Ranking Asymmetric Auctions using the Dispersive Order*

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Abstract

The revenue ranking of asymmetric auctions with two heterogenous bidders is examined. The main theorem identifies a general environment in which the first-price auction is more profitable than the second-price auction. By using mechanism design techniques, the problem is simplified and several extensions are made possible. Roughly speaking, the first-price auction is more profitable when the strong bidder's distribution is flatter and more disperse than the weak bidder's distribution. These sufficient conditions turn out to have appealing geometric and economic interpretations. The theorem applies to certain environments with multi-dimensional types. It is also possible, for the first time, to extend the ranking to auctions with reserve prices and to auctions with more bidders. Implications for contests architecture and other auction formats are also pursued.

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1 Introduction

Many, if not most, auctions involve bidders that are heterogeneous ex ante. For example, procurement auctions may involve domestic and foreign firms; an auction for a new licence or technology may pit an incumbent against a prospective entrant; an art collector may be vying for a synergy not relevant to a bidder with unit demand, and so on. Even if bidders are initially homogenous, asymmetries may be created over time, as willingness-to-pay in the last auction in a sequence likely depends on how many items have been won at that time.

The interest in asymmetric auctions dates back to the inception of modern auction theory, with Vickrey (1961). As Vickrey (1961) first discovered, and Myerson (1981) and Riley and Samuelson (1981) later proved more generally, the first-price auction (FPA) and second-price auction (SPA) yield the same expected revenue in the independent private values model when bidders are homogenous. However, Vickrey (1961) proved by example that revenue equivalence does not hold when bidders are heterogeneous. Thus, the question of which auction is more profitable with heterogenous bidders is an old and fundamental question.

The objective of this paper is to contribute to the literature on asymmetric auctions in three principal ways. The first is methodological; a method that simplifies the analysis of the problem is proposed. Second, substantially more general results are obtained and a more robust intuition is developed. Finally, the pivotal role of a specific stochastic order, the dispersive order, is emphasized and its economic interpretation and application is pursued.

In a seminal paper on auctions with two heterogeneous bidders, Maskin and Riley (2000) present three seemingly separate classes of environments in which it is possible to rank the FPA and the SPA in terms of revenue. In two of the cases – if the strong bidder's type distribution is either a "shifted" or a "stretched" version of the weak bidder's type distribution – the FPA yields higher expected revenue than the SPA.[1]

[1] Maskin and Riley's (2000) paper is the most general treatment of the revenue effects of bidder heterogeneity in the existing literature. In an earlier paper, Maskin and Riley (1985) assume type distributions are discrete and prove that the revenue ranking is also ambiguous in that case. The remaining literature uses analytical or numerical examples to shed light on the problem, almost always with the conclusion that the FPA is more profitable. These include Vickrey (1961), Lebrun (1996), and Cheng (2006), all of which analytically examine special cases of power distributions. Likewise, Greismer et al (1967), Plum (1992), and Kaplan and Zamir (2010) derive bidding strategies in such environments but without comparing revenue. Power distributions have also been examined in the numerical literature, starting with Marshall et al (1994). More recent examples of numerical

In this paper, I will argue that the latter results should not be viewed as two separate propositions but rather as corollaries of a more general theorem. Thus, Maskin and Riley's (2000) results are unified and extended. Roughly speaking, the FPA dominates if the strong bidder's distribution is *flatter* and more *disperse* than the weak bidder's distribution; "shifting" and "stretching" are special cases. Indeed, Maskin and Riley's (2000) assumptions are unnecessarily strong even within each of these two models. As explained later, the sufficient conditions have appealing geometric and economic interpretations.

Maskin and Riley (2000) use arguments from mechanism design to demonstrate why the problem is non-trivial. They then abandon the approach with the conclusion that "mechanism design considerations do not settle the matter of which auction generates more revenue".[2] Instead, they use the system of differential equations that describe bidding behavior to derive two technical lemmata that quantify revenue in the two auctions. Using these lemmata, they prove the superiority of the FPA in the two environments described above.

In contrast, I will demonstrate that mechanism design can in fact not only be fruitfully used to address the problem, it also greatly simplifies the analysis itself. Thus, the first contribution of the paper is methodological. The starting point is Myerson's (1981) result that expected revenue is largely determined by the expected value of the winner's virtual valuation or marginal revenue. The key step in developing the main result is to formulate the expected value of the winner's virtual valuation as the expected value of a conditional expectation, where the latter is the expected value of the winner's virtual valuation *conditional* on the weak bidder's type. For the simple case where bidders' supports have the same lowest end-point, the assumptions in the theorem ensure that this conditional expectation is greater for the FPA than the SPA for *all* types in the weak bidder's support. When the lower end-points are different, the proof is completed by exploiting the fact that the FPA extracts more rent from the strong bidder than the SPA. Compared to other uses of Myerson's (1981) results, the important feature of the method is that it allows comparison of two auctions when neither auction dominates for all combinations of types.[3]

revenue comparisons include Fibich and Gavious (2003), Li and Riley (2007), and Gayle and Richard (2008). Vickrey (1961) and Cheng (2010) provide analytical examples in which the SPA is superior. See also Gavious and Minchuk (2010).

[2] To clarify, Maskin and Riley (2000) construct one environment in which they use mechanisms design to prove the SPA is more profitable. Their proofs are different when the FPA dominates.

[3] Mares and Swinkels (2010a) is a recent example. They compare auctions with handicaps and

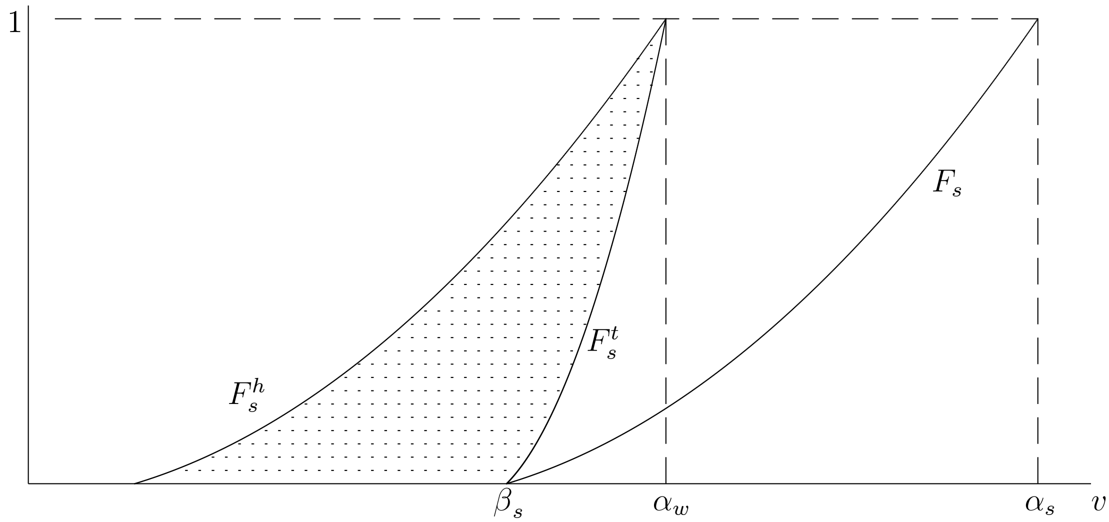


Figure 1: Truncations and horizontal shifts.

Note: F_s has support $[\beta_s, \alpha_s]$. Consider some F_w whose support ends at $a_w \in (\beta_s, \alpha_s)$. Generate the appropriate truncation, F_s^t, and horizontal shift, F_s^h, of F_s, such that these end at α_w. The FPA dominates if F_w is more disperse than F_s^t but less disperse than F_s^h.

As in Maskin and Riley (2000), I consider auctions with two bidders. The strong bidder draws a type or valuation from a distribution, F_s, that dominates the distribution, F_w, of the weak bidder in terms of the reverse hazard rate. With this assumption, some inferences concerning bidding behavior in the FPA are possible. If the strong bidder's density function is monotonic, the FPA is shown to be more profitable than the SPA if it is also the case that F_s is flatter and more disperse (more "spread out") than F_w. The main theorem allows the density function to be non-monotonic by appropriately strengthening these conditions.

To illustrate, consider for the moment the most stringent assumptions in Maskin and Riley (2000), namely that F_s is convex and log-concave. One way of thinking about their "stretch" model is that F_w is a truncation of F_s, which I denote F_s^t. In their "shift" model, F_w is a horizontal, left-ward shift of F_s, denoted F_s^h. These cases are depicted in Figure 1. Of course, F_w can take many other forms. The FPA is shown to dominate if F_w "lies between" F_s^t and F_s^h and satisfies certain regularity conditions. These conditions are satisfied if F_w is more disperse than F_s^t, but less disperse F_s^h. Similar results obtain if F_s is concave, but with a vertical shift of F_s

show that one auction format dominates another because it is better for *all* pairs of types. The difficulty in the current paper is that the FPA is not superior for all combinations of types, as Maskin and Riley (2000) points out.

taking the place of the horizontal shift. A related result for non-monotonic densities is also derived.

The sufficient conditions have appealing geometric and economic interpretations. Reverse hazard rate dominance implies that the *ratio* of the two bidders' distribution functions move closer to one the larger the stakes are (the larger types are). At the same time, the assumptions that F_s is flatter and more disperse than F_w imply, respectively, that the vertical and horizontal *differences* between the two distributions grow larger the larger the stakes are. In summary, the main theorem addresses situations where the relative asymmetry between bidders decreases but the absolute asymmetry increases as the stakes get higher.

The dispersive order has recently attracted some attention in the theoretical auction literature. Jia et al (2010) and Katzman et al (2010) examine comparative statics in symmetric auctions when bidders' distributions become more disperse. Ganuza and Penalva (2010) consider symmetric auctions in which the seller can influence the precision of bidders' information by making their signals more or less disperse. Johnson and Myatt (2006) examine a related question in the context of a monopoly. Their "rotation order" is also used to compare how spread out two distributions are. In asymmetric auctions, the dispersive order plays a role in determining the qualitative features of revenue-enhancing interventions into particular auction formats, as demonstrated by Kirkegaard (2010) and Mares and Swinkels (2010a, 2010b). The results in these papers are particularly strong when densities are monotonic. Hopkins (2007) describe qualitative features of bidding behavior in auctions where distribution functions cross and one is smaller than the other in the dispersive order. However, the current paper is the first to explicitly use the dispersive order to rank revenue across standard auctions with asymmetric bidders.

To further illustrate the usefulness of notions of dispersion or spread to auction theory, consider the following generalization of Maskin and Riley's (2000) model of distribution shifts to permit "stochastic shifts" of the distribution function. Assume the strong bidder's valuation is the sum of two components which are independent draws from distribution functions F and G, respectively. Formally, his valuation is determined by the convolution of F and G. Assume the weak bidder's valuation is an independent draw from F. Maskin and Riley's (2000) "shift" model fits this set-up if G is degenerate. However, the revenue ranking extends to the more interesting case in which G is non-degenerate. One of the main assumptions needed to ensure this result is that F is *dispersive*. The distribution F is dispersive if and only if the convolution of F and G is more disperse than F regardless of the distribution of G. A related model in which the uncertainty is multiplicative rather than additive is also examined. The results are similar, for both the degenerate and non-degenerate case. An alternative interpretation of these models is that one bidder suffers an externality if the other bidder wins, but the magnitude of this externality is his private information. See Jehiel et al (1999) for a discussion of the difficulties that arise in such environments. Their revenue results are confined to symmetric environments, however.

The main result can be extended in several directions. Mares and Swinkels (2010a) point out that it is not known whether reserve prices affect the ranking. An application of the main theorem effortlessly demonstrates that reserve prices in fact do not affect the ranking in any model satisfying the assumptions outlined before. However, the ranking may be affected in other models. Other implications of the main result are examined in Section 7.

