being equal.

(e) Other things being equal and with the same lower strike price k_1, the higher the contract's coverage (i.e. $k_2 - k_1$), the higher the option price.

For contracts traded during the loss period, (a) and (b) imply that if time effects dominate event effects, then theoretical prices decline as $t \to T$; if event effects dominate time effects, then theoretical prices jump up after catastrophe event occurs. For contracts traded prior to loss period, since there are neither event effects nor time effects, theoretical prices should be constant. Four call spread contracts for 3rd quarter E contracts are plotted in Figure 8, which plots the number of events, PCS loss index, market prices/theoretical prices, and ratios of theoretical prices over market prices against transaction times.[12] Since none of those call-spread contracts are triggered, theoretical prices decline to zero gradually with time. Figure 8 confirms the above arguments. For example, for argument (c), I calculated theoretical prices under different loss distribution models: one is Weibull distribution – the optimal choice based on historical loss claim data; the other is Log-logistic distribution, which has heavier tail than weibull distribution. The third row in Figure 8 indicates that prices are higher under Loglogistic distribution model (the cross signs) than those under Weibull distribution model (the small circles).

[12] If the transaction time (in month) is less than zero, the contract is traded prior to the loss period; if the time is between 0 and 3, the contract is traded during the loss period; if the time is beyond 3, then the contract is traded during the development period. In this paper, "9609" refers to the third quarter of the year 1996. A contract whose loss period is 9609 is called a 9609 contract. Other quarterly contracts are defined in the similar way.

- **Market prices.**

Market prices are symbolized by solid points in the third row of Figure 8. For contracts traded prior to loss period, market prices do decrease in k_1 and increase in $k_2 - k_1$, which are consistent with (d) and (e) mentioned above. For contracts traded during loss period, market prices jump up and down along with theoretical prices, though at different levels. It implies that similar time effects and event effects are identified. However, the differences between market prices and theoretical prices are also distinguishable. Three puzzles which can not be explained by zero beta theory are listed as follows:

(i) ***Puzzle 1*.** *Market prices are higher than theoretical prices in general, and theoretical prices are not good approximation of market prices.*

(ii) ***Puzzle 2*.** *For contracts traded prior to loss period, theoretical prices are flat, while market prices are very volatile*. Multiple market prices are observed while no information comes at all. *Multiple prices are also observed during the loss period*.

(iii) ***Puzzle 3.*** *Although market prices are closer to theoretical prices with time in general, but not always.*

Define ***price ratio*** for a specific contract to be $\hat{P}_t / P_t$, i.e. theoretical price over market price. These ratios are plotted in the fourth row of Figure 8. Under optimal loss distribution, price ratios are less than one in general. Average price ratios for all 3rd quarter E contracts are reported in the third and fourth columns of Table 21. Prior to loss period, the ratio is 43% for 9609 contracts and around 60% for 9709 and 9809 contracts, and the ratio is higher than 50% after event occurred. Therefore, theoretical prices are not good approximation of market prices. Even though Figure 8 shows that *Puzzle 1* is sensitive to the assumption of underlying loss distribution model, *Puzzle 2* and *Puzzles 3* are not. Theoretical prices are always flat no matter which loss distribution model is assumed, so there must be some fundamental reasons that lead to multiple market prices. For *Puzzle 3*, intuitively, as time closes to maturity and more information is available with event occurrence, market prices are expected to close to theoretical prices. However, the average price ratio for 9709 contracts is smaller when $N = 0$ than when $N > 0$, which contradicts with our intuition. Wrongly selected model is not an issue here since price ratios do have same shapes under different loss severity distribution models (see the fourth row of Figure 8). However, the existence of multiple prices may cause a problem.

- **Robustness**

Do the three puzzles observed from the 3rd quarter E contracts characterize catastrophe-linked securities in general? And will market prices behave the same across region and different length in loss periods? For robustness, I estimate theoretical prices for all contracts with different regional coverage and loss periods. Estimations under the alternative loss severity distributions for quarterly contracts are also provided in order to check the sensitivity of model selection. In the following section, I define the ratio of market price to theoretical price as its ***unit price,*** which is the price per unit of losses covered by a call spread option contract. Figure 9 plots National annual contracts and the above properties derived from the 3rd quarter E contracts are consistently observed. But the price ratios are much lower for these annual contracts. Further evidences can be found in Table 21 and Table 22. Table 21 is calculated for the 3rd quarter contract covering east, southeast and the whole nation in order to identify regional effects. Table 22 is calculated for all annual contracts and quarterly contracts, separately, in order to identify the effect of the length of loss period. Several features are observed:

(1) Unit prices are always greater than one.

(2) The *unit prices are higher prior to any occurrence of catastrophe events, i.e.* $N = 0$ *than those during the loss period and* $N > 0$ *in general.* One possible explanation might be like this: without the occurrence of catastrophe events, there are more uncertainties about the future losses. After an event occurs, investors update his subjective probability and modify their expectations. The more information is available, the less uncertainty investors are, and the closer market prices are to theoretical prices.

(3) *The larger the geographic region, the less expensive the contracts or more accurate are the theoretical prices.* It makes sense that investors could pre-estimate the aggregate losses better for a larger region than a smaller one. For example, it is more difficult to predict whether a hurricane will land in southeast or northeast region than to say it will land in the eastern region. However, for a contract covering a larger geographic region, the basis risks are noticeable for a small insurer (see Major in Chap.10 in Froot (1999b)).

(4) *The longer the loss period, the more expensive the contracts, or less accurate the theoretical prices are.* With shorter loss period, the probability of losses can be updated relatively more frequently than contracts having longer loss period. Also, as I mentioned in Section II, quarterly contracts are more liquid than the annual contracts.

To sum, the expected discount value of catastrophe losses covered by a contract is not a good approximation of its market price. This value can capture the market price's movement during loss period; however, the differences in price level are significant. The volatility of the market prices prior to loss period implies non-uniqueness of pricing. So further research is required to explain the multiple prices observed and narrow the gaps between theoretical prices and the market prices. However, it is beyond the scope of this paper. Two thoughts might be helpful, though: one is related to the incompleteness of catastrophe reinsurance market and less liquidity. Due to the nature of catastrophe-linked securities, such securities can hardly be traded as liquid as other securities and liquidity premium might not be ignored. The other is related to Epstein and Wang (1994)'s work. Epstein and Wang derived a formal intertemproal asset pricing model under Knightian uncertainty, and they show that the model with uncertainty may lead to equilibria that are indeterminate, that is, there may exist a continuum of equilibria for given fundamentals and sizable volatility may be observed. In this chapter, due to limited historical records of catastrophe events, option participants may not have unique prior to the distributions of frequency and loss severity. Epstein and Wang (1994)'s argument may be applicable.

V. Catastrophe-linked security market versus traditional excess-of-loss catastrophe reinsurance market

Froot (2001) examines the traditional excess-of-loss catastrophe reinsurance market, using data from Guy Carpenter & Company. Defining the **reinsurance price per unit of loss** as the ratio of the reinsurance premium over the corresponding expected losses, he finds three features: *(i) The average reinsurance prices across years are always greater than one; (ii) The reinsurance prices jumped up significantly right after the occurrence of Hurricane Andrew in August 1992, and fell strong only in 1998; (iii) The average prices for lower probability layers are considerably higher than fair value.*

The first feature is consistent with the unit prices of option contracts. For the second point, I do observe unit prices decrease and price ratios rise with year (see Table 21 to Table 24). However, it is not sure whether we observe the reinsurance cycle or not since we have not seen another catastrophe events like Hurricane Andrew occurred during the period of 1996 and 1999. On the other hand, less uncertainty in PCS index and familiarity might also be reasonable explanations. For the last point, Froot linked the average reinsurance prices with the probabilities of layers and found a negative relationship between the two variables. Intuitively, it is the relationship between average prices and average catastrophe risks transferred. This result is counter-intuitive since a contract covering lower probability of layer contains lower risks, so its price per unit of losses should be lower, not higher. He listed six possible explanations, including limited capacity of reinsurance industry, reinsurance market power, frictional costs, and individual behavior factors, and so on. Among these reasons, he is inclined to the limited capacity theory (see Froot and O'Connell, Chap. 5 in Froot (1999b)). He argues that the imperfect capital market causes the marginal cost of capital for the protection of layers with lower probabilities is higher, so does the reinsurance prices per unit of losses. However, limited capacity is likely to be an issue with exchange-based PCS options; do we still observe the same negative relationship between unit prices and catastrophe risks transferred? A new measure of catastrophe risks is defined in the following paragraph and detailed arguments follow.