The revenue ranking can also be extended to certain environments with more bidders. The FPA is more profitable when there are many weak bidders, but only one strong bidder. Such a situation occurs when an incumbent monopolist bids against several potential entrants for a licence to operate on a market or use a new technology. However, the method proposed here and in Maskin and Riley (2000) are not powerful enough to generate a revenue ranking in the general case with many strong bidders. Nevertheless, a ranking can be obtained if the asymmetry is "large enough".

2 Small auctions: Preliminaries

There are two risk neutral bidders. Bidder s is strong and bidder w is weak; bidder s is more likely to value the object being sold more highly. Bidder i draws a type or valuation from a distribution function, F_i, which is continuously differentiable on its support, $S_i = [\beta_i, \alpha_i]$, $i = s, w$. The density, f_i, is strictly positive on $(\beta_i, \alpha_i]$, with $\alpha_i > \beta_i \geq 0$, $i = s, w$. Mass points are ruled out. It is assumed that $\beta_w \leq \beta_s$ and $\alpha_w < \alpha_s$. In the following, it will be necessary to compare the density functions over certain intervals. To facilitate this comparison, let $F_i(v) = f_i(v) = 0$ for all $v < \beta_i$, such that $f_w(v) \geq f_s(v)$ for all $v < \beta_s$. Let $C = S_s \cap S_w$ denote common support. Thus, $C = [\beta_s, \alpha_w]$ if the supports overlap.

The relationship between F_s and F_w plays two distinct roles. The first is to determine the properties of the interaction between bidders in the FPA. The second is to enable a comparison between the revenue that different mechanisms generate. In this respect, the relative strength of the bidders and the relative slope and dispersion of their distributions, respectively, are key.

2.1 Strength and bidding strategies

The sense is which bidder s is stronger than bidder w needs to be made more precise. There are at least four ways in which bidder heterogeneity can be modelled, depending on how F_s and F_w are related on C:

1. *F_s dominates F_w i.t.o. the likelihood ratio, $F_w \leq_{lr} F_s$:* $\frac{f_s(v)}{f_w(v)}$ is increasing on C.[4]

2. *F_s dominates F_w i.t.o. the reverse hazard rate, $F_w \leq_{rh} F_s$:* $\frac{f_s(v)}{F_s(v)} \geq \frac{f_w(v)}{F_w(v)}$, $\forall\, v \in C$.

3. *F_s dominates F_w i.t.o. the hazard rate, $F_w \leq_{hr} F_s$:* $\frac{f_s(v)}{1-F_s(v)} \leq \frac{f_w(v)}{1-F_w(v)}$, $\forall\, v \in C$.

4. *F_s first order stochastically dominates F_w, $F_w \leq_{st} F_s$:* $F_s(v) \leq F_w(v)$, $\forall\, v \in C$.

See Krishna (2002) for an introduction to these stochastic orders and their use in auction theory. See Shaked and Shanthikumar (2007) for an in-depth treatment. The first order implies the other orders. The second and third both imply the fourth.

Maskin and Riley (2000) assume that $F_w \leq_{rh} F_s$.[5] In the first part of their paper, this assumption enables them to derive rather tight bounds on equilibrium strategies.

To be more specific, let $r(v) = F_s^{-1}(F_w(v))$, $v \in S_w$. By definition, bidder s is just as likely to have a type below $r(v)$ as bidder w is to have a type below v; the two bidders have the same *rank*, or $F_s(r(v)) = F_w(v)$. Since $F_w \leq_{st} F_s$, $r(v) \geq v$ for all $v \in S_w$. Given $F_w \leq_{rh} F_s$, Maskin and Riley (2000) show that in a FPA, bidder w with type v either submits a non-serious bid (one that is so low that it never wins) or he submits a bid of the same magnitude as a bid submitted by the strong bidder

[4] In this paper, increasing is taken to mean non-decreasing; decreasing means non-increasing. The abbreviation i.t.o. stands for "in terms of".

[5] Since $F_w(v)/F_s(v)$ is decreasing and strictly larger than one at $v = \alpha_w$, it follows that F_s strictly first order stochastically dominates F_w in the sense that $F_s(v) < F_w(v)$ for all $v \in (\beta_w, \alpha_w]$. Maskin and Riley (2000) assume something slightly stronger than reverse hazard rate dominance. However, reverse hazard rate dominance is strong enough to deliver the key implication on bidding strategies, namely that the weak bidder is at least as aggressive as the strong bidder for comparable types in the first-price auction (see e.g. Kirkegaard (2009)). Hence, reverse hazard rate dominance is assumed in this paper.

of some type, $k_1(v)$, somewhere in the interval $[v, r(v)]$.[6] In other words, the weak bidder is more aggressive than the strong bidder, but not aggressive enough to make up for the difference in strength. Using three different techniques, these properties have been proven in *(i)* Lebrun (1999) and Maskin and Riley (2000), *(ii)* Milgrom (2004) and Hopkins (2007), and *(iii)* Kirkegaard (2009), respectively.

The bid is strictly increasing in type for those that submit serious bids. Moreover, bidder w with type α_w submits the same bid as bidder s with type α_s. Hence, $k_1(\alpha_w) = \alpha_s = r(\alpha_w)$.

In a SPA, it is a weakly dominant strategy to submit a bid equal to the bidder's type. Since the auction is efficient, bidder w with type v wins if and only if bidder s has a type below $k_2(v) = \max\{\beta_s, v\}$.

2.2 Dispersion and price sensitivity

In the second part of Maskin and Riley (2000), the bounds on bidding strategies help them to infer that the FPA is more profitable than the SPA if F_s is either a "shifted" or a "stretched" version of F_w. What is less obvious is that their additional assumptions in fact imply

$$f_w(v) \geq f_s(x) \text{ for all } x \in [v, r(v)] \text{ and all } v \in S_w. \tag{1}$$

Coupled with first order stochastic dominance, $f_w(v) \geq f_s(v)$ implies $F_w \leq_{hr} F_s$. Thus, hazard rate dominance is implicit in Maskin and Riley (2000).

I will explicitly assume that (1) is satisfied. As just mentioned, (1) is weaker than the assumptions in Maskin and Riley (2000). Nevertheless, it is sufficient for the main result. Moreover, the assumption has a clear and intuitive economic interpretation.

Bulow and Roberts' (1989) analogy to monopoly pricing is useful to interpret (1) and understand its implications. Thinking of v as a price, the survival function $q_i(v) \equiv 1 - F_i(v)$ has the properties of a demand curve in a market with a continuum of consumers of mass one, distributed on S_i, $i = s, w$. Since $F_w \leq_{st} F_s$, $q_s(v) \geq q_w(v)$.

The slope of the demand curve is $q_i'(v) = -f_i(v)$, $i = s, w$. Thus, condition (1) implies that the weak bidder's demand curve is steeper than the strong bidder's

[6] A non-serious bid is made only if bidder w's type, v, is sufficiently far below β_s. A non-serious bid wins with probability $0 = F_s(\beta_s)$. Since $\beta_s \in [v, r(v)]$ when $v \leq \beta_s$, letting $k_1(v) = \beta_s$ for all v that submit non-serious bids implies that $k_1(v) \in [v, r(v)]$ for all $v \in S_w$.

demand curve. At the other end of the $[v, r(v)]$ interval, recall that at price $r(v)$ and v, respectively, demand would be the same in the markets described by demand curves q_s and q_w. Hence, (1) also means that the weak bidder's demand curve is steeper at comparable quantities. Since $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$, the implication is that the weak bidder's demand curve at a given price is steeper than the strong bidder's demand curve on a range of prices above v.[7] Figure 2 illustrates the assumption.

In summary, the weak bidder's demand responds more in *absolute* terms to a marginal price change. In this sense, the market with the lowest willingness-to-pay is also the most price sensitive. The *relative* change in demand following a marginal price increase can be measured by

$$\left|\frac{q_i'(v)}{q_i(v)}\right| = \frac{f_i(v)}{1-F_i(v)} \text{ and } \varepsilon_i(v) = \left|\frac{vq_i'(v)}{q_i(v)}\right| = \frac{vf_i(v)}{1-F_i(v)}. \tag{2}$$

Since $F_w \leq_{hr} F_s$, the weak bidder is also more price sensitive than the strong bidder in relative terms, at comparable prices. For future reference, define marginal revenue evaluated at price v as

$$J_i(v) = v\left[1 - \frac{1}{\varepsilon_i(v)}\right] = v - \frac{1-F_i(v)}{f_i(v)}.$$

The interpretation of J_i as marginal revenue is due to Bulow and Roberts (1989). Myerson (1981) refers to J_i as bidder i's virtual valuation. Hazard rate dominance implies that $J_w(v) \geq J_s(v)$ for all $v \in C$.

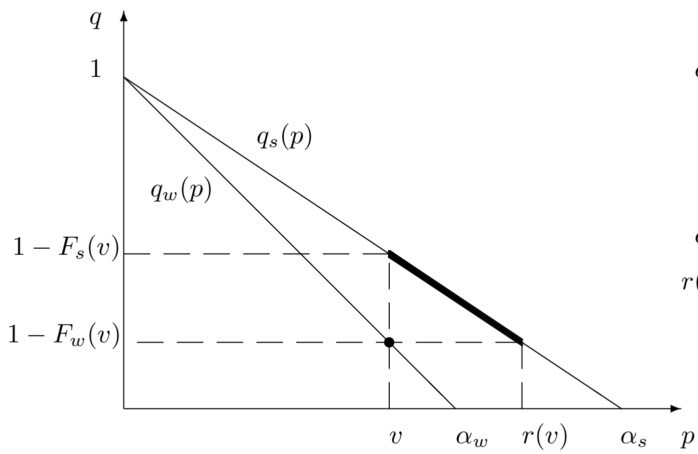
It is also instructive to examine the implications of $f_w(v) \geq f_s(r(v))$. To do so, another small detour into order statistics is useful. The following orders of "spread" are relevant for the current paper, in descending order of importance:

1. *F_w is smaller than F_s in the dispersive order*, $F_w \leq_{disp} F_s$: $r(v) - v$ is increasing on S_w.

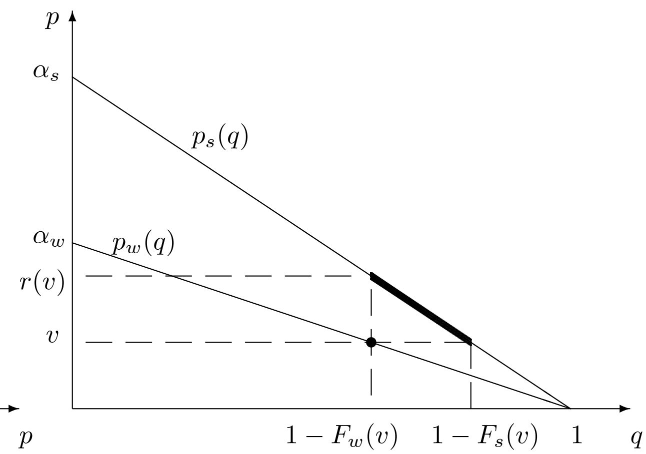
2. *F_w is smaller than F_s in the star order*, $F_w \leq_* F_s$: $\frac{r(v)}{v}$ is increasing on S_w.

3. *F_w is smaller than F_s in the convex transform order*, $F_w \leq_c F_s$: $r(v)$ is convex on S_w.

[7] Recall that both markets has a set of consumers of mass one, so $q \in [0,1]$ on both markets. With the assumption that demand in the strong market is more "spread out", or $\alpha_s - \beta_s \geq \alpha_w - \beta_w$, it must necessarily be the case that p_s is steeper than p_w locally, for some q. Thus, (1) can be viewed as a regularity condition, roughly saying that p_s is steeper than p_w globally.



(a) Demand curves



(b) Inverse demand curves

Figure 2: Demand curves for $\beta_w = \beta_s = 0$.

Note: The demand curve of bidder s is flatter on the fat segment than bidder w 's demand curve at the highlighted point (Panel a). Bidder s has a steeper inverse demand curve (b).

Shaked and Shanthikumar (2007) review these stochastic orders.[8] In words, $F_w \leq_{disp} F_s$ if the distance between the types that are at the same percentile is increasing. Thus, geometrically, the dispersive order means that the horizontal difference between F_s and F_w (or $q_w(p)$ and $q_s(p)$) is increasing. Since $r(v) = F_s^{-1}(F_w(v))$,

$$r'(v) = \frac{f_w(v)}{f_s(r(v))},$$

and $F_w \leq_{disp} F_s$ if and only if $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$, or $r'(v) \geq 1$. Thus, (1) implies dispersion, which is therefore also implicit in Maskin and Riley (2000). The dispersive order has some intuitive properties. If F_s is more disperse than F_w then it has larger variance and wider support, $\alpha_s - \beta_s \geq \alpha_w - \beta_w$. I write $F_w =_{disp} F_s$ if $r(v) - v$ is constant.

The dispersive order, star order, and convex transform order have natural economic interpretations, all related to various notions of price sensitivity. A discussion of these interpretations are postponed until Section 6, however.

[8] The literature on the star order and convex transform order should be read with some care. In this literature, it is often assumed that $\beta_s = \beta_w = 0$, and a number of results rely on this assumption (for example, if $\beta_s = \beta_w = 0$ then $\leq_c \Longrightarrow \leq_*$). In the current paper, β_s and β_w are allowed to be strictly positive.

Although $F_w \leq_{disp} F_s$ is required, it will sometimes be assumed that F_s is not too much more disperse than F_w. For instance, it is possible that $F_w \leq_{disp} F_s$ and yet $F_s \leq_* F_w$. The assumption that $F_s \leq_* F_w$ plays a role in Kirkegaard's (2010) analysis of favoritism in asymmetric all-pay auctions. Mares and Swinkels (2010) consider favoritism in procurement auctions in which the buyer has a preference for a specific bidder (seller). Translating their procurement setting into a standard auction, one of their assumptions is that $F_s \leq_c F_w$.

2.3 An interpretation of the joint assumptions

Since condition (1) implies that F_w is steeper than F_s on C, the vertical distance between the distributions, $F_w - F_s$, is increasing on C. Dispersion implies that the horizontal difference between the distributions is increasing as well. Thus, the absolute difference between bidders is larger the larger the stakes are.

First order stochastic dominance implies that $\frac{F_w(v)}{F_s(v)} \geq 1$ for all $v \in C$. Reverse hazard rate dominance requires the ratio $\frac{F_w(v)}{F_s(v)}$ to be decreasing on C. Thus, as the stakes get higher, the relative difference between the bidders diminishes.

In summary, the auction environment is one in which the absolute difference between bidders grows but the relative difference diminishes as the stakes get higher.

Given the interpretation of (1) in terms of price sensitivity, the joint assumptions are similar to the assumption used in the textbook explanation of third degree price discrimination that the market with the lowest willingness-to-pay is also the most price sensitive. Recall that the reason for student discounts, say, is not that students have lower demand or willingness-to-pay per se, but rather that their demand responds more to a price change. The common assumption is that low demand and high price sensitivity go hand-in-hand. It is an assumption of this nature that is made here.

3 Analysis and the main result

A mechanism that "favors" the more price sensitive bidder a bit might be expected to be more profitable than one that does not. In a SPA, the bidder with the highest type or willingness-to-pay wins. In contrast, the weak bidder wins more often in a FPA, which is reminiscent of a monopolist offering a price discount on the weak, price sensitive, market. However, the price discount may be too steep.

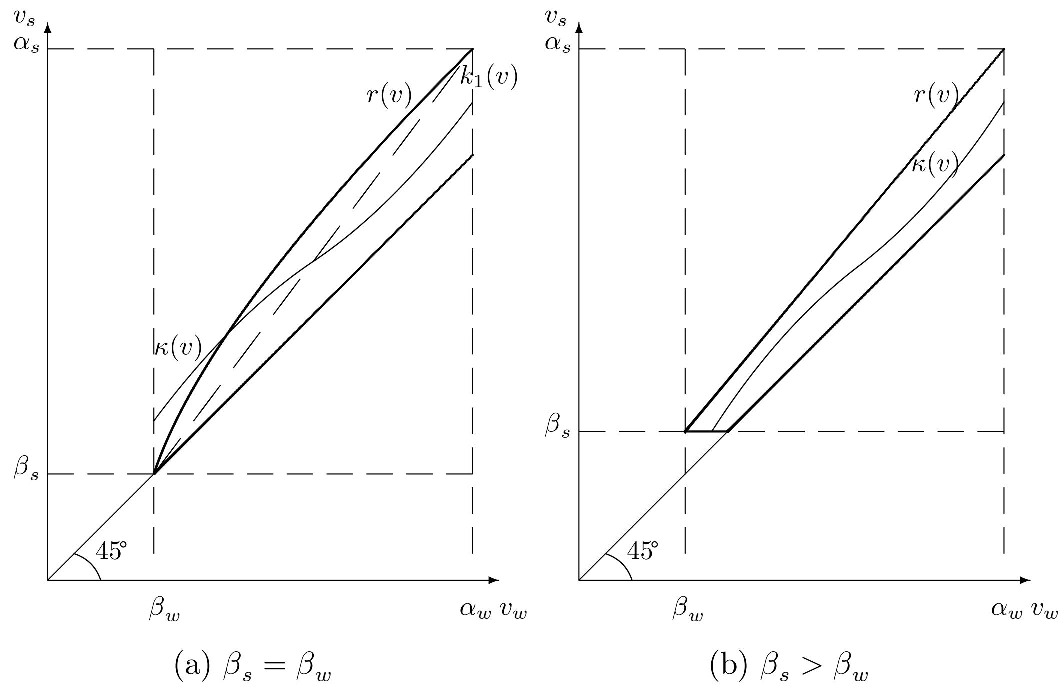


Figure 3: Comparing different mechanisms.

Note: The allocation in the FPA, $k_1(v)$, (the dashed curve in Panel a) falls between the two thick curves, the lower of which describes the allocation in the SPA, $k_2(v)$.

Maskin and Riley (2000) discuss this point, using a figure similar to Figure 3, above. To begin, consider the simpler case with $\beta_w = \beta_s = \beta$, as in Figure 3a. Recall that the weak bidder has higher marginal revenue than the strong bidder for comparable types, or $J_w(v) \geq J_s(v)$ for all $v \in C$. However, when v is close to α_w, it is also the case that $J_w(v) < J_s(r(v))$. For instance, $J_w(\alpha_w) = \alpha_w < \alpha_s = J_s(\alpha_s) = J_s(r(\alpha_w))$. Assume for the moment that J_i is strictly increasing, $i = s, w$, and let $\kappa(v) = J_s^{-1}(J_w(v))$ (whenever it exists) denote the type the strong bidder must have for his marginal revenue to coincide with that of the weak bidder. By hazard rate dominance, $\kappa(v) \geq v$, but when v is sufficiently large, $\kappa(v) < r(v)$. In an optimal auction (subject to the constraint that the object is sold), the weak bidder should win if and only if the strong bidder's type is below $\kappa(v)$, thereby ensuring that the object is allocated to the bidder with the highest marginal revenue.

In a FPA, bidder w with type v wins if his opponent's type is below $k_1(v)$, which is somewhere in the interval $[v, r(v)]$. In contrast, the SPA is efficient, $k_2(v) = \max\{\beta_s, v\}$. Thus, *fixing bidder w's type* at v, the difference between a FPA and SPA is that the weak bidder wins in the former but loses in the latter if the strong bidder's type is in $[v, k_1(v)] \subset [v, r(v)]$ (a vertical distance in Figure 3). If $k_1 \leq \kappa$, the allocation in the FPA is more profitable; the weak bidder is eating his way into an area where his marginal revenue exceeds the strong bidder's marginal revenue. However, if $k_1 > \kappa$, the trend reverses – the weak bidder is now winning too often, beating the strong bidder even though he has a comparably low marginal revenue. This occurs when v is close to α_w, since $k_1(\alpha_w) = r(\alpha_w) = \alpha_s$, but $\kappa(\alpha_w) < \alpha_s$. Hence, depending on the strong bidder's *actual* type, switching from the SPA to the FPA may or may not increase the winner's marginal revenue. Therefore, Maskin and Riley (2000) conclude that mechanism design is of no use in determining which auction generates more revenue.

However, the concern is with expected revenue. All that is required is to determine which of the two conflicting effects dominates *in expectation*. Mechanism design can in fact be used to address this question, as follows.

Myerson (1981) shows that expected revenue in any mechanism is equal to the expected value of the winning bidder's marginal revenue, less any rent earned by the lowest types. Starting with this principle, the key step is to formulate expected revenue to capture the trade-off discussed above. Specifically, the trick is to write the expected value of the winner's marginal revenue as the expected value of a conditional expectation.

Consider some mechanism where bidder w with type v wins if and only if bidder s's type is below $k(v)$, and let $u_i^k(\beta_i)$ denote bidder i's expected utility if his type is β_i, $i = s, w$. If bidder w loses (wins) with probability one, then $k(v) = \beta_s$ ($k(v) = \alpha_s$). Then, expected revenue can be written

$$ER^k = \int_{\beta_w}^{\alpha_w} \left(J_w(v) F_s(k(v)) + \int_{k(v)}^{\alpha_s} J_s(x) dF_s(x) \right) dF_w(v) - u_w^k(\beta_w) - u_s^k(\beta_s). \quad (3)$$

The term in parenthesis is the expected value of the winning bidder's marginal revenue, conditional on the weak bidder's type being v.

Equation (3) is the counterpart to Lemma 4.1 and Lemma 4.2 in Maskin and Riley (2000), in which they derive expressions for revenue in the two particular auctions they study. The proofs of these lemmata are somewhat technical and offer little economic insight.

Whether the auction is a FPA or SPA, the weak bidder wins with probability zero and earns zero rent if his type is β_w. The same is true for the strong bidder with type

β_s in the case where $\beta_s = \beta_w$.

However, the strong bidder earns positive rent if $\beta_s > \beta_w$. In this case,

$$u_s^2(\beta_s) = \int_{\beta_w}^{\beta_s} (\beta_s - v) dF_w(v) \tag{4}$$

in the SPA, assuming that bidders use the weakly dominant strategy of submitting a bid that equals the bidder's type. In a FPA, the strong bidder with type β_s receives expected payoff of

$$u_s^1(\beta_s) = (\beta_s - b_*) F_w(b_*) \tag{5}$$

where b_* is the bid submitted by the strong bidder with type β_s, where $b_* \in (\beta_w, \beta_s)$. Such a bid wins with probability $F_w(b_*)$ because the weak bidder does not bid above b_* if his type is at or below b_*. Thus, the strong bidder with type β_s prefers the SPA if $\beta_s > \beta_w$. Subtracting (5) from (4) yields

$$u_s^2(\beta_s) - u_s^1(\beta_s) = \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\beta_s} (\beta_s - v)\, dF_w(v). \tag{6}$$

Since bidder w wins more often in the FPA than the SPA, $k_1(v) \geq k_2(v)$. Let

$$D(v|k_1, k_2) = \int_{k_2(v)}^{k_1(v)} (J_w(v) - J_s(x))\, dF_s(x). \tag{7}$$

$D(v|k_1, k_2)$ measures the consequences of the change in allocation for a fixed value of v – the seller obtains $J_w(v)$ by sacrificing $J_s(x)$ when he moves from a SPA to a FPA.