The catastrophe-linked security market works in such a way that a willing buyer's demand for a particular layer of protection meets up with a willing seller at mutually agreeable price, no limited capital capacity and no market power. Except for the investor's preference and behavior factors, option prices should be determined by the underlying catastrophe risks transferred. Probabilities of layer of protections are highly correlated with catastrophe risks transferred by contracts; however, as my understanding, these probabilities are defined as $\Pr(k_1 \le L \le k_2)$ and do not consider the absolute levels of $\Pr(L > k_1)$ and $\Pr(L > k_2)$. Contract information is not fully employed. Therefore, by holding the call spread $k_2 - k_1$ of contracts to be a constant, I use ***exceedence probability*** $\Pr(L > k_1)$ as a measure of catastrophe risks transferred. The higher k_1, the smaller $\Pr(L > k_1)$, or the less catastrophic risks transferred. Table 5 summaries the distribution of contract spreads, $k_2 - k_1$, and we notice that around 60 percent of contracts have spread 20, i.e. $2 billion catastrophe loss coverage. Therefore, the relationship between unit prices and catastrophe risks is analyzed for contracts with call spread equal to 20. The average exceedence probabilities and unit prices are reported in Table 23 and Table 24. Table 23 is for the three annual contracts: NA, W, and CA. Table 24 is for E, SE, and FL contracts. Since California is one of the western states; Florida is one of southeastern states, southeastern region is a part of whole eastern region; the regional effects argued above are confirmed again. In addition, comparing Table 23 to Table 24, I observe that annual contracts have higher unit prices than those quarterly contracts under consideration. The fourth property under robustness is observed clearly. In all, these tables indicate a negative relationship between catastrophe risks and the unit prices across years.

<u>Puzzle 4.</u> *The lower the exceedence probability, the lower catastrophe risks transferred. As a result, the higher the unit price is, or the more expensive a contract is.*

This result is consistent with the result in Froot (2001), and this negative relationship between unit prices and exceeddence probabilities are more significant for annual contracts tracing NA index. Compared catastrophe-linked security market to the traditional excess-of-loss catastrophe reinsurance market, the former has no strong market power, no limited capacity; and has high transparency and low transaction cost. Therefore, above result can not be explained by limited capacity and market power as mentioned in Froot (1997, 2001), and Froot and O'Connell (chap. 5 in Froot (1999b)). The reasons for the negative relationship lie elsewhere. The negative relationship is explained intuitively by Figure 10 and Figure 11. Figure 10 contains five columns: annual contracts, 3rd quarter contracts, 3rd quarter **E** contracts, 3rd quarter **SE** contracts, and 2nd quarter **MW** contracts. Call spreads equal 20 for the first four columns and 10 for the last column.

(a) *Prior to loss periods*. There is no probability update at all. The smaller k_1, the larger $\Pr(L \geq k_1)$. Therefore, contracts with smaller lower price k_1 will lie to the right of those with higher k_1. Due to the existence of multiple market prices, multiple unit prices are observed at the same exceedence probability value. However, the three graphs for annual contracts, 3rd quarter contracts, and 2nd quarter MW contracts are roughly downward sloping. Intuitively, for a contract with very small probability to be triggered, its actuarial fair price is close to zero; however, its market price can not be too low, at least it should be larger than the transaction cost per trade in order to have a deal made. Therefore, with the decline of exceedence probability, transaction costs have heavier weights in market prices, and unit prices as ratio of market prices to actuary-fair prices rise. As for those **E** and **SE** contracts, no clear trend is observed. Both multiple prices and narrow range of their exceedence probabilities may contaminate any trend, if existed.

(b) *During loss periods.* The exceedence probability of a contract increases with the number of events due to ***event effects***; and decreases with time as $t \rightarrow T$ due to ***time***

effects. However, unit prices decline with number of events and time (as $t \to T$).[13] Therefore, the unit prices will be negatively related to exceedence probabilities if event effects dominate time effects; and positively related to exceedence probabilities, otherwise; and mixed relationship (or roughly concave curve) will exist if the two effects appear intangibly. Figure 10 shows that the negative relationship is more significant for all chosen annual contracts. For annual contracts most of them covering losses in whole US (i.e. NA contracts) and western regions (W contracts), number of events grew steadily with time; then event effects dominated time effects, and unit prices declined quickly with more occurrence of catastrophe events. For each state-specific contract, the number of catastrophe events grew slowly with time in most of years, and then event effects were not strongly enough to dominate time effects. Hence, negative relationship is more likely observed at the right end of horizon axis, and positive relationship is more likely at the left end of the horizon axis. Thus, based on my argument, the negative relationship observed by Froot (2001)—all of the reinsurance contracts are annual contracts, may be explained (at least some percentage of the negative relationship may be explained) without considering the limited capacity in the reinsurance industry.

(c) *Robustness of model selection.* Figure 11 plots under the alternative loss severity distributions for above quarterly contracts. All the graphs have similar shapes to those in Figure 10. Once again, the results show that incorrect selection of distribution models is not a main issue here. The wrongly selected distribution model will give wrong estimates in price level, but it does not contaminate above general conclusions. Therefore, limited capacity and reinsurance market power may help to explain above

[13] Table 21 and Table 22; the last rows of Figure 8 and Figure 9 could help to explain this statement.

results, but they are not essential reasons. It is the special risks – low probability with extremely high losses, transferred through catastrophe-linked securities or reinsurance contracts that determine the specialties of this new asset class.

VI. Conclusion

Based on PCS options traded at the CBOT between 1996 and 1999, I find that half of these contracts were traded prior to loss periods and others were traded around the occurrence of catastrophe events, only few contracts were traded during the development period. The less likely a contract is triggered, the less liquid the contract is.

By assuming that the theoretical price of a contract equals its actuarial fair price, I find that theoretical prices are not good approximation of market prices. Market prices are quite volatile prior to loss period, while theoretical prices are flat; and theoretical prices can only explain 25%-45% and 8.5%-28% of market prices for quarterly contracts and annual contracts, respectively, when transactions were made with $N = 0$. Even though theoretical prices move up and down along with market prices when catastrophe events occurred during the loss period, the differences in price level are still significant. In addition, the gaps between theoretical prices and market observations are larger for contracts covering larger regions or/and with shorter loss periods. New theory is required to explore what fundamental prices of catastrophe-linked securities are. In addition, in light of the results in this chapter, there is no reason to believe that cat bonds should generate a risk free rate, either. A higher than risk free rate would be more reasonable.

Comparing PCS call spread option contracts to the traditional excess-of-loss catastrophe reinsurance examined by Froot (2001), I find that the pricing patterns of the two assets are similar. Especially, unit prices (or reinsurance prices per unit of losses) are negatively correlated with catastrophe risks transferred in most of cases. However, detailed analyses on PCS options show that the negative relationship is true only when event effects dominated time effects during loss periods. Limited capacity or reinsurance market power may exaggerate the negative relationship and reinsurance cycle problems, but not the essential reasons. The special risks embedded in catastrophe-linked securities — the lower probability with extremely high losses or the strong event effects are.

Finally, as a supplement market of the traditional reinsurance market, the development of catastrophe-linked security market was slow since there were no extremely damaged catastrophe events occurred during that period and reinsurers' undercutting activities. At the same time, the ratio of losses reinsured to the total world economic losses was 23% in 1999; and the ratio was only 16% in 1998. The total world economic losses increased to $US 100 billion in 1999 from US $94 billion in 1998 (see, "The world catastrophe reinsurance market", 1999 and 2000). These numbers show that both the total world economic losses and percentage of catastrophe losses reinsured are increasing with time; however, the current catastrophe reinsurance is far from enough. With limited catastrophe reinsurance and the increasing world demand for reinsurance protection with the development of world economy, the catastrophic security market will flourish.

Appendix I. Zero Beta

There are several existing papers showing that the correlation between the insured catastrophe losses and returns on financial assets is zero, based on a calculation of the correlation between insured catastrophe losses and market returns, which are usually annually recorded. This zero correlation is referred to as zero beta theory (Froot, 1995). Within a regression model framework, this paper tries to determine whether or not mega-catastrophic events have any impact on stock market returns. First, based on the catastrophe losses, three dummy variables are defined. The raw data used here encompasses the daily catastrophic event losses from year 1950 to 1999 and is provided by PCS.[14] Data are grouped to construct a sequence of monthly national event losses. Let D_i be the i^{th} dummy variable and $D_{1t} = 1$ if the losses are greater than 2 billion dollars, 0 otherwise; $D_{2t} = 1$ if the losses are between 1 billion and 2 billion dollars, 0 otherwise; and $D_{3t} = 1$ if the losses are between 500 million and 1 billion dollars, 0 otherwise. Second, monthly value-weighted stock index returns from the NYSE/AMEX are used as the proxy of market returns. As most recent papers show that dividend yield on those indexes, short term interest rates and long term-short term bond yield spread have predictive power for the real asset returns (Campbell, 1996), a VAR model is set to catch their forecasting power to the market returns. Finally, the dummies are added into the VAR model to determine whether those variables have any explanatory power. The lag effects

[14] Loss claim data are provided by the Property Claims Service. They reflect PCS's best judgment of the total industry net insurance payment for personal and commercial property lines of insurance covering fixed property, personal property, time-element losses, vehicles, boats, and related property items, but do not include losses involving uninsured property, including uninsured publicly owned property and utilities, agriculture, aircraft, and property insured under the National Flood Insurance Program or Write-Your-Own-Program.