From (3),

$$ER^1 - ER^2 = \int_{\beta_w}^{\alpha_w} D(v|k_1, k_2) dF_w(v) + u_s^2(\beta_s) - u_s^1(\beta_s). \tag{8}$$

The allocation is the same in both auctions if bidder w's type falls below b_* since he loses with probability one ($k_1(v) = k_2(v) = \beta_s$). Thus, $D(v|k_1, k_2) = 0$ for $v \in [\beta_w, b_*]$. Using (6), (8) can then be expanded to

$$\begin{aligned} ER^1 - ER^2 \;=\; & \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\beta_s} (\beta_s - v + D(v|k_1, k_2))\, dF_w(v) \\ & + \int_{\beta_s}^{\alpha_w} D(v|k_1, k_2) dF_w(v). \end{aligned} \tag{9}$$

The final step of the proof is to show that each term is positive by showing that the function under the integration sign is positive for every value of v in the relevant range. This is trivially true for the first term. It turns out to also be true for the last term, because (1) implies that D is positive in this case. However, for $v \in [b_*, \beta_s)$, D may be negative. The reason is illustrated in Figure 3b. When $\beta_w < \beta_s$ it is possible that $J_w(\beta_w) < J_s(\beta_s)$, meaning that the weak bidder should ideally never win if his type is close to β_w, yet he may do so in a FPA. However, the extra rent the seller appropriates from the strong bidder with type β_s in the FPA is more than enough to compensate for this particular drawback. The case with $\beta_s > \alpha_w$ (no overlap) is handled in a similar manner.

Theorem 1 *Assume $F_w \leq_{rh} F_s$ and condition (1) holds. Then, the FPA generates strictly higher expected revenue than the SPA.*

Proof. Assume first that $\beta_s \leq \alpha_w$. If $v \in [\beta_s, \alpha_w]$ then $k_1(v) \in [v, r(v)]$ and $k_2(v) = v$, in which case

$$\begin{aligned}
D(v|k_1, k_2) &= J_w(v)(F_s(k_1) - F_s(v)) + k_1(1 - F_s(k_1)) - v(1 - F_s(v)) \\
&= -\frac{1 - F_w(v)}{f_w(v)}(F_s(k_1) - F_s(v)) + (k_1 - v)(1 - F_s(k_1)) \\
&= \frac{1 - F_s(k_1)}{f_w(v)} \left[f_w(v)(k_1 - v) - \frac{1 - F_w(v)}{1 - F_s(k_1)} (F_s(k_1) - F_s(v)) \right] \\
&\geq \frac{1 - F_s(k_1)}{f_w(v)} \left[f_w(v)(k_1 - v) - (F_s(k_1) - F_s(v)) \right] \\
&= \frac{1 - F_s(k_1)}{f_w(v)} \int_v^{k_1} (f_w(v) - f_s(x))\, dx \geq 0,
\end{aligned}$$

where the first inequality follows from $F_s(k_1) \leq F_w(v)$ when $k_1(v) \leq r(v)$, and the second inequality from condition (1). Since $k_1(v) < r(v)$ almost always, the first inequality is strict almost always. If $v \in [b_*, \beta_s]$ then $k_2(v) = \beta_s$, while $k_1(v) \in [\beta_s, r(v)]$. Hence,

$$\begin{aligned}
\beta_s - v + D(v|k_1, k_2) &= \beta_s - v + J_w(v)F_s(k_1) + k_1(1 - F_s(k_1)) - \beta_s \\
&= \frac{1 - F_s(k_1)}{f_w(v)} \left[f_w(v)(k_1 - v) - \frac{1 - F_w(v)}{1 - F_s(k_1)} F_s(k_1) \right] \\
&\geq \frac{1 - F_s(k_1)}{f_w(v)} \int_v^{k_1} (f_w(v) - f_s(x))\, dx,
\end{aligned}$$

which is positive, by assumption. The last step uses the fact that $f_s(x) = 0$ for $x \in [v, \beta_s)$. In conclusion, every term in (9) is positive (and strictly positive almost always). Hence, the FPA is strictly more profitable than the SPA.

If $\beta_s > \alpha_w$ (no overlap),

$$\begin{aligned} u_s^2(\beta_s) - u_s^1(\beta_s) &= \int_{\beta_w}^{\alpha_w} (\beta_s - v) dF_w(v) - (\beta_s - b_*) F_w(b_*) \\ &= \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\alpha_w} (\beta_s - v)\, dF_w(v). \end{aligned}$$

Since $k_2(v) = \beta_s$ and $k_1(v) \in [\beta_s, r(v)]$, with $k_1(v) = \beta_s$ for $v \leq b_*$,

$$\begin{aligned} ER^1 - ER^2 &= \int_{b_*}^{\alpha_w} D(v|k_1, k_2) dF_w(v) + u_s^2(\beta_s) - u_s^1(\beta_s) \\ &= \int_{\beta_w}^{b_*} (b_* - v)\, dF_w(v) + \int_{b_*}^{\alpha_w} (\beta_s - v + D(v|k_1, k_2))\, dF_w(v). \end{aligned}$$

The remainder of the proof is identical to the $\beta_s \leq \alpha_w$ case. Maskin and Riley (2000) point out that if β_s is much larger than α_w, the equilibrium of the FPA has the strong bidder always bidding α_w and winning. This case corresponds to $b_* = \alpha_w$. ■

4 Intermediate dispersion, mixtures, and rank-mixtures

The purpose of this section is to illustrate both the scope and limitations of Theorem 1. Maskin and Riley's (2000) two propositions in which the FPA dominates are shown to be corollaries of Theorem 1. Their propositions turn out to be useful benchmarks, because they lie on opposite boundaries of Theorem 1's "domain".

The section begins with some preliminary observations related to the conditions in Theorem 1. First, condition (1) is particularly simple to check if f_s is monotonic. The proof of the following Lemma is trivial and is therefore omitted.

Lemma 1 *Condition (1) is satisfied if:*

1. *f_s is increasing on S_s and $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$, or*

2. *f_s is decreasing on S_s, $f_w(v) \geq f_s(v)$ $\forall v \in C$, and f_w is decreasing on $[\beta_w, \beta_s)$.*

For any $v \in C$,

$$\frac{d}{dv}\left(\frac{F_w(v)}{F_s(v)}\right) = \frac{d}{dv}\left(\frac{F_s(r(v))}{F_s(v)}\right) \propto \frac{f_s(r(v))r'(v)}{F_s(r(v))} - \frac{f_s(v)}{F_s(v)}. \tag{10}$$

Dispersion requires $r'(v) \geq 1$. Thus, if F_s is locally log-convex ($\frac{f_s(v)}{F_s(v)}$ is locally increasing) the right hand side may easily be positive, thereby violating $F_w \leq_{rh} F_s$. Thus, the dual assumption of $F_w \leq_{disp} F_s$ and $F_w \leq_{rh} F_s$ is more likely to be satisfied when F_s is log-concave, when $C \neq \emptyset$. Incidentally, Lebrun (2006) has shown that equilibrium in the FPA is essentially unique if $\beta_s > \beta_w$, or if $\beta_s = \beta_w$ and F_i is strictly log-concave close to β_s, $i = s, w$. Maskin and Riley (2000) assume F_s is log-concave in their examples in which the FPA dominates. This assumption will typically also be imposed here, but the following example shows that it is not necessary.

Example 0 (concave vs. convex): Assume F_w is concave and F_s is convex, with $f_w(\alpha_w) \geq f_s(\alpha_s)$. F_s need not be log-concave. Note that $r(v)$ must be concave, or $F_s \leq_c F_w$. The curvature assumptions imply $F_w \leq_{lr} F_s$ and therefore $F_w \leq_{rh} F_s$. Condition (1) is satisfied since densities are monotonic and $f_w(\alpha_w) \geq f_s(\alpha_s)$. ▲

At times, a stronger assumption will be imposed, namely that $F_s(e^v)$ is log-concave. Since $F_s(e^v)$ is log-concave if and only if the function $v\frac{f_s(v)}{F_s(v)}$ is decreasing, log-concavity of $F_s(e^v)$ requires that the reverse hazard rate falls sufficiently rapidly. Both $F_i(e^v)$ and $F_i(v)$, $i = s, w$, play a role in this paper, in part because

$$F_w(v) \leq_* F_s(v) \Longleftrightarrow F_w(e^v) \leq_{disp} F_s(e^v).^9 \tag{11}$$

For any $\alpha \in (\beta_s, \alpha_s)$, define

$$F_s^t(v|\alpha) = \frac{F_s(v)}{F_s(\alpha)},\ v \in [\beta_s, \alpha]. \tag{12}$$

Since $F_s^t(\cdot|\alpha)$ has the same reverse hazard rate as F_s, $F_s^t(\cdot|\alpha) \leq_{rh} F_s$. Thus, $F_s^t(\cdot|\alpha)$ may serve as a benchmark against which other distributions can be compared. Indeed, in one of Maskin and Riley's (2000) examples, F_w can be thought of as a truncation of F_s (see below). The next result links dispersion and reverse hazard rate dominance.

[9] Consider two random variables, X and Y. By Theorem 4.B.1 in Shaked and Shanthikumar (2007), $X \leq_* Y \Longleftrightarrow \log X \leq_{disp} \log Y$. The relationship in (11) comes from the fact that if X is distributed according to $F(x)$ then $\log X$ is distributed according to $F(e^x)$.

Lemma 2 *Assume the upper end-point of F_w's support is $\alpha_w \in (\beta_s, \alpha_s)$. Then:*

1. *If F_s is log-concave then $F_s^t(\cdot|\alpha_w) \leq_{disp} F_s$; if $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$ then $F_w \leq_{rh} F_s$.*

2. *If $F_s(e^v)$ is log-concave then $F_s^t(\cdot|\alpha_w) \leq_* F_s$; if $F_s^t(\cdot|\alpha_w) \leq_* F_w$ then $F_w \leq_{rh} F_s$.*

Proof. Assume first that F_s is log-concave. F_s^t can be written in one of two ways, $F_s^t(v|\alpha_w) = \frac{F_s(v)}{F_s(\alpha_w)}$ or $F_s^t(v|\alpha_w) = F_s(r^t(v))$. Thus, $F_s(r^t(v)) = \frac{F_s(v)}{F_s(\alpha_w)}$ and so

$$r^{t\prime}(v) = \frac{1}{F_s(\alpha_w)} \frac{f_s(v)}{f_s(r^t(v))} = \frac{f_s(v)}{F_s(v)} \frac{F_s(r^t(v))}{f_s(r^t(v))} \geq 1$$

by log-concavity, as $r^t(v) \geq v$. Thus, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_s$. Next, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w \Longrightarrow F_w \leq_{st} F_s^t(\cdot|\alpha_w)$ since the upper bound of the supports, α_w, are the same. Thus, $S_w \supset C$. Since $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$, $f_w(v) \leq f_s^t(x|\alpha_w)$ must hold for any $v \in S_w$, where x satisfies $F_w(v) = F_s^t(x|\alpha_w)$. Since $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$, $x \geq v$. Thus, for any $v \in C$,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_w(v)}{F_s^t(x|\alpha_w)} \leq \frac{f_s^t(x|\alpha_w)}{F_s^t(x|\alpha_w)} = \frac{f_s(x)}{F_s(x)} \leq \frac{f_s(v)}{F_s(v)},$$

where the second inequality comes from the log-concavity of F_s. This proves the first part of the Lemma. By (11) and the assumed log-concavity of the function $F_s(e^v)$, the proof of the first part can be applied to prove the second part. ■

4.1 Examples on the boundary of Theorem 1

To provide a first illustration of Theorem 1, consider the following three examples. Each is at a "boundary" of Theorem 1. Examples 1 and 3 are somewhat generalized versions of the models examined in Maskin and Riley (2000).

EXAMPLE 1 (HORIZONTAL SHIFTS): Assume that F_s is convex and that F_w is obtained by shifting F_s to the left. That is, $F_w(v) = F_s(v + a)$, for $v \in [\beta_w, \alpha_w]$, where $a = \beta_s - \beta_w = \alpha_s - \alpha_w > 0$ and $\beta_w \geq 0$. Since $r(v) = v + a$, $F_w =_{disp} F_s$. By Lemma 1, (1) is satisfied. Assume that F_s is log-concave, which implies that

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v + a)}{F_s(v + a)} \leq \frac{f_s(v)}{F_s(v)}, \text{ for all } v \in C.$$

Hence, both assumptions of the theorem are satisfied. ▲

REMARK 1: Maskin and Riley (2000) assume that $J_w(v) \leq 0$ for v close to β_w. This assumption is not necessary here; the bounds on revenue are tighter because (9) makes better use of the fact that the seller appropriates more rent from the strong bidder with type β_s in the FPA. On the other hand, Maskin and Riley (2000) allow for a mass point at β_i, or $F_i(\beta_i) \geq 0$. $\triangle$

EXAMPLE 2 (VERTICAL SHIFTS): Assume F_s and F_w are concave and that F_w is a vertical shift of F_s on $C \neq \emptyset$. That is, $F_w(v) = F_s(v) + 1 - F_s(\alpha_w)$ for $v \in [\beta_s, \alpha_w]$, where $\alpha_s > \alpha_w > \beta_s > 0$. On $[\beta_w, \beta_s)$, F_w is some (unspecified) concave function, with $\beta_w \geq 0$. For $v \in C$,

$$\frac{F_w(v)}{F_s(v)} = \frac{1 - F_s(\alpha_w)}{F_s(v)} + 1,$$

which is decreasing. Hence, $F_w \leq_{rh} F_s$. By concavity, $f_w(v) \geq f_s(x)$ for all $x \in [v, \alpha_s]$, implying that (1) is satisfied as well. $\blacktriangle$

REMARK 2: Comparing Examples 1 and 2, the former satisfies $f_w(v) = f_s(r(v))$ and the latter $f_w(v) = f_s(v)$ on C. Hence, (1) is satisfied "with equality" at one of the endpoints of the interval $[v, r(v)]$. $\triangle$

EXAMPLE 3 (TRUNCATIONS AND STRETCHES): Assume F_s is log-concave and that F_w is a truncation of F_s, i.e. $F_w = F_s^t$ as defined earlier. It has already been established that $F_s^t \leq_{rh} F_s$. By log-concavity,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(v)}{F_s(v)} \geq \frac{f_s(x)}{F_s(x)}$$

for any $x \in [v, r(v)]$. For any x in this range, $F_s(x) \leq F_w(v)$. The above inequality then necessitates that $f_s(x) \leq f_w(v)$ for all $x \in [v, r(v)]$, implying (1). $\blacktriangle$

REMARK 3: Maskin and Riley's (2000) set-up is slightly different. They say that F_s is obtained by "stretching" F_w, so that $F_s(v) = \lambda F_w(v)$ on $[\beta_w, \alpha_w]$, for some $\lambda \in (0, 1)$. This leaves the problem of what F_s looks like on $[\alpha_w, \alpha_s]$, and they are then forced to make restrictive assumptions on this as well (compare (4.13) in their paper to (1) in the current paper). Unfortunately, their conditions rule out the much-studied class of convex power distributions, where $F_s(v) = (v/\alpha_s)^{\gamma_s}$ for some $\gamma_s > 1$, $v \in [0, \alpha_s]$, and all other distribution functions for which $f_s(\beta_s) = 0$. In contrast, the power distribution satisfies all the assumptions in Example 3 of the current paper. $\triangle$

The more interesting and challenging case is when $C \neq \emptyset$ since both conditions in Theorem 1 then comes into play. Therefore, consider F_s fixed and assume $\alpha_w > \beta_s$.

For any $\alpha \in (\beta_s, \alpha_s)$, let $F_s^h(v|\alpha) = F_s(v+\alpha_s-\alpha)$ denote a horizontal and left-ward shift of F_s, such that the new distribution's support ends at α. The weak bidder's distribution in Example 1 takes this form. Let β^h denote the lowest end-point of this distribution's support. Technically, it is possible that $\beta^h < 0$ (depending on α), in which case the distribution does not satisfy the assumptions made in Section 2. Nevertheless, it remains a useful benchmark. Let $r^h(v|\alpha) = F_s^{-1}(F_s^h(v|\alpha))$. Similarly, let $F_s^v(v|\alpha)$ denote a vertical shift of F_s (as in Example 2), with the added requirement that $f_s^v(v|\alpha) = f_s(\beta_s)$ for all $v \in [\beta^v, \beta_s]$. Here, it is also possible that $\beta^v < 0$. Finally, as in Lemma 2, let $r^t(v|\alpha) = F_s^{-1}(F_s^t(v|\alpha))$.

Whenever F_s is log-concave, $F_s^h(\cdot|\alpha) \leq_{rh} F_s^t(\cdot|\alpha)$ and $F_s^v(\cdot|\alpha) \leq_{rh} F_s^t(\cdot|\alpha)$. Since the upper end-points of the supports coincide, $F_s^t(v|\alpha) \leq \min\{F_s^h(v|\alpha), F_s^v(v|\alpha)\}$, for all $v \in [\beta_s, \alpha]$. Next, consider a distribution function, $F_w(v)$, with support $[\beta_w, \alpha_w]$, where $\alpha_w \in (\beta_s, \alpha_s)$ and $\beta_w \in [\max\{0, \beta^h, \beta^v\}, \beta_s]$. Assume that $F_w(v) \leq \min\{F_s^h(v|\alpha_w), F_s^v(v|\alpha_w)\}$ for all $v \in S_w$ and $F_w(v) \geq F_s^t(v|\alpha_w)$ for all $v \in C$. Let $\mathcal{F}_w(\alpha_w)$ denote the set of distributions with these properties. It is now possible to more precisely characterize Theorem 1's "domain".

Proposition 1 *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume F_s is log-concave. If the conditions in Theorem 1 are satisfied then $F_w \in \mathcal{F}_w(\alpha_w)$.*

Proof. Assume $F_w(x) < F_s^t(x|\alpha_w)$ for some $x \in [\beta_s, \alpha_w)$. Then, $F_w(\alpha_w) = F_s^t(\alpha_w|\alpha_w)$ necessitates $f_w(v) > f_s^t(v|\alpha_w)$ for some $v \in [x, \alpha_w]$ where $F_w(v) < F_s^t(v|\alpha_w)$. Since $F_s^t(\cdot|\alpha_w)$ has the same reverse hazard rate as F_s, F_w violates the first condition in Theorem 1.

If $F_w(x) > F_s^h(x|\alpha_w)$ for some $x \in (\beta_w, \alpha_w)$ then F_w cannot be less disperse than F_s^h and still satisfy $F_w(\alpha_w) = F_s^h(\alpha_w|\alpha_w)$. Since $F_s^h =_{disp} F_s$, condition (1) is then violated. For similar reasons, condition (1) is violated if $F_w(x) > F_s^v(x|\alpha_w)$ for some $x \in (\beta_w, \alpha_w)$. ■

Proposition 1 means that Theorem 1 cannot be used to compare revenue in the FPA and SPAs unless F_w "lies between" a truncation and a shift of F_s.[10] The remainder of the section describes environments where a ranking can be obtained.

[10]There are two distinct reasons why it is difficult to extend the revenue ranking to $F_w \notin \mathcal{F}_w(\alpha_w)$. First, if F_w lies anywhere below $F_s^t(\cdot|\alpha_w)$ then reverse hazard rate dominance is violated, and behavior in the first-price auction becomes harder to describe. Second, if F_w is anywhere above

4.2 f_s is monotonic

If f_s is increasing then $F_s^v(v|\alpha_w) \leq_{st} F_s^h(v|\alpha_w)$. The opposite holds if f_s is decreasing. In either case, Lemma 1 makes it simpler to check condition (1).

Corollary 1 (Intermediate dispersion) *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume F_s is convex but log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w \leq_{disp} F_s^h(\cdot|\alpha_w) =_{disp} F_s$.*[11]

Proof. Lemma 1 implies (1) is satisfied. Lemma 2 establishes $F_w \leq_{rh} F_s$. ■

Figure 1 in the introduction illustrates Corollary 1. It applies if, for instance,

$$F_w(v) = \frac{F_s(v + \beta_s - \beta_w)}{F_s(\alpha_w + \beta_s - \beta_w)}, \; v \in [\beta_w, \alpha_w], \tag{13}$$

such that F_w is obtained by first shifting F_s leftward, and then truncating it.[12] Maskin and Riley (2000, p. 423) allude to this possibility, but do not provide any details or proof.

Using the logic of Proposition 1, Theorem 1 is violated if either $F_w \leq_{disp} F_s^t(\cdot|\alpha_w)$ or $F_s^h(\cdot|\alpha_w) \leq_{disp} F_w$. Thus, Corollary 1 signifies that intermediate dispersion of F_w, compared to the benchmarks, are "almost" necessary and sufficient for the conditions of Theorem 1 to hold when F_s is convex and log-concave. The qualifier is due to the fact that the dispersive order is not a complete order.

Corollary 1 relates F_w to the benchmark distributions $F_s^h(v|\alpha_w)$ and $F_s^t(v|\alpha_w)$. It is also of interest to compare F_w directly to F_s. In the following, the assumption that F_s is log-concave is strengthened. It is then possible to describe qualitative relationships between F_w and F_s that are sufficient for the FPA to dominate the SPA. Recall that if $F_w \leq_{disp} F_s$, then the absolute distance between $r(v)$ and v is increasing. The implication of the next result is that the FPA is superior if the asymmetry between bidders do not increase too fast with type, or $\frac{r(v)}{v}$ is decreasing.

$\min\{F_s^h(v|\alpha), F_s^v(v|\alpha)\}$ then condition (1) is not strong enough. This is a more manageable problem, as (1) only needs to be satisfied "on average". See Section 4.3 for a strengthening of Theorem 1.

[11] The assumptions in the proposition imply that $\beta_w \in [\beta_w^h, \beta_s]$. It should be understood from the description of the model in Section 2 that it is also required that $\beta_w \geq 0$.

[12] Write $F_w(v) = F_s^t(\widehat{r}(v)|\alpha_w)$. If F_w is described by (13) then $\widehat{r}(v) \leq v + \beta_s - \beta_w$ and

$$\widehat{r}'(v) = \frac{f_s(v + \beta_s - \beta_w)}{F_s(v + \beta_s - \beta_w)} \frac{F_s(\widehat{r}(v))}{f_s(\widehat{r}(v))} \leq 1,$$

by log-concavity. Thus, $F_s^t(\cdot|\alpha_w) \leq_{disp} F_w$. The proof that $F_w(v) \leq_{disp} F_s^h(\cdot|\alpha_w)$ is similar.

Corollary 2 *Assume $F_s(v)$ is convex and $F_s(e^v)$ is log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $F_w \leq_{disp} F_s \leq_* F_w$.*[13]

Proof. Assume $\alpha_w \geq \beta_s$. By Lemma 2, $F_s \leq_* F_w$ implies $F_s^t(\cdot|\alpha_w) \leq_* F_s \leq_* F_w$ and therefore $F_w \leq_{rh} F_s$. Since $F_w \leq_{disp} F_s$ and F_s is convex, condition (1) is satisfied as well. The result also holds if $\alpha_w < \beta_s$, since only condition (1) is required in that case. ■

A counterpart to Corollary 2 exists when F_s is concave.

Corollary 3 *Assume F_s is concave and that $F_w(v) = G(F_s(v))$, $v \in [\beta_s, \alpha_w]$, with $\alpha_w > \beta_s$ and $G'(\cdot) \geq 1$. If $G(x)/x$ is decreasing then the FPA yields strictly higher expected revenue than the SPA. $G(x)/x$ is decreasing if G is concave.*[14]

Proof. $\frac{F_w(v)}{F_s(v)} = \frac{G(F_s(v))}{F_s(v)}$ is decreasing by assumption, implying $F_w \leq_{rh} F_s$. Since $f_w(v) = G'(F_s(v))f_s(v) \geq f_s(v)$, Lemma 1 ensures condition (1) is satisfied. ■

For both concave and convex F_s, the first price auction dominates when F_w satisfies a natural regularity condition. By Lemma 2, $r^{t\prime}(v|\alpha_w) \geq 1 = r^{h\prime}(v|\alpha_w)$ for all $v \in C$ when F_s is log-concave. Likewise, $f_s^t(v|\alpha_w) \geq f_s^v(v|\alpha_w)$ for all $v \in C$.