of mega-catastrophe events are also considered by including the lagged dummy variables.[15] The mathematical model is as follows:

$$x_t = Ax_{t-1} + \sum_{i=1}^{3} \alpha_i D_{i,t} + \sum_{i=1}^{3} \beta_i D_{i,t-1} + \varepsilon_t,\ E_{t-1}(\varepsilon_t) = 0,\ E_{t-1}(\varepsilon_t^2) = h_t$$

where $x_t = (x_{1,t}, x_{2,t}, x_{3,t}, x_{4,t})$ and $x_{1,t}$ is the value weighted stock index returns, $x_{2,t}$ is the dividend yield, $x_{3,t}$ and $x_{4,t}$ are the U.S. 3-month Treasury Bill rate less than its 1-year backward moving average and 10-year bond yield spread over 3-month Treasury bill rates at time *t*, respectively. All these variables are demeaned before estimation. A is a 4 by 4 coefficient matrix. Both α_i and β_i, $i = 1,2,3,4$, are 4 by 1 vectors, which measure the impact of catastrophic events on $x_{i,t}$. The zero beta theory here means that $\alpha_i = \beta_i = 0$ for all $i = 1,2,3,4$ will not be rejected at the significant level of 5%, for instance. The estimation results are shown in Table 3 with standard deviations in parentheses. The patterns of coefficients that reported in Table 3 are similar to many that have been estimated in the literature. The coefficient for the lagged dividend yield in dividend yield equation is 1.0029 with standard deviation 0.0029. It seems that a non-stationarity is observed in the dividend yield series. Actually, the series is not non-stationary. The time series analysis shows that the dividend yield is better modeled by ARMA(1,1) model with coefficient for AR(1) 0.99713 (std=0.0032) and MA(1) -0.02686 (std=0.0427). The autocorrelation is persistent, but still stationary. Since the focus of this paper is the forecasting power of the mega-catastrophic losses on the market excess returns, simple VAR model is used here. *t*-statistics show that none of the coefficients of the dummies are significantly different from zero at the significance level of 5%. In addition, the second last row of the table contains the *F-test* (joint

[15] The regression model is also estimated with 1-year lagged dummies and same conclusions are drawn.

test for zero coefficients in front of the dummies in each equation), and all of these are smaller than the critical value $\alpha_{0.5} = 2.10$. So, the zero beta theory is not rejected at the significance level of 5%. Also, most of the PCS option contracts cover losses over $4 billion, which have a much smaller probability of event occurrence than the threshold values used here to generate the three dummy variables. Thus, I expect higher threshold values will not change my conclusion. In fact, I have run the regression models based on different threshold values, and the results are consistent with my findings here.

Appendix II. Relevant Statistics

Suppose the number of occurrences of catastrophe events follows Negative Binomial distribution $NB(\alpha, p)$ with the probability density as

$$p_k(T) = \Pr(N(0,T] = k) = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)k!}(1-p)^{\alpha} p^k, k = 0,1,2,\cdots,$$

where T is the length of loss period. I assume the point process is stationary and independent among the non-overlapped time period. Suppose in the time interval $(0,t], t \le T$, $N(0,t] \sim NB(\alpha_t, p_t)$, then we have

Theorem 1. $\alpha_t = \frac{t}{T}\alpha,$ and $p_t = p$.

Proof: Define $N(x) = N(0,x]$ **to** be the number of catastrophe events occurred during the period $(0,x]$. Due to independency and stationarity, it is easy to obtain following two equations:

$$E[N(x+y)] = E[N(x)] + E[N(y)]$$
$$Var[N(x+y)] = Var[N(x)] + Var[N(y)]$$

Then we have

$$E[N(x)] = E[N(1)]x; \quad Var[N(x)] = Var[N(1)]x$$

$$E[N(x)] = \frac{x}{y}E[N(y)]; \quad Var[N(x)] = \frac{x}{y}Var[N(y)]$$

For the Negative Binomial process, we have $E[N(T)] = \alpha p/(1-p)$ and $Var[N(T)] = \alpha p/(1-p)^2$. Then

$$E[N(t)] = \frac{\alpha_t p_t}{1-p_t} = \frac{t}{T}E[N(T)] = \frac{t}{T}\frac{\alpha p}{1-p}$$

$$Var[N(t)] = \frac{\alpha_t p_t}{(1-p_t)^2} = \frac{t}{T} Var[N(T)] = \frac{t}{T} \frac{\alpha p}{(1-p)^2}$$

The theorem can be concluded easily.

In addition, changing in the threshold value d will affect event frequency. In the following two theorems, I will give the transformation formula among different threshold values. Both Negative Binomial and Poisson distribution are considered.

Theorem 2. *Let U be the event loss claims and $N \sim NB(\alpha, p)$. Denote N and $\tilde{N}$ to be the number of events with losses over $d > 0$ and $\tilde{d}$ $(\tilde{d} > d)$, respectively. Then $\tilde{N} \sim NB(\alpha, \tilde{p})$, where $\tilde{p} = pq/[1 - p(1-q)]$ and $q = \Pr(U > \tilde{d})/\Pr(U > d)$.*

Proof: For any $k = 0,1,\cdots,$ we have

$$\tilde{p}_k = \Pr\left(\tilde{N} = k \middle| U_i > \tilde{d}, i = 0,\cdots,k\right)$$

$$= \sum_{n=k}^{\infty} p_k C_n^k q^k (1-q)^{n-k}$$

$$= \sum_{n=k}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!} (1-p)^{\alpha} p^n C_n^k q^k (1-q)^{n-k}$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!} \left(\frac{1-p}{1-p(1-q)}\right)^{\alpha} \left(\frac{pq}{1-p(1-q)}\right)^{k}$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)k!} (1-\tilde{p})^{\alpha} \tilde{p}^{k}$$

❑

Theorem 3. *Let* U *be the event loss claims and* $N \sim Poi(\lambda)$. *Denote* N *and* $\tilde{N}$ *to be the number of events with losses over* $d > 0$ *and* $\tilde{d}$ $\left(\tilde{d} > d\right)$, *respectively. Then* $\tilde{N} \sim Poi(\tilde{\lambda})$, *where* $\tilde{\lambda} = \lambda q$ *and* $q = \Pr(U > \tilde{d}) / \Pr(U > d)$.

Proof: For any $k = 0,1,\cdots,$ we have

$$
\begin{aligned}
\tilde{p}_k &= \Pr\left(\tilde{N} = k \middle| U_i > \tilde{d}, i = 0,\cdots,k\right) \\
&= \sum_{n=k}^{\infty} p_k C_n^k q^k (1-q)^{n-k} \\
&= \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} C_n^k q^k (1-q)^{n-k} \\
&= \frac{e^{-\lambda q}(\lambda q)^k}{k!}
\end{aligned}
$$

❑

- **Construction cost index and population regression models (1950-1994)**

In this section, the construction cost index and population growth models are estimated for 3rd quarter eastern region contracts. Other regional contracts with different loss periods are calculated similarly. Standard deviations are in the parenthesis. *cci* stands for quarterly construction cost index, and it is demeaned first. *pop* stands for population.

1. *Construction cost index model (1950:3-1994:3):* trend fit vs. AR(1) model

$$\log(cci_t) = -2.4806 + 0.0598t + \varepsilon_t, \quad R^2 = 0.9798$$
(0.0335) (0.0013)

$$\log(cci_t) = 0.0535 + 0.9947\log(cci_{t-1}) + \varepsilon_{t-1}, \quad R^2 = 0.9988$$
(0.0042) (0.0054)

In order to choose between the trend model and the AR(1) model, I forecast the construction cost index for year 1995 to 2000. In Figure 2, the circles stand for the true data and the cross signs trace the fitted values before 1995 and forecasted values after 1995. It shows that AR(1) model is better fitted than trend model. Similar results are derived for other 28 different contracts. The estimated model is used to estimate cost index for the sake of pricing.

2. *Population growth model(1950:3-1994:3)*

$$\log(pop_t) = \underset{(0.0024)}{18.0207} + \underset{(0.0005)}{0.01952}t - \underset{(0.0000)}{0.00035}t^2 + \underset{(0.0000)}{3.478\times10^{-6}}t^3 + \varepsilon_t, \quad R^2 = 0.9991$$

Figure 3 plots the true data along with fitted and predicted data. It turns out to be a good fit. Similar estimates have done for other 28 contracts.

Appendix III. Tables and Figures

Table 3 Monthly PCS National Loss Claim data for period 01/1954—12/1999[16]

$x_{1,t}$: value weighted stock index returns; $x_{2,t}$: dividend yield; $x_{3,t}$: U.S. 3-month treasure bill rates less than its 1-year backward moving average; $x_{4,t}$: 10-year bond yield spread over 3-month T-bill rates;

D_i be the i^{th} dummy variable. $D_{1t} = 1$ if the losses are greater than 2 billion dollars, 0 otherwise; $D_{2t} = 1$ if the losses are between 1 billion and 2 billion dollars, 0 otherwise; and $D_{3t} = 1$ if the losses are between 500 million and 1 billion dollars, 0 otherwise. Estimates in boldface are significantly different from zero at the 5% significance level.