Proposition 2 (Mixtures and rank-mixtures) *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Then, the FPA yields strictly higher expected revenue than the SPA if either:*

1. *F_s is convex but log-concave and $r(v|\alpha_w)$ is steeper than $r^h(v|\alpha_w)$ but flatter than $r^t(v|\alpha_w)$; $r'(v|\alpha_w) \geq r^{h\prime}(v|\alpha_w)$ for all $v \in S_w$ and $r'(v|\alpha_w) \leq r^{t\prime}(v|\alpha_w)$ for all $v \in C$.*

2. *F_s is concave and F_w is steeper than F_s^v but flatter than F_s^t; $f_w(v) \geq f_s^v(v|\alpha_w)$ for all $v \in S_w$ and $f_w(v) \leq f_s^t(v|\alpha_w)$ for all $v \in C$.*

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[13] If F_s is a convex power distribution with $\beta_s = 0$ then $\ln F_s(e^v)$ is linear and the conditions in Theorem 1 are satisfied *if and only if* $r(0) = 0$, $r'(v) \geq 1$, and $r(v)/v$ is decreasing. It is possible to construct examples where f_s is increasing but f_w has a peak. Assume $f_s(v) = 2v$, $v \in [0, 1]$ and $r(v) = 4ve^{-v}$, $v \in [0, 0.357]$. Then, $f_w(v) = 2r(v)r'(v)$ is non-monotonic. This result should be contrasted with Maskin and Riley's (2000) two models, in which f_w never has more peaks than f_s (see remark 4, below).

[14] The transformation in Example 3 is linear and thus a special case of the one in Corollary 3. However, with the linear transformation in Example 3 it is possible to weaken the assumption that F_s is concave and instead assume only that it is log-concave.

Proof. For the first part, the assumptions that $r'(v|\alpha_w) \geq r^{h\prime}(v|\alpha_w) = 1$ and F_s is convex mean that condition (1) is satisfied, by the first part of Lemma 1. The assumptions in the proposition also imply that $r^t(v|\alpha_w) \leq r(v|\alpha_w) \leq r^h(v|\alpha_w)$ on C. Thus, for $v \in C$,

$$\frac{f_w(v)}{F_w(v)} = \frac{f_s(r(v|\alpha_w))}{F_s(r(v|\alpha_w))} r'(v|\alpha_w) \leq \frac{f_s(r^t(v|\alpha_w))}{F_s(r^t(v|\alpha_w))} r'(v|\alpha_w) \leq \frac{f_s(r^t(v|\alpha_w))}{F_s(r^t(v|\alpha_w))} r^{t\prime}(v|\alpha_w) = \frac{f_s(v)}{F_s(v)},$$

where the first inequality comes from $r(v|\alpha_w) \geq r^t(v|\alpha_w)$ and the log-concavity of F_s. The second inequality comes from $r'(v|\alpha_w) \leq r^{t\prime}(v|\alpha_w)$.

By Lemma 1, the concavity of F_s and $f_w(v) \geq f_s^v(v|\alpha_w) \geq f_s(\max\{v, \beta_s\})$ for all $v \in S_w$ ensures condition (1) is satisfied. Since $f_w(v) \leq f_s^t(v|\alpha_w)$ for all $v \in C$, $F_w \leq_{st} F_s^t(\cdot|\alpha_w)$ and therefore

$$\frac{f_w(v)}{F_w(v)} \leq \frac{f_s^t(v|\alpha_w)}{F_s^t(\cdot|\alpha_w)} = \frac{f_s(v)}{F_s(v)},$$

thereby proving $F_w \leq_{rh} F_s$. ■

The second part of Proposition 2 applies if F_w is a convex combination (a mixture) of F_s^v and F_s^t on C, (with an appropriate differentiable extension on S_w/C). Similarly, the first part applies if F_w is a "rank-mixture" of F_w^h and F_w^t. Clearly, this part of Proposition 2 has a similar flavor as Corollary 1. However, neither implies the other.

4.3 f_s is non-monotonic

Example 3 does not require f_s to be monotonic. In this sub-section it is demonstrated that a strengthening of Theorem 1 can be used in other environments with non-monotonic densities. A counterpart to Corollary 1 is also derived.

To begin, it is clear from the proof of Theorem 1 that condition (1) is not necessary; it can be replaced with the weaker condition that

$$\int_v^{k_1(v)} (f_w(v) - f_s(x))\, dx \geq 0 \text{ for all } v \in S_w.^{15} \tag{14}$$

Assume from now on that $F_s(e^v)$ is convex but log-concave.[16] Hence, $vf_s(v)$ is

[15] The inequality is satisfied for any $v < b_*$, since $k_1(v) = \beta_s$ and $f_s(x) = 0$ for all $x \in [v, k_1(v))$ in that case.

[16] If F is the uniform distribution then $F(e^v)$ is both strictly convex and strictly log-concave

increasing but $\frac{vf_s(v)}{F_s(v)}$ is decreasing. The density need not be monotonic.

Lemma 3 *Assume $F_s(e^v)$ is convex, $F_w \leq_* F_s$, and $F_w \leq_{rh} F_s$ if $C \neq \emptyset$. Then, condition (14) is satisfied.*

Proof. Assume $v \in C$. Since $f_w(v) = f_s(r(v))r'(v)$, the left hand side of (14) equals

$$f_s(r(v))r(v)\left[\frac{r'(v)}{r(v)}(k_1(v)-v) - \int_v^{k_1(v)} \frac{f_s(x)x}{f_s(r(v))r(v)}\frac{1}{x}dx\right],$$

where the first term under the integration is less than one because $vf_s(v)$ is increasing (since $F_s(e^v)$ is convex) and $x \leq k_1(v) \leq r(v)$ (since $F_w \leq_{rh} F_s$). The sign of the above expression is determined by the terms inside the square brackets, which is no smaller than

$$\begin{aligned}\frac{r'(v)}{r(v)}(k_1(v)-v) - \int_v^{k_1(v)} \frac{1}{x}dx &= (k_1(v)-v)\left[\frac{r'(v)}{r(v)} - \frac{\ln k_1(v) - \ln v}{k_1(v)-v}\right] \\ &\geq (k_1(v)-v)\left[\frac{r'(v)}{r(v)} - \frac{1}{v}\right] \geq 0,\end{aligned}$$

where the first inequality is due to concavity of the ln function and the second to $F_w \leq_* F_s$. Hence, condition (14) is satisfied. The argument is similar if $v \notin C$. ■

Example 4 (rescaling): Assume $F_s(e^v)$ is convex but log-concave. Assume $S_w = \left[\frac{\beta_s}{\gamma}, \frac{\alpha_s}{\gamma}\right]$, where $\gamma > 1$. Thus, either $\beta_s = \beta_w = 0$ or $\beta_s > \beta_w > 0$. Finally, assume $r(v) = \gamma v$, which implies $F_w =_* F_s$. If $C \neq \emptyset$, then by Lemma 2, $F_s^t \leq_* F_s =_* F_w$ and therefore $F_w \leq_{rh} F_s$. Lemma 3 implies condition (14) is satisfied. Thus, the FPA dominates the SPA. ▲

Remark 4: The difference between Examples 3 and 4 is significant. In the former, the truncation changes the shape of the density. For example, f_w may be monotonic even if f_s is not. In contrast, the rescaling in Example 4 preserves the shape of the

whenever $\beta_s > 0$. Write $F_s(v) = (1-\varepsilon)F(v) + \varepsilon H(v)$, $\varepsilon \in (0,1)$, where $F(v)$ is the uniform distribution and $H(v)$ is a distribution on S_s. Assume H has finite density. If $\beta_s > 0$, $F_s(e^v)$ is then convex and log-concave when ε is sufficiently small. Note that f_s may have arbitrarily many peaks. As another example, assume F_s is obtained by truncating a Normal distribution with mean $.5\beta_s + .5\alpha_s$. If $\beta_s > 0$ and the variance is sufficiently large, then $F_s(e^v)$ is convex and log-concave (the truncated Normal distribution converges to the uniform distribution as the variance increases).

density. The examples coincide only if F_s is a power distribution with $\beta_s = 0$. There are two ways of transforming F_s to get F_w; F_w can be written $F_w(v) = G(F_s(v))$ or $F_w(v) = F_s(r(v))$. G is a linear transformation in Example 3. On the other hand, it is r that is linear in Example 4.[17]

In Maskin and Riley's (1985) first paper on asymmetric auctions, each bidder has one of two types, β_i or α_i, $\alpha_i > \beta_i \geq 0$, $i = s, w$. The probability that bidder i has type α_i is p_i. They show that the FPA dominates the second price auction if $\beta_w = \beta_s = 0$, $\alpha_s > \alpha_w$ and $p_s = p_w$. Since a truncation would change the probabilities, this model is inconsistent with a truncation. Instead, the model (and the conclusion) is consistent with Example 4. As Maskin and Riley (1985) state, the FPA dominates "when bidders have distributions with the same shape (but different supports)."

However, Examples 1 and 4 are intimately related. The difference between bidders' valuations is an additive term in Example 1, $r(v) = v+a$. Thus, $F_w =_{disp} F_s$; $r(v)-v$ is constant because the dispersive order is location free. In Proposition 3, the difference is a multiplicative term, $r(v) = \gamma v$. Therefore, $F_w =_* F_s$; $\frac{r(v)}{v}$ is constant because the star order is scale free. An extension of these models is considered in the next section. For completeness, note that the convex transform order is scale and location free; if $r(v) = \gamma v + a$ then $F_w =_c F_s$. $\triangle$

Let $F_s^r(v|\alpha_w)$ denote a rescaling of F_s such that $\frac{\alpha_s}{\gamma} = \alpha_w$. A counterpart to Corollary 1 can now be derived. Recall from Example 4 that $F_s^t \leq_* F_s^r =_* F_s$.

Proposition 3 *Fix F_s and $\alpha_w \in (\beta_s, \alpha_s)$. Assume $F_s(e^v)$ is convex but log-concave. Then, the FPA yields strictly higher expected revenue than the SPA if $F_s^t(\cdot|\alpha_w) \leq_* F_w \leq_* F_s^r(\cdot|\alpha_w)$.*

Proof. Since $F_s^t(\cdot|\alpha_w) \leq_* F_w$, Lemma 2 ensures $F_w \leq_{rh} F_s$. Lemma 3 guarantees condition (14) is satisfied. ■

Proposition 3 applies if F_w is obtained by first scaling down F_s, and then truncating it, $F_w(v) = \frac{F_s(\gamma v)}{F_s(\gamma \alpha_w)}$, where $\beta_w = \frac{\beta_s}{\gamma}$, $\alpha_w \in \left(\frac{\beta_s}{\gamma}, \frac{\alpha_s}{\gamma}\right]$, and $\gamma \geq 1$.

[17] Bagnoli and Bergstrom (2005, Section 5.3) write that a truncation is a "linear transformation" of the original distribution function. Unfortunately, they appeal to a result that relies on r, not G, being linear in order to "prove" their Theorem 9, which is erroneous. It is easy to construct examples where their claims concerning log-convex functions are untrue.

4.4 Other rankings

Obviously, the conditions in the Theorem are sufficient, but not necessary. Cheng (2006) provides an example in which the FPA dominates, even though (1) is not satisfied. In Cheng's (2006) example, distribution functions are power distributions of the form $F_i(v) = (v/\alpha_i)^{\gamma_i}$, $v \in [0, \alpha_i]$, with $\gamma_s > \gamma_w > 0$ and $\alpha_s > \alpha_w > 0$. When $\gamma_s \geq 1$, Theorem 1 applies whenever $\alpha_s \gamma_w \geq \alpha_w \gamma_s$. This restriction is violated in Cheng's (2006) model. However, by imposing another restriction on the parameters, he is able to characterize bidding strategies explicitly and thus calculate expected revenue. In particular, he hypothesizes linear bidding strategies and then "backwards engineer" to get the conditions that provide such an outcome. Cheng (2006) also extends the revenue ranking to situations with many weak and strong bidders.