Regressors	**Dependent Variable**			
	$x_{1,t}$	$x_{2,t}$	$x_{3,t}$	$x_{4,t}$
$x_{1,t-1}$	0.0171 (0.0432)	0.0031 (0.0000)	0.0005 (0.0005)	-0.0002 (0.0006)
$x_{2,t-1}$	**-7.3810 (2.5318)**	**1.0002 (0.0029)**	0.0320 (0.0270)	**-0.0012 (0.0355)**
$x_{3,t-1}$	**-8.4056 (2.6569)**	-0.0078 (0.0031)	**0.8122 (0.0285)**	**0.2428 (0.0373)**
$x_{4,t-1}$	1.9601 (2.1303)	-0.0005 (0.0024)	-0.0564 (0.0229)	0.9172 (0.0299)
$D_{1,t}$	-0.0079 (0.0142)	0.0000 (0.0000)	0.0001 (0.0002)	-0.0001 (0.0002)
$D_{2,t}$	0.0043 (0.0128)	0.0000 (0.0000)	0.0001 (0.0001)	-0.0001 (0.0002)
$D_{3,t}$	-0.0021 (0.0085)	0.0000 (0.0000)	0.0000 (0.0001)	0.0001 (0.0001)
$D_{1,t-1}$	-0.0108 (0.0145)	0.0000 (0.0000)	0.0000 (0.0002)	0.0001 (0.0002)
$D_{2,t-1}$	-0.0043 (0.0130)	0.0000 (0.0000)	0.0000 (0.0001)	0.0000 (0.0002)
$D_{3,t-1}$	0.0081 (0.0082)	0.0000 (0.0000)	0.0000 (0.0001)	0.0001 (0.0001)
F(6,n-10)	0.3798	1.1183	0.0706	0.4775
R^2	0.05818	0.996	0.7167	0.6775

[16] 10-year bond rates and 3-month Treasure-bill rates are from the Federal Reserve Board. Value-weighted stock index returns are from CRSP. Since the 10-year bond yields are available from April 1953. The sample period for the VAR model is between 01/1954 and 12/1999.

Table 4 Average values of lower strike prices and contract limits (in $100 million)

	1^{st}		2^{nd}		3^{rd}		4^{th}	
	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2
NA	21	35	14	25	40	60	40	60
E	18	30	---	---	60	82	37	57
SE	---	---	---	---	76	99	188	226
NE	10	20	---	---	72	109	89	135
FL	5	15	---	---	76	96	80	100
MW	---	---	9	19	---	---	---	---
TX	---	---	8	18	60	84	---	---

Annual Contracts		
	k_1	k_2
NA	93	168
W	107	274
CA	73	110

Table 5 Distribution of Contract Spreads

Loss period	Spread ($k_2 - k_1$) ($100 million)						
	5	10	15	20	30	50	150 call
Annual	1.96%	---	---	71.24%	---	3.27%	23.53%
Quarter	0.68%	29.83%	3.73%	50.85%	0.68%	14.23%	---
3^{rd}+4^{th}	---	---	---	65.96%	---	34.04%	---
T**otal**	1.01%	17.78%	2.22%	58.59%	0.40%	12.73%	7.27%

Table 6 Frequency Analysis, Annual contracts

Region	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
National	4.1861 (1.0736)	0.8014 (0.0428)	163.2145	16.8913 (0.6060)	217.0852
West	1.6401 (0.6448)	0.5819 (0.1025)	91.9322	2.2826 (0.2228)	101.9376
CA	1.7740 (1.1639)	0.3937 (0.1622)	67.8828	1.1521 (0.1583)	70.2238

Table 7 Frequency Analysis, Quarterly contracts, National

Quarter	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
1st	2.5694 (1.1054)	0.5577 (0.1104)	103.4734	3.2391 (0.2654)	110.9931
2nd	4.6683 (1.6526)	0.6150 (0.0863)	129.8491	7.4565 (0.4026)	142.9603
3rd	3.9941 (1.8355)	0.4731 (0.1176)	103.8772	3.5870 (0.2792)	109.4409
4th	1.7658 (0.6964)	0.6042 (0.1003)	98.0686	2.6957 (0.2421)	108.2747
3rd+4th	3.4092 (1.1140)	0.6467 (0.0780)	125.6708	6.2391 (0.3683)	143.5199

Table 8 Frequency Analysis, Quarterly contracts, Eastern

Quarter	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
1st	2.4244 (1.4643)	0.3753 (0.1462)	74.8150	1.4565 (0.1779)	77.6513
3rd	1.9986 (1.3294)	0.3700 (0.1601)	68.2624	1.1739 (0.1597)	70.5836
4th	1.8486 (1.1821)	0.3928 (0.1580)	69.0051	1.1957 (0.1622)	71.5127
3rd+4th	2.4821 (1.1288)	0.4861 (0.1185)	91.2824	2.3478 (0.2259)	97.4655

Table 9 Frequency Analysis, Quarterly contracts, Northeast

Quarter	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
1st	0.6032 (0.3027)	0.5578 (0.1387)	54.9514	0.7609 (0.1286)	62.2881
3rd	3.7680 (5.6631)	0.1433 (0.1862)	49.1142	0.6304 (0.1171)	49.4283
4th	---	---	---	0.5870 (0.1130)	45.2569
3rd+4th	4.6040 (4.8776)	0.2002 (0.1714)	65.8021	1.1522 (0.1583)	66.4580

Table 10 Frequency Analysis, Quarterly contracts, Southeast

Quarter	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
1st	6.6601 (12.164)	0.1206 (0.1944)	58.7503	0.9130 (0.1409)	58.9385
3rd	3.8858 (6.1721)	0.1478 (0.2016)	50.9900	0.6739 (0.1210)	51.2644
4th	---	---	---	0.6522 (0.1191)	59.6673

Table 11 Frequency Analysis, Quarterly contracts, FL

Quarter	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
1^{st}	0.2087 (1.558)	0.5555 (0.2131)	28.1271	0.2609 (0.0753)	31.9960
3^{rd}	1.4609 (2.9094)	0.1407 (0.2439)	27.9364	0.2391 (0.0721)	28.1245
4^{th}	0.5027 (0.8522)	0.2060 (0.2872)	18.5381	0.1304 (0.0533)	18.9144
$3^{rd}+4^{th}$	0.5450 (0.4396)	0.4041 (0.2084)	36.5136	0.3696 (0.0896)	38.8562

Table 12 Frequency Analysis, Quarterly contracts, MidWest and Texas

Quarter	$NB(\alpha, p)$			$Poisson(\lambda)$	
	$\hat{\alpha}$	$\hat{p}$	$-Ln(L)$	$\hat{\lambda}$	$-Ln(L)$
2^{nd}	2.8578 (1.5285)	0.4195 (0.1343)	86.2421	2.0652 (0.2119)	90.1433
3^{rd}	0.9819 (0.8648)	0.3069 (0.1959)	40.4848	0.4348 (0.0972)	41.9157
$3^{rd}+4^{th}$	1.0624 (0.6045)	0.4622 (0.1508)	60.9002	0.9130 (0.1409)	64.8246
MidWest					
2^{nd}	7.3677 (4.3369)	0.3780 (0.1400)	108.1080	4.4783 (0.3120)	110.8206

Table 13 Loss severity distribution estimates, Annual contracts

Region	Distribution Family	Parameter estimates
National	***Pareto*** (α, λ)	$\hat{\alpha}$ =1.2299 (0.0793) $\hat{\lambda}$ =0.1053 (0.0132)
West	***Loglogistic*** (δ, λ)	$\hat{\delta}$ =1.0608 (0.0869) $\hat{\lambda}$ =0.2451 (0.0377)
California	***Loglogistic*** (δ, λ)	$\hat{\delta}$ =1.1373 (0.1475) $\hat{\lambda}$ =1.1615 (0.2985)

Table 14 Loss severity distribution estimates, Quarterly contracts, MidWest

Quarter	Distribution Family	Parameter estimates
2^{nd}	***Lognormal*** (μ, σ)	$\hat{\mu}$ =-1.3411 (0.1244) $\hat{\sigma}$ = 1.3473 (0.0855)
	Loglogistic (δ, λ)	$\hat{\delta}$ =1.2759 (0.0833) $\hat{\lambda}$ =0.1849 (0.0230)

Table 15 Loss severity distribution estimates, Quarterly contracts, National

Quarter	Distribution Family	Parameter estimates
1st	***Lognormal*** (μ, σ)	$\hat{\mu}$ =-2.7302 (0.2316) $\hat{\sigma}$ = 1.6924 (0.1427)
	Loglogistic (δ, λ)	$\hat{\delta}$ =1.0494 (0.0823) $\hat{\lambda}$ =0.0631 (0.0113)
2nd	***Lognormal*** (μ, σ)	$\hat{\mu}$ =-2.1709 (0.0819) $\hat{\sigma}$ = 1.3061 (0.0598)
	Loglogistic (δ, λ)	$\hat{\delta}$ =1.3048 (0.0640) $\hat{\lambda}$ =0.0585 (0.0077)
3rd	***Pareto*** (α, λ)	$\hat{\alpha}$ =0.8805 (0.0931) $\hat{\lambda}$ =0.0369 (0.0106)
	Loglogistic (δ, λ)	$\hat{\delta}$ =0.9292 (0.0740) $\hat{\lambda}$ =0.0525 (0.0089)
4th	***Loglogistic*** (δ, λ)	$\hat{\delta}$ =1.0796 (0.09609) $\hat{\lambda}$ =0.0743 (0.01493)
	Pareto (α, λ)	$\hat{\alpha}$ =1.1060 (0.1669) $\hat{\lambda}$ =0.0981 (0.0296)
3rd+4th	***Pareto*** (α, λ)	$\hat{\alpha}$ =0.9477 (0.0826) $\hat{\lambda}$ =0.0544 (0.0113)
	Loglogistic (δ, λ)	$\hat{\delta}$ =0.9779 (0.0583) $\hat{\lambda}$ =0.0621 (0.0079)

Table 16 Loss severity distribution estimates, Quarterly contracts, Texas

Quarter	Distribution Family	Parameter estimates
2nd	***Lognormal*** (μ, σ)	$\hat{\mu}$ =-0.5788 (0.3107) $\hat{\sigma}$ = 1.5786 (0.1641)
	Loglogistic (δ, λ)	$\hat{\delta}$ =1.1619 (0.1062) $\hat{\lambda}$ =0.6415 (0.1394)
3rd	***Pareto*** (α, λ)	$\hat{\alpha}$ =0.7998 (0.2573) $\hat{\lambda}$ =0.7954 (0.6146)
	Lognormal (μ, σ)	$\hat{\mu}$ =-0.1681 (1.1018) $\hat{\sigma}$ = 2.1276 (0.6147)
3rd+4th	***Pareto*** (α, λ)	$\hat{\alpha}$ =0.9496 (0.2283) $\hat{\lambda}$ =0.6447 (0.3852)
	Loggamma (α, λ)	$\hat{\alpha}$ =0.7292 (0.3954) $\hat{\lambda}$ =0.9228 (0.3072)

Table 17 Loss severity distribution estimates, Quarterly contracts, East

Quarter	Distribution Family	Parameter estimates
1st	***Weibull*** (c,δ)	$\hat{c}$ =3.0829 (0.4978) $\hat{\delta}$ =0.2999 (0.0692)
	Loggamma (α,λ)	$\hat{\alpha}$ =0.0365 (0.1522) $\hat{\lambda}$ =1.0729 (0.3351)
3rd	***Weibull*** (c,δ)	$\hat{c}$ =5.0113 (2.7639) $\hat{\delta}$ =0.1039 (0.0621)
	Loglogistic (δ,λ)	$\hat{\delta}$ =0.5290 (0.0963) $\hat{\lambda}$ =0.0600 (0.0732)
4th	***Loggamma*** (α,λ)	$\hat{\alpha}$ =0.2435 (0.1862) $\hat{\lambda}$ =0.9891 (0.3190)
	Pareto (α,λ)	$\hat{\alpha}$ =0.9791 (0.4795) $\hat{\lambda}$ =0.1929 (0.0738)
3rd+4th	***Weibull*** (c,δ)	$\hat{c}$ =3.3271 (0.7782) $\hat{\delta}$ =0.1755 (0.0464)
	Loglogistic (δ,λ)	$\hat{\delta}$ =0.6863 (0.0798) $\hat{\lambda}$ =0.1491 (0.0430)

Table 18 Loss severity distribution estimates, Quarterly contracts, Northeastern

Quarter	Distribution Family	Parameter estimates
1st	***Gamma*** (α,λ)	$\hat{\alpha}$ =0.0655 (0.1795) $\hat{\lambda}$ =0.2337 (0.1133)
	Lognormal (μ,σ)	$\hat{\mu}$ =-1.1957 (0.4967) $\hat{\sigma}$ = 1.7318 (0.3197)
3rd	***Weibull*** (c,δ)	$\hat{c}$ =3.4116 (1.6257) $\hat{\delta}$ =0.1993 (0.1022)
	Loggamma (α,λ)	$\hat{\delta}$ =0.0000549 (0.0032) $\hat{\lambda}$ =0.6573 (0.1925)
4th	***Loglogistic*** (δ,λ)	$\hat{\delta}$ =1.2049 (0.2363) $\hat{\lambda}$ =0.7042 (0.2353)
	Lognormal (μ,σ)	$\hat{c}$ =-0.3636 (0.3855) $\hat{\delta}$ =1.5657 (0.2833)
3rd+4th	***Weibull*** (c,δ)	$\hat{c}$ =2.0676 (0.5138) $\hat{\delta}$ =0.2977(0.0763)
	Loglogistic (δ,λ)	$\hat{\delta}$ =0.8623 (0.1296) $\hat{\lambda}$ =0.2973 (0.0911)

Table 19 Loss severity distribution estimates, Quarterly contracts, Southeastern

Quarter	Distribution Family	Parameter estimates
1st	***Lognormal*** (μ,σ)	$\hat{\mu}$ =-2.9376 (1.2346) $\hat{\sigma}$ = 2.1382 (0.4671)
	Pareto (α,λ)	$\hat{\alpha}$ =0.9684 (0.1411) $\hat{\lambda}$ =0.1178 (0.0438)
3rd	***Weibull*** (c,δ)	$\hat{c}$ =4.7061 (4.1880) $\hat{\delta}$ =0.0906 (0.0786)
	Loglogistic (δ,λ)	$\hat{\delta}$ =0.4289 (0.1134) $\hat{\lambda}$ =0.0058 (0.1861)
4th	***Weibull*** (c,δ)	$\hat{c}$ =2.7556 (1.0114) $\hat{\delta}$ =0.2487 (0.0966)
	Loggamma (α,λ)	$\hat{\alpha}$ =0.1599 (0.2761) $\hat{\lambda}$ =0.8119 (0.3369)

Table 20 Loss severity distribution estimates, Quarterly contracts, FL

Quarter	Distribution Family	Parameter estimates
1st	***Loggamma*** (α,λ)	$\hat{\alpha}$ =0.0000341 (0.001648) $\hat{\lambda}$ =0.9129123 (0.294514)
	Loggamma (α,λ)	$\hat{\alpha}$ =0.0000164 (0.0003792) $\hat{\lambda}$ =0.0631220 (0.0270537)
3rd	***Weibull*** (c,δ)	$\hat{c}$ =1.1303 (1.0363) $\hat{\delta}$ =0.2082 (0.1268)
	Loggamma (α,λ)	$\hat{\alpha}$ =0.6021 (0.5880) $\hat{\lambda}$ =0.3442 (0.2134)
4th	***Loggamma*** (α,λ)	$\hat{\alpha}$ =0.2435 (0.1862) $\hat{\lambda}$ =0.9891 (0.3190)
	Gamma (α,λ)	$\hat{\alpha}$ =0.9791 (0.4795) $\hat{\lambda}$ =0.1929 (0.0738)
3rd+4th	***Weibull*** (c,δ)	$\hat{c}$ =260.5851 (17921.6365) $\hat{\delta}$ =0.00268 (0.1841)
	Loggamma (α,λ)	$\hat{\alpha}$ =0.0000396 (0.0005232) $\hat{\lambda}$ =0.0308052 (0.0155530)

Table 21 Third Quarter Contracts

Variables	Contract	Eastern		Southeastern		National	
		$N=0$	$N>0$	$N=0$	$N>0$	$N=0$	$N>0$
Market price /theoretical prices	9609	2.43	2.19	5.23	4.88	7.06	2.69
	9709	1.70	1.84	4.10	4.35	2.60	1.69
	9809	1.69	---	3.79	1.25	---	---
Theoretical price /market price	9609	0.43	0.50	0.21	0.30	0.35	0.52
	9709	0.60	0.56	0.25	0.29	0.49	0.60
	9809	0.61	---	0.40	0.80	---	---
Rate on line	9609	0.20	0.12	0.14	0.11	0.11	0.17
	9709	0.14	0.11	0.13	0.11	0.16	0.18
	9809	0.16	---	0.07	0.13	---	---

Table 22 Annual contracts vs. Quarterly contracts

Variables	Contract	Annual		Quarter	
		$N=0$	$N>0$	$N=0$	$N>0$
Market price /theoretical prices	1996	11.875	10.9656	4.3068	3.3376
	1997	11.957	7.2575	5.5310	4.0456
	1998	4.5980	3.4561	3.4902	2.9644
Theoretical price /market price	1996	0.0850	0.1638	0.2936	0.4170
	1997	0.0898	0.1464	0.2691	0.3602
	1998	0.2830	1.0561	0.4533	0.8811
Rate on line	1996	0.1438	0.1617	0.1570	0.1555
	1997	0.1845	0.2329	0.1391	0.1125
	1998	0.2063	0.2442	0.1267	0.2421

Table 23 Annual contracts with call spread=20

Variables	Contract	NA	W	CA
Market price /theoretical prices	1996	2.9286	12.7576	---
	1997	8.1040	10.5370	17.8
	1998	3.633	---	11.875
$\Pr(L>k_1)$	1996	0.3857	0.01182	---
	1997	0.0456	0.01258	0.0048
	1998	0.2861	---	0.0109

Table 24 3rd quarter contracts with call spread=20 under the optimal distributions

Variable	Contract	E	SE	FL
Market price /theoretical prices	9609	2.9055	5.4841	7.50
	9709	1.6596	4.2157	5.75
	9809	1.6254	2.8500	---
$\Pr(L>k_1)$	9609	0.0783	0.0301	0.0111
	9709	0.0800	0.0338	0.0094
	9809	0.1167	0.0342	---

Figure 1. Raw and Adjusted losses (3rd quarter, 01/1950-12/1995, East)