Lebrun (1996) also considers power distributions, but he assumes $\alpha_s = \alpha_w$. In this case it is immediately clear that Theorem 1 does not apply; the combination of first order stochastic dominance and dispersion necessitates $\alpha_s > \alpha_w$. In fact, when $\alpha_s = \alpha_w$, Lebrun (1996) shows that it is always the case that $D(v|k_1, k_2)$ is negative for v close to $\alpha_s = \alpha_w$, but positive for small values of v. Remarkably, Lebrun (1996) is able to prove that, on balance, the FPA is more profitable than the SPA whenever $2\gamma_s\gamma_w \geq 1$, as it is when both distributions are convex. However, his proof relies heavily on the specific functional form of the distribution functions, and does unfortunately not generalize.

Vickrey (1961) derives strategies and expected revenue in a model where bidder 1 draws a type from a uniform distribution with support $[0, 1]$ while bidder 2's type is known to be $a > 0$ (his distribution is degenerate). The FPA is more profitable than the SPA as long as $a \geq 0.43$, with the reverse ranking obtaining otherwise. Vickrey's (1961) paper remains the only one to obtain theoretical results for cases where first order stochastic dominance does not apply. Theorem 1 does not work because $k_1(v)$ and $k_2(v)$ will cross in such cases.

Maskin and Riley (2000) presents a class of models in which the SPA dominates. In that model, the two distribution functions share the same support, F_w has a mass point at β_w, and it is flatter than F_s. Maskin and Riley (1985) contains the discrete counterpart to this model, with the same conclusion. Clearly, (1) is violated. Returning to the monopoly analogy, F_w's mass point in a sense implies that it is no longer the case that two "markets" with the same mass of consumers are being compared; the mass at β_w never wins the auction and so might as well not have been there. In Figure 2(b), where $\beta_w = 0$, the weak bidder's inverse demand curve would be steeper and hit the horizontal axis before the strong bidder's inverse demand curve. That is, the market with the lowest willingness-to-pay is the market that is the least price sensitive. Favoring the weaker market would then be a mistake, which explains why the FPA performs poorly in that set-up. See Cheng (2010) and Gavious and Minchuk (2010) for more examples of situations in which the SPA dominates.

5 Multi-dimensional types

Example 1 can be interpreted as follows: Both bidders draw a valuation, v, from the same distribution, F_w. For the weak bidder, this valuation constitutes his type. For the strong bidder, however, it is only a component of his type. For instance, the strong bidder may have additional uses of the object, or he may expect synergies between existing objects and the object for sale. Alternatively, he may suffer a negative externality should the weak bidder win. Let the additional component be worth a known constant a. The strong bidder's willingness-to-pay is then $u_s(v, a) = v + a$, $a \geq 0$. The model in Example 4 can be interpreted in a similar way, except the difference is multiplicative. Here, the strong bidder's willingness-to-pay is $u_s(v, \gamma) = v\gamma$, $\gamma \geq 1$. Examples 1 and 4 identify conditions under which the FPA dominates the SPA. However, it is natural to ask whether this ranking extends if a or γ is not known. What if a or γ is privately known and drawn from some distribution, G?

Here, the ranking will be shown to remain unchanged when G is non-degenerate. However, a technical point must be addressed first. For the additive case, what matters when describing the strong bidder is the distribution of the sum of the two components, v and a. The distribution, F_s, of this summary type, $x = v + a$, is characterized by the convolution of F_w and G. For the remainder of the section, assume G is a non-degenerate distribution with no mass-points, density g, and support $[\beta, \alpha]$, $0 \leq \beta < \alpha < \infty$. Then, the density of the convolution of F_w and G is

$$f_s(x) = \int_{\beta}^{\alpha} f_w(x - z)g(z)dz, \quad x \in [\beta_s, \alpha_s] \tag{15}$$

where $\beta_s = \beta_w + \beta$, $\alpha_s = \alpha_w + \alpha$, and $f_w(v) = 0$ if $v \notin S_w$. It is important to note that $f_s(\beta_s) = f_s(\alpha_s) = 0$, which has two implications that are reviewed in turn.

First, $f_s(\alpha_s) = 0$ violates the assumptions that are typically imposed to analyze the FPA. Nevertheless, this complication affects only the known proofs of uniqueness, such as that in Lebrun (2006). Existence of an equilibrium is still guaranteed, since the proofs in Athey (2001) and Lebrun (1996) are unaffected. Likewise, since equilibrium can be characterized by a set of differential equations (see e.g. Athey (2001)) the properties of any equilibrium are unaffected; it remains the case that $k_1(v) \in [v, r(v)]$ in any equilibrium. Thus, although the property that $f_s(\alpha_s) = 0$ is unusual it does not invalidate Theorem 1.[18,19]

Second, $f_s(\beta_s) = f_s(\alpha_s) = 0$ implies $f_s(v)$ and $vf_s(v)$ are non-monotonic. Thus, Lemma 1 and Lemma 3 are inadequate. In the following, a function is said to be unimodal if it is monotonic or has an inverse U shape (it may have regions where it is flat). If f_s is unimodal but not monotonic, let $\widehat{v}$ denote the smallest type at the peak, such that $f_s(\widehat{v}) > f_s(v)$ for all $v < \widehat{v}$.

Lemma 4 *Condition (1) is satisfied if:*

1. *F_w is increasing on S_w and $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$, or*
2. *$\beta_w = \beta_s$, f_s and f_w are unimodal on S_s and S_w, respectively, $f_w(v) \geq f_s(v)$ and $f_w(v) \geq f_s(r(v))$ for all $v \in S_w$, and $\widehat{v} \leq \alpha_w$.*

Proof. The assumptions in the first part imply $f_w(v) \geq f_w(x) \geq f_s(r(x))$ for all $x \in [\beta_w, v]$ and any $v \in S_w$. Hence, $f_w(v) \geq f_s(z)$ for all $z \leq r(v)$. The second part follows from Lemma 1 if f_s is monotonic. Hence, assume f_s is not monotonic; $\widehat{v}$ is in the interior of S_s. By assumption, $\widehat{v} \in S_w$. Since $F_w \leq_{disp} F_s$, $f_w(v) \geq f_s(r(v))$ for all $v \in [\beta_w, r^{-1}(\widehat{v})]$, an interval on which f_s is increasing since $r^{-1}(\widehat{v}) < \widehat{v}$. Thus, recalling Lemma 1, condition (1) is satisfied for all $v \in [\beta_w, r^{-1}(\widehat{v})]$. Since $f_w(r^{-1}(\widehat{v})) \geq f_s(\widehat{v})$, $f_w(\widehat{v}) \geq f_s(\widehat{v})$, and f_w is itself unimodal, it must hold that $f_w(v) \geq f_s(\widehat{v})$ for all $v \in [r^{-1}(\widehat{v}), \widehat{v}]$, and since f_s attains its peak at $\widehat{v}$, condition (1) must also be satisfied on the interval $[r^{-1}(\widehat{v}), \widehat{v}]$. As in Lemma 1, for $v > \widehat{v}$, $f_w(v) \geq f_s(v)$ combines with the monotonicity of f_s on that region to ensure that (1) is satisfied here as well. ■

[18] There are several ways in which the model could be perturbed to obtain $f_s(\alpha_s) > 0$. One is to introduce a negligible mass-point into G at $a = \alpha$. Doing so causes the density f_s to become strictly positive at $x = \alpha_s$, but without introducing a mass-point into F_s. In the same vein, imagine that the strong bidder's preferences are derived as above with probability $1 - \varepsilon$, but that his type is drawn from some other distribution function on $[\beta_s, \alpha_s]$ with probability ε, where ε is an arbitrarily small but strictly positive number.

[19] However, the weak bidder's type must be one-dimensional. If his valuation is obtained by convoluting two distributions the resulting density will be zero at α_w, making it impossible to satisfy condition (1).

A density is unimodal if it is log-concave, for example.[20] As documented by Bagnoli and Bergstrom (2005) and An (1998), many common distributions have log-concave densities. Mares and Swinkels (2010a) assume densities are log-concave throughout their paper.

Since log-concavity is preserved under integration (Prékopa (1971, 1973), An (1998) and Bagnoli and Bergstrom (2005)), the distribution F is log-concave if its density is log-concave. Moreover, if $f(e^v)$ is log-concave, the function

$$F(e^v) = \int_{\beta}^{e^v} f(x)dx = \int_{\ln\beta}^{v} f(e^z)e^z dz$$

is log-concave as well since it integrates a log-concave function. If $f(e^v)$ is log-concave, then $f(v)$ is also log-concave if it is increasing, but not necessarily if it is decreasing (for instance, let $f(v) = e^{-\sqrt{v}}$). In any case, $f(v)$ is unimodal if $f(e^v)$ is log-concave.

There is a deep and well-known connection between log-concave density functions and the dispersive order. Likewise, there is a less well-known but equally important relationship between the log-concavity of $f(e^v)$ and the star order. These relationships are explored next, in the additive and multiplicative models, respectively.

5.1 The additive model

A random variable, F, is said to be *dispersive* if the convolution of F and *any* other distribution is more dispersive than F is. In other words, F is dispersive if adding more noise makes the resulting distribution more disperse. A fundamental result due to Droste and Wefelmeyer (1985), building on Lewis and Thompson (1981), says that F is dispersive if and only if its density is log-concave. Thus, in the following it will be assumed that f_w is log-concave. It is then automatic that $F_w \leq_{disp} F_s$.

If $C \neq \emptyset$ then for any $v \in C = [\beta_w + \beta, \alpha_w]$,

$$\frac{F_s(v)}{F_w(v)} = \int_{\beta}^{\min\{\alpha, v-\beta_w\}} \frac{F_w(v-z)}{F_w(v)} g(z)dz,$$

is increasing in v because $F_w(v)$ is log-concave. Thus, $F_w \leq_{rh} F_s$. Using tools from

[20] A function f is log-concave if $f(\lambda x_1 + (1-\lambda)x_2) \geq f(x_1)^{\lambda} f(x_2)^{1-\lambda}$ for all x_1, x_2 in the support, S, of f and all $\lambda \in [0,1]$. As An (1998) emphasizes, if f is log-concave on S then the "extended" function (with $f(x) = 0$ if $x \in \mathbb{R} \backslash S$) is also log-concave on $\mathbb{R}$. In the following, any reference to the support of a log-concave function will therefore be suppressed

total positivity, Miravete (2010) provides a different proof of this property under slightly different conditions. A first extension of Example 1 can now be made.

Proposition 4 *The FPA yields strictly higher expected revenue than the SPA in the additive model if f_w is increasing and log-concave.*

Proof. $F_w \leq_{rh} F_s$ has already been established. Since $F_w \leq_{disp} F_s$, Lemma 4 implies condition (1). ■

In order to generalize Example 1 to permit G to be non-degenerate, the assumption that F_w is log-concave need only be replaced with the slightly stronger assumption that f_w is log-concave. By imposing more conditions on g, the assumption that f_w is increasing can also be relaxed. The intention is to use the second part of Lemma 4.

However, the convolution of two unimodal densities is not necessarily unimodal. Ibragimov (1956) has shown that the convolution of a log-concave density with any unimodal density is itself unimodal. Hence, a log-concave function is sometimes referred to as strongly unimodal. Since it has already been assumed that f_w is log-concave, unimodality of f_s is then guaranteed if g is unimodal.[21] It will also be assumed that $\beta = 0$, implying that $\beta_w = \beta_s$. Log-concavity of f_w also implies that $F_w \leq_{disp} F_s$, as mentioned above. However, to apply the second part of Lemma 4, it is also necessary that F_w is steeper than F_s on S_w. Unfortunately, a convolution may increase the density locally. Thus, additional assumptions are required.