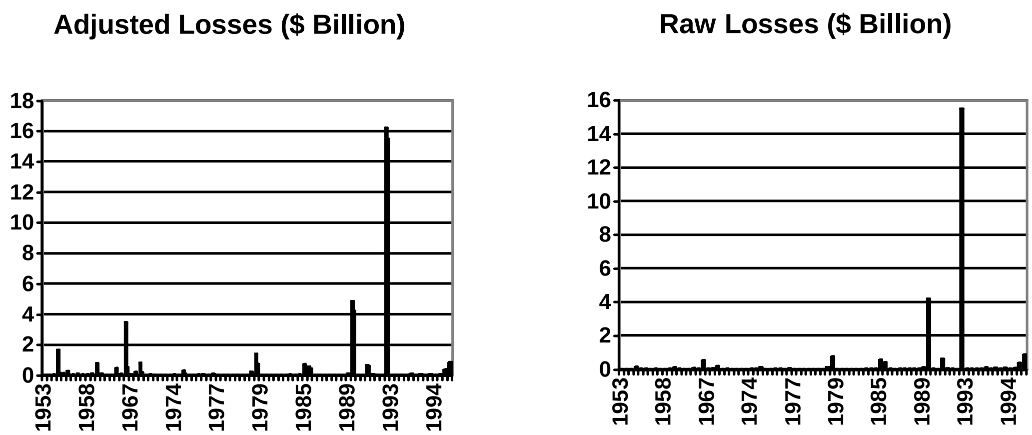


Figure 2. Construction Cost Index Regression Model

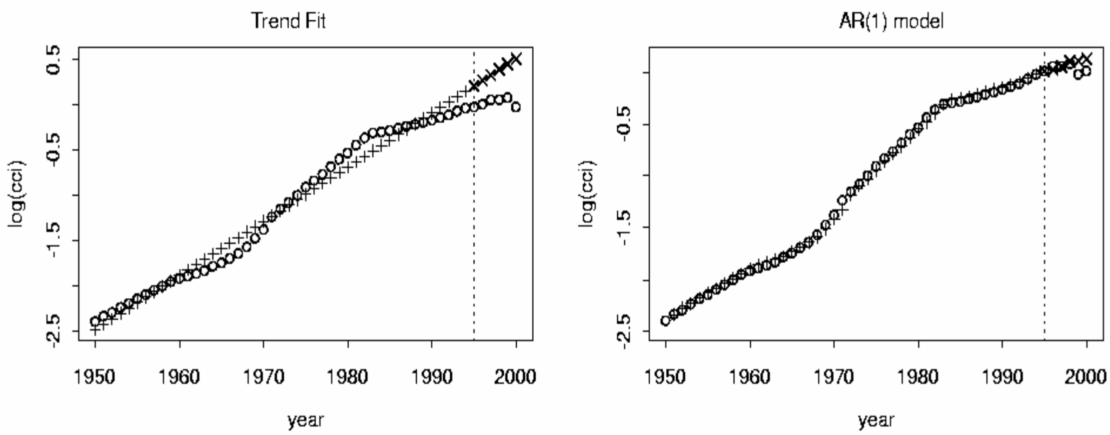


Figure 3. Population Growth Regression Model

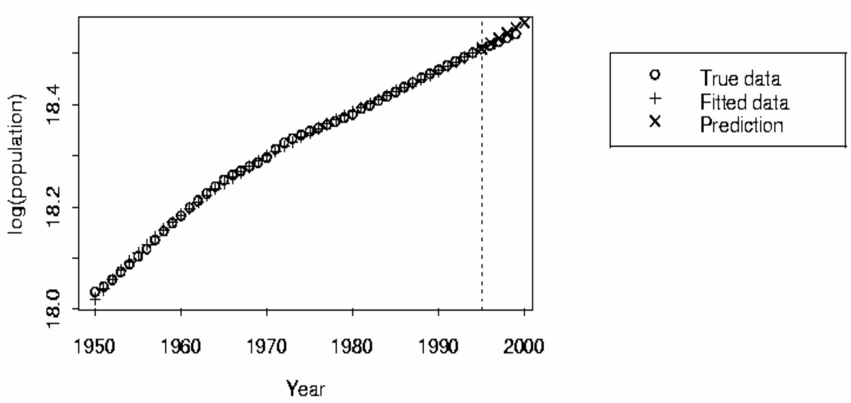


Figure 4. c.d.f of event number, 1950-1995

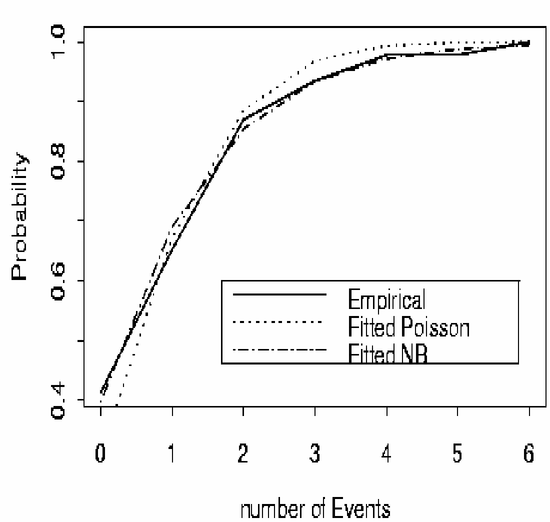


Figure 5. Loss Severity Density

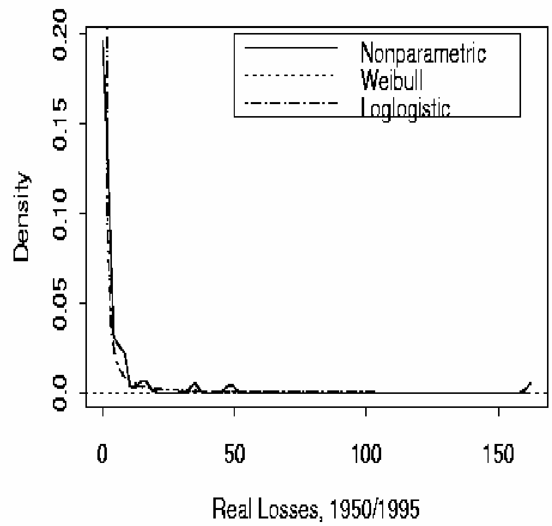


Figure 6. C.D.F. of the Loss Severity

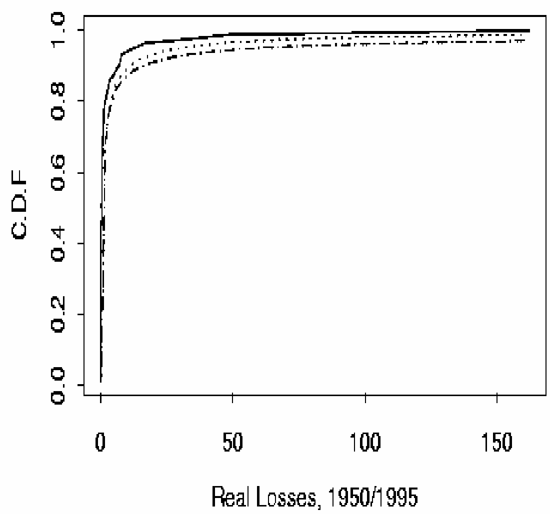


Figure 7. Right Tail

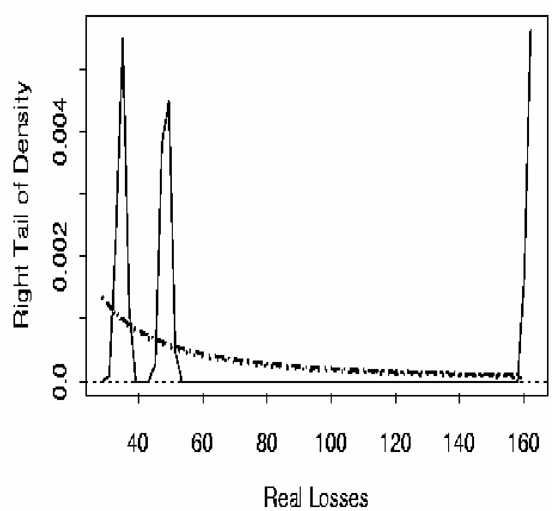


Figure 8. 3rd quarter Eastern contracts

Time < 0: prior to loss period; *Time* $\in [0,3]$: during loss period; *Time* > 3: during development period. Third row is transaction time vs. price. Solid points are market prices; small circles are estimated theoretical prices under the weibull distribution; cross signs are estimated theoretical prices under the loglogistic distribution. The vertical axis of the fourth row is the ratio of estimated theoretical price over market price.

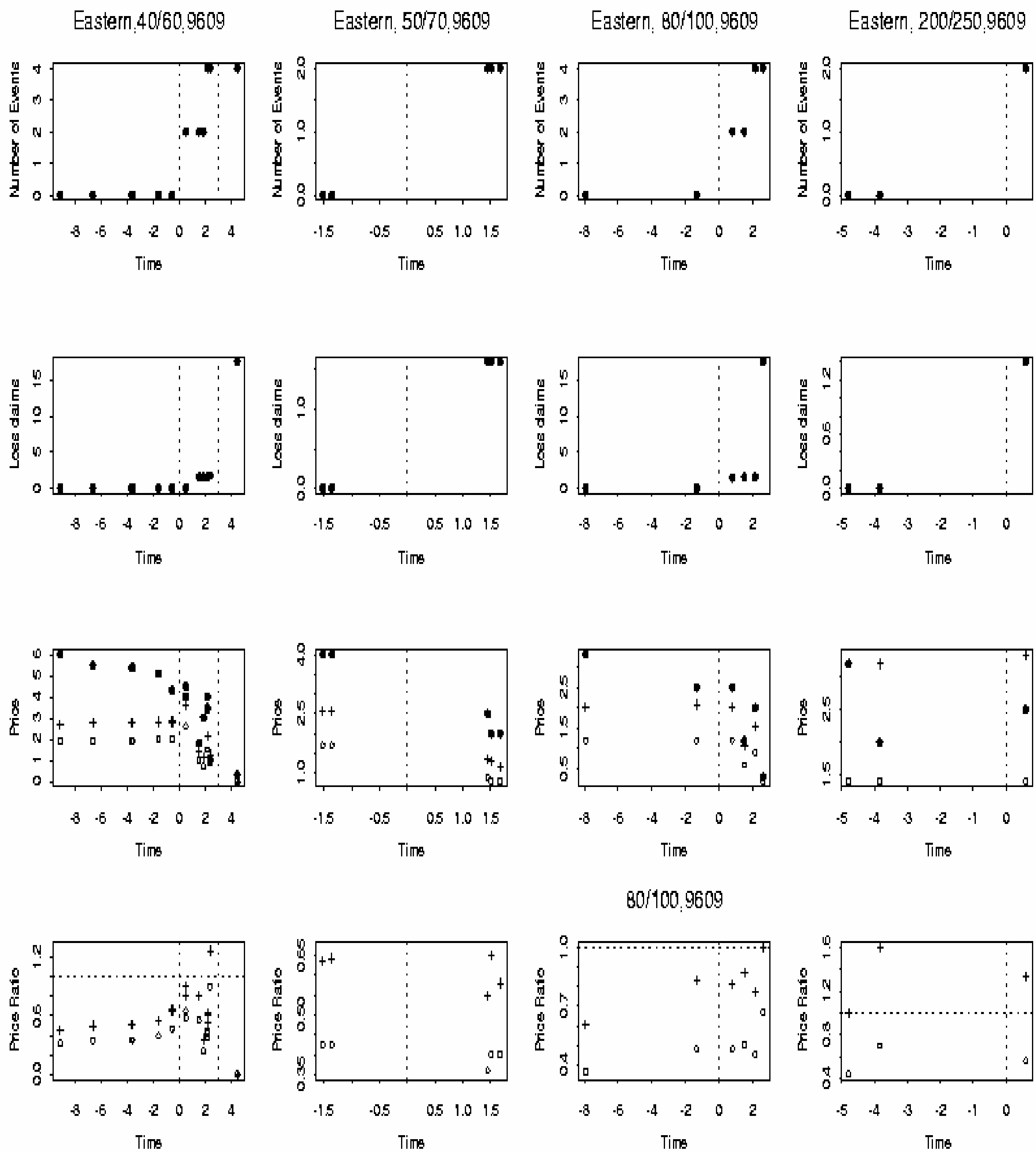


Figure 9. Annually National Contracts

$Time < 0$: prior to loss period; $Time \in [0,12]$: during loss period; $Time > 12$: during development period. Third row is transaction time vs. price. Solid points are market prices; small circles are estimated theoretical prices under Pareto distribution. The vertical axis of the fourth row is the ratio of estimated theoretical price over market price.

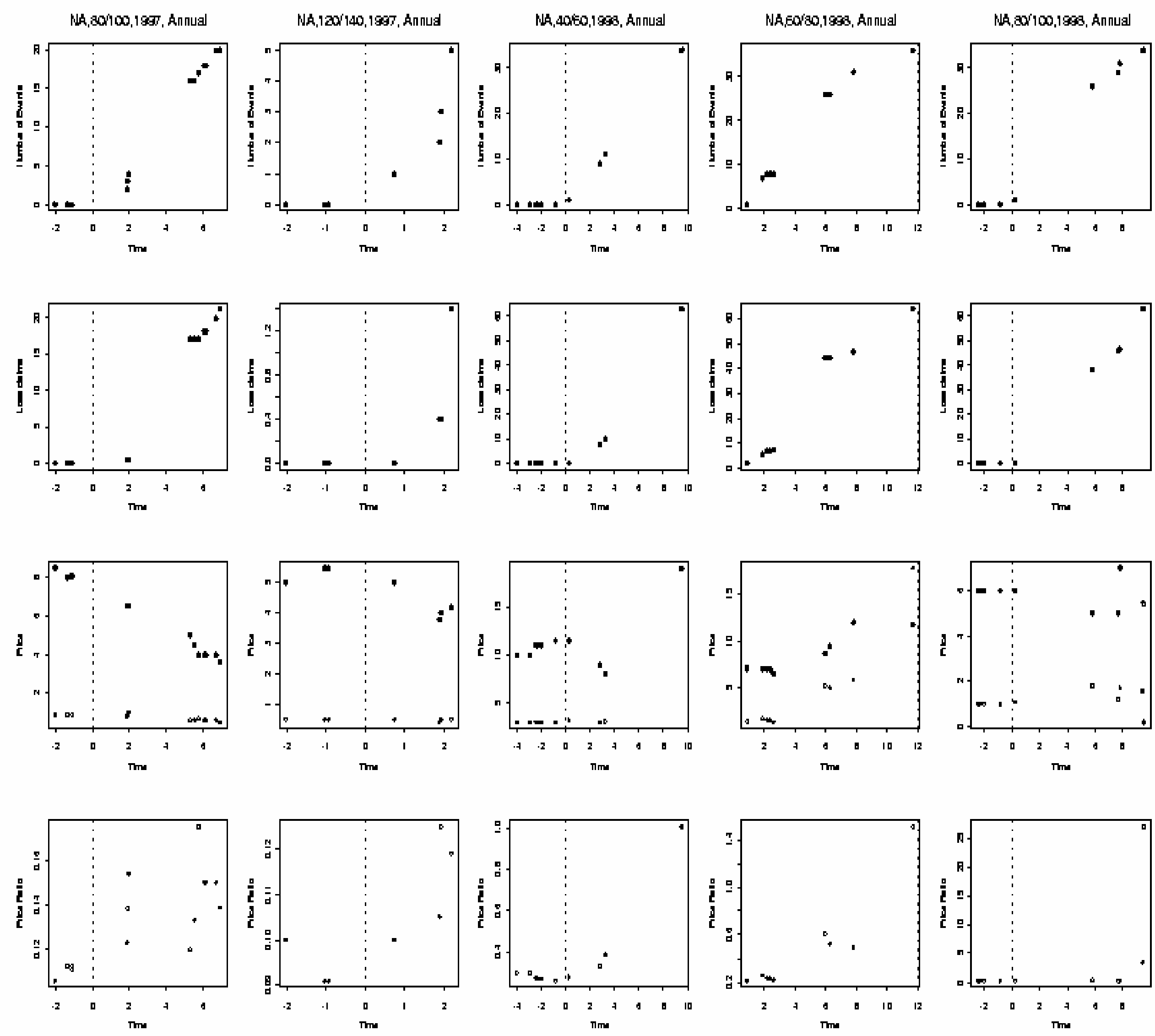


Figure 10. Unit prices calculated under the best-fitted loss severity distributions

The first column is plotted for annual contracts; the second column is for quarterly contracts covering the 3rd quarter losses; the third column is for the eastern contracts covering the 3rd quarter losses; the fourth column is for south-eastern contracts covering the 3rd quarter losses; and the last column is for mid-western contracts covering the 2nd quarter losses. Call spreads equal 20 in the first four columns, and 10 for the last column. 'Prior' means that contracts were traded prior to loss period; 'During' means that contracts were traded during loss periods.

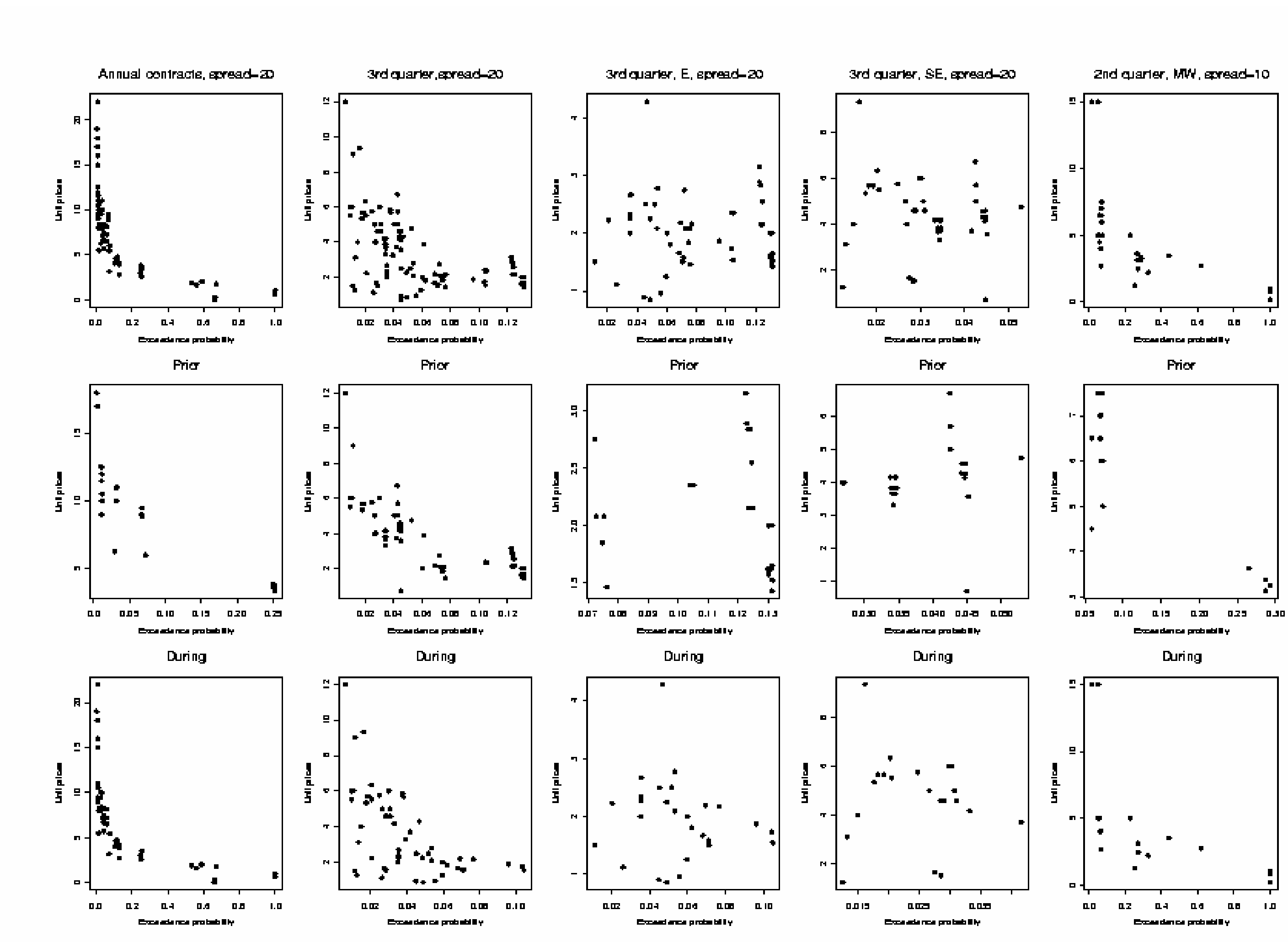
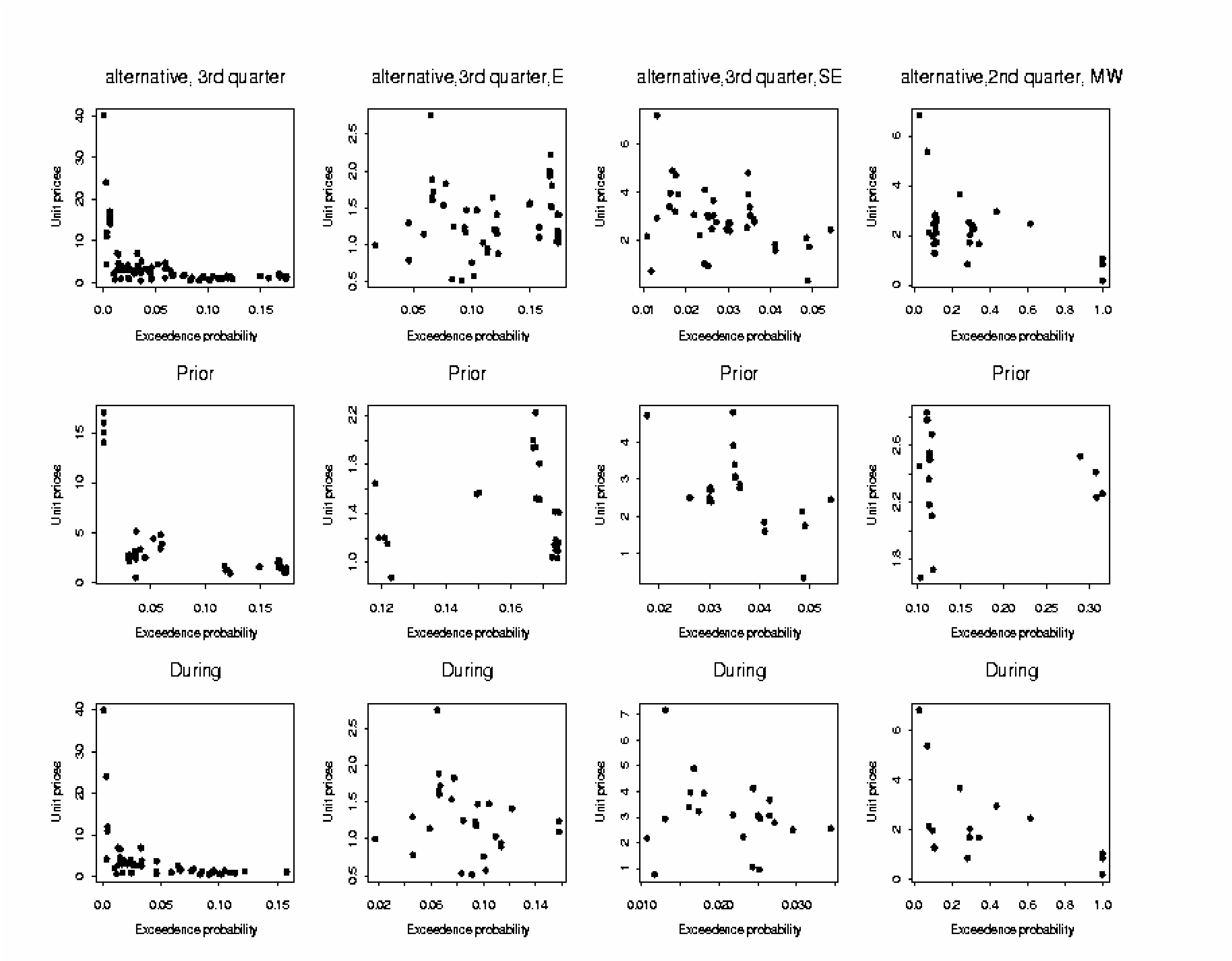


Figure 11. Unit prices calculated under the alternative loss severity distributions

The first column is for quarterly contracts covering the 3rd quarter losses; the second column is for the eastern contracts covering the 3rd quarter losses; the third column is for south-eastern contracts covering the 3rd quarter losses; and the last column is for mid-western contracts covering the 2nd quarter losses. Call spreads equal 20 in the first three columns, and 10 for the last column. 'Prior' means that contracts were traded prior to loss period; 'During' means that contracts were traded during loss periods.



Appendix IV. Eight density functions

1. **Burr distribution**: $Burr(\alpha,\lambda,\delta)$ with $\alpha > 0, \lambda > 0, \delta > 0$

   c.d.f. : $F(x) = 1 - \left(\frac{\lambda}{\lambda + x^{\delta}}\right)^{\alpha}$

   p.d.f.: $f(x) = \alpha\delta\lambda^{\alpha} x^{\delta-1}\left(\lambda + x^{\delta}\right)^{-\alpha-1}$

2. **Pareto distribution**: $Pareto(\alpha,\lambda)$ is $Burr(\alpha,\lambda,1)$ and $GPareto(\alpha,\lambda,1)$

3. **Generalized Pareo distribution**: $GPareto(\alpha,\lambda,\delta)$ with $\alpha > 0, \lambda > 0, \delta > 0$

   c.d.f. : $F(x) = \beta\left(\delta,\alpha;\frac{x}{x+\lambda}\right)$, $\beta(a,b;x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^x y^{a-1}(1-y)^{b-1}dy$

   p.d.f. : $f(x) = \frac{\Gamma(\alpha+\delta)\lambda^{\alpha} x^{\delta-1}}{\Gamma(\alpha)\Gamma(\delta)(\lambda+x)^{\alpha+\delta}}$

4. **Gamma distribution**: $Gamma(\alpha,\lambda)$ with $\alpha > 0, \lambda > 0$.

   c.d.f. : $F(x) = \Gamma(\alpha;\lambda x)$, where $\Gamma(\alpha;x) = \int_0^x y^{\alpha-1}e^{-y}dy \Big/ \Gamma(\alpha)$

   p.d.f. : $f(x) = \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x} / \Gamma(\alpha)$

5. **Loggamma distribution**: $Lgamma(\alpha,\lambda)$ with $x > 1$ and $\alpha > 0, \lambda > 0$.

   c.d.f. : $F(x) = \Gamma(\alpha;\lambda \ln x)$

   p.d.f. : $f(x) = \frac{\lambda^{\alpha}(\ln x)^{\alpha-1}}{x^{\lambda+1}\Gamma(\alpha)}$

6. **Lognormal distribution**: $Lnorm(\mu,\sigma)$ with $\sigma > 0$

c.d.f. : $F(x) = \Phi\left(\dfrac{\ln x - \mu}{\sigma}\right)$ where $\Phi(x) = \int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy \Big/ \sqrt{2\pi}$

p.d.f. : $f(x) = \dfrac{\exp\left[-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right]}{x\sigma\sqrt{2\pi}}$

7. **Loglogistic distribution**: $L\log istic(\lambda,\delta)$ is $Burr(1,\lambda,\delta)$

8. **Weibull distribution**: $Weibull(c,\delta)$ with $c > 0, \delta > 0$.

c.d.f. : $F(x) = 1 - e^{-cx^{\delta}}$

p.d.f. : $f(x) = c\delta x^{\delta-1} e^{-cx^{\delta}}$

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