Proposition 5 *The FPA yields strictly higher expected revenue than the SPA in the additive model if f_w is log-concave, $\beta = 0$, g is decreasing and satisfies*

$$f_w(\alpha_w) \geq \int_0^{\alpha} f_w(\alpha_w - z) g(z) dz. \tag{16}$$

Proof. For $v \in C = S_w$,

$$\frac{f_s(v)}{f_w(v)} = \frac{\int_0^{\alpha} f_w(v-z)g(z)dz}{f_w(v)} = \int_0^{\alpha} \frac{f_w(v-z)}{f_w(v)} g(z)dz,$$

where $f_w(v - z) = 0$ if $v - z \leq \beta_w$. Thus,

$$\frac{d}{dv}\left(\frac{f_s(v)}{f_w(v)}\right) \geq \int_0^{\alpha} \left(\frac{f'_w(v-z)}{f_w(v-z)} - \frac{f'_w(v)}{f_w(v)}\right) \frac{f_w(v-z)}{f_w(v)} g(z)dz,$$

[21] Miravete (2010) examines the properties of convolutions of two log-concave densities. He emphasizes their relevance to models of asymmetric information, including multidimensional screening.

which is positive since f_w is log-concave. Hence, $F_w \leq_{lr} F_s$ (this conclusion relies on $\beta = 0$). Condition (16) is equivalent to $\frac{f_s(\alpha_w)}{f_w(\alpha_w)} \leq 1$. Thus, since $\frac{f_s(v)}{f_w(v)}$ is increasing on S_w, $f_w(v) \geq f_s(v)$ for all $v \in S_w$. Since the convolution of f_w and g is unimodal, Lemma 4 applies if f_s peaks to the left of α_w. However, when $v > \alpha_w$,

$$f_s(v) = \int_{\max\{\beta_w, v-\alpha\}}^{\alpha_w} g(v-z) f_w(z) dz,$$

which in decreasing in v when g is decreasing. Consequently, f_s peaks at or before α_w, and the proof is now complete. ■

Condition (16) requires that $f_w(\alpha_w)$ exceeds a weighted average of f_w over the interval $[\alpha_w - \alpha, \alpha_w]$ (on which f_w may be zero if $\alpha_w - \alpha \leq \beta_w$). Thus, if α is small it rules out that f_w is decreasing. However, f_w may be non-monotonic as long as it does not "dip down" too much after it has passed its peak. The condition is less restrictive if α is large, such that the asymmetry between bidders is large.

5.2 The multiplicative model

Cuculescu and Theodorescu (1998) examine multiplication of random variables. For non-negative random variables, they show that log-concavity of f_w must be replaced by log-concavity of $f_w(e^v)$ to obtain results that mirrors those for addition of random variables.[22] That is, if a random variable with this property is multiplied with another random variable with unimodal density, then the resulting variable also has unimodal density. Likewise, the multiplicative convolution of F_w and a non-degenerate random variable G is more star disperse than F_w itself.

Assume $\beta \geq 1$, such that $\beta_s \geq \beta_w$ and F_s (the multiplicative convolution of F_w and G) first order stochastically dominates F_w. Then, $F_w \leq_* F_s$ implies $F_w \leq_{disp} F_s$.

Turning to reverse hazard rate dominance, note first that for any $v \in C$,

$$\frac{F_s(v)}{F_w(v)} = \int_{\beta}^{\alpha} \frac{F_w\left(\frac{v}{z}\right)}{F_w(v)} g(z) dz.$$

Since $f_w(e^v)$ is log-concave, $F_w(e^v)$ is log-concave as well. Equivalently, $v f_w(v)/F_w(v)$

[22] Jewitt (1987, footnote 15) makes a related observation in a model of risk aversion with two sources of uncertainty. Jewitt (1987) uses tools from total positivity, as in Miravete (2010).

is decreasing. Thus, since $z \geq 1$,

$$\frac{d}{dv}\left(\frac{F_w\left(\frac{v}{z}\right)}{F_w(v)}\right) = \frac{1}{v}\frac{F_w\left(\frac{v}{z}\right)}{F_w(v)}\left(\frac{\left(\frac{v}{z}\right) f_w\left(\frac{v}{z}\right)}{F_w\left(\frac{v}{z}\right)} - \frac{v f_w(v)}{F_w(v)}\right) \geq 0$$

and $F_w \leq_{rh} F_s$ follows. A counterpart to Proposition 4 is now immediate.

Proposition 6 *The FPA yields strictly higher expected revenue than the SPA in the multiplicative model if $f_w(e^v)$ is increasing and log-concave.*

Proof. Identical to the proof of Proposition 4. ■

To relax the assumption that f_w is monotonic it is necessary to impose more restrictions on g instead. As in the additive model, the second part of Lemma 4 is used. The proof of the following proposition is omitted since it is analogous to the proof of Proposition 5.

Proposition 7 *The FPA yields strictly higher expected revenue than the SPA in the multiplicative model if $f_w(e^v)$ is log-concave, $\beta = 1$, g is decreasing and satisfies*

$$f_w(\alpha_w) \geq \int_1^{\alpha} f_w\left(\frac{\alpha_w}{z}\right)\frac{g(z)}{z}dz \tag{17}$$

6 Interpretation & application of $\leq_{disp}$, $\leq_*$, and $\leq_c$

The three orders of dispersion are related to various notions of price sensitivity, as summarized in the next proposition. For the third part it is assumed that densities are differentiable. Most of the results in this section do not require first order stochastic dominance.

Proposition 8 *For any $v \in S_w$:*

1. *$r(v) - v$ increasing $\Longleftrightarrow \left|\frac{q_w'(v)}{q_w(v)}\right| \geq \left|\frac{q_s'(r(v))}{q_s(r(v))}\right|$.*
2. *$\frac{d}{dv}\left(\frac{r(v)}{v}\right) \geq 0 \Longleftrightarrow \varepsilon_w(v) \geq \varepsilon_s(r(v))$.*
3. *$r''(v) \geq 0 \Longleftrightarrow J_w'(v) \geq J_s'(r(v))$.*

Proof. The first part follows directly from (2) and $f_w(v) \geq f_s(r(v))$. The second part follows from

$$\frac{d}{dv}\left(\frac{r(v)}{v}\right) \propto r'(v)v - r(v) = \frac{f_w(v)}{f_s(r(v))}v - r(v) \propto f_w(v)v - f_s(r(v))r(v) \propto \varepsilon_w(v) - \varepsilon_s(r(v)),$$

while the third part is due to

$$r''(v) \geq 0 \iff \frac{f_w'(v)}{(f_w(v))^2} \geq \frac{f_s'(r(v))}{(f_s(r(v)))^2} \iff J_w'(v) \geq J_s'(r(v)).$$

■

In the monopoly interpretation, Proposition 8 implies that, starting at comparable quantities, a marginal price increase would have a greater impact on the less disperse market.

$F_w \leq_{disp} F_s$ implies that the inverse demand curve $p_w(q) = F_w^{-1}(1-q)$ is flatter than $p_s(q) = F_s^{-1}(1-q)$, as in Figure 2(b). In contrast, $\frac{p_s(q)}{p_w(q)}$ is decreasing if $F_w \leq_* F_s$.

Expressing marginal revenue as a function of quantity,

$$MR_i(q) \equiv J_i(F_i^{-1}(1-q)) = \frac{f_i(F_i^{-1}(1-q))F_i^{-1}(1-q) - q}{f_i(F_i^{-1}(1-q))}, \tag{18}$$

for $q \in [0,1]$, it follows that

$$MR_i'(q) = J_i'(F_i^{-1}(1-q))\frac{-1}{f_i(F_i^{-1}(1-q))}. \tag{19}$$

Assuming $F_w \leq_{disp} F_s$, cases in which either $F_s \leq_* F_w$ or $F_s \leq_c F_w$ also hold have interesting interpretations.

Corollary 4 *If $F_w \leq_{disp} F_s \leq_c F_w$ then $|MR_s'(q)| \geq |MR_w'(q)|$. If $F_w \leq_{disp} F_s \leq_* F_w$ then $MR_s(q) \geq MR_w(q)$ whenever $MR_w(q) \geq 0$.*[23]

Proof. For the first part, if $r_s(v) = F_s^{-1}(F_w(v))$ satisfies $r'(v) \geq 1$ and $r''(v) \leq 0$ then $f_1(F_1^{-1}(1-q)) \leq f_2(F_2^{-1}(1-q))$ and $J_1'(F_1^{-1}(1-q)) \geq J_2'(F_2^{-1}(1-q))$, respectively. The result then follows from (19). For the second part, $F_s \leq_* F_w$

[23] Mares and Swinkels (2010a) consider procurement auctions in which bidders' costs, c, are private information. Bidder i's virtual cost is $\omega_i(c) = c + \frac{F_i(c)}{f_i(c)}$. Counterparts to Corollary 4 exist for "marginal costs" if (i) $F_2 \leq_{disp} F_1$ and $F_2 \leq_c F_1$ or (ii) $F_2 \leq_{disp} F_1$ and $F_2 \leq_* F_1$. Mares and Swinkels (2010a) assume (i).

implies $f_1(F_1^{-1}(1-q))F_1^{-1}(1-q) \geq f_2(F_2^{-1}(1-q))F_2^{-1}(1-q)$. The result then follows from (18). ■

Wang (1993) compares auctions and posted-price selling in a model where otherwise symmetric bidders arrive sequentially. One of his comparative statics results is that if the marginal revenue curve is steeper for distribution F_1 than distribution F_2, then auctions are more likely to dominate posted-price selling when all bidders draw types from F_1 rather than F_2. Wang (1993) proves that for this to hold, F_1 must necessarily be more disperse than F_2. Corollary 4 implies that $F_2 \leq_{disp} F_1$ combined with $F_1 \leq_c F_2$ is sufficient. First order stochastic dominance is not required for this result.

Although Johnson and Myatt's (2006) focus is very different from Wang's (1993), many of the "ingredients" in their analysis are similar. Two distributions satisfy Johnson and Myatt's (2006) rotation order if they cross precisely once.[24] They are also interested in distributions whose marginal revenue curves coincide at most once, which is obviously the case if one marginal revenue curve is steeper than the other. Both Wang (1993) and Johnson and Myatt (2006) explicitly mention variance ordered distributions, where F_i can be written $F_i(v) = F\left(\frac{v-\mu_i}{\sigma_i}\right)$. In this case, $r(v)$ is linear, with $r'(v) > 1$ whenever $\sigma_1 > \sigma_2$. Thus, $F_2 \leq_{disp} F_1$ but $F_1 =_c F_2$. The result in the previous paragraph therefore applies.

Moreover, assuming non-negative marginal costs, the important comparison of marginal revenues is at quantities where they are positive. It is irrelevant how many times marginal revenue curves cross below zero. Corollary 4 implies that marginal revenues are ordered in the positive quadrant if $F_2 \leq_{disp} F_1 \leq_* F_2$. However, $F_2 \leq_{disp} F_1 \leq_* F_2$ implies $F_2 \leq_{st} F_1$.

7 Extensions and implications

Theorem 1 can be extended to auctions with reserve prices and to certain other auction formats. These extensions are presented below. Implications of Theorem 1 for contest architecture are also discussed. Auctions with more bidders are considered in Section 8.

[24] The rotation order and the dispersive order are related. If two distributions cross and $r(v) - v$ is strictly increasing then they cross exactly once. However, as Ganuza and Penalva (2010) point out, dispersion only requires $r(v) - v$ to be weakly increasing, which permits the two distributions to coincide on an interval.

7.1 Reserve prices

Reserve prices are often employed in practice. However, as Mares and Swinkels (2010a, footnote 33) recently note, the "degree to which Maskin and Riley's ranking depends on the absence of a reserve price is open". The method developed in Section 3 can be used to address the question. Let τ denote the reserve price.

Assume $\beta_s = \beta_w$. Then, the proof of Theorem 1 establishes the superiority of the FPA for all values of the weak bidder's type. The only thing a reserve price does is to "shut out" some types, but this effect is the same in both types of auctions.[25] For those types that are not excluded, it remains the case that the FPA dominates contingent on the weak bidder's type. Thus, the revenue ranking is intact.[26]

More generally, assume that $\beta_w \leq \beta_s < \alpha_w$, or $C \neq \emptyset$. The FPA is then strictly more profitable than the SPA for a fixed reserve price as long as the reserve price has "bite", or $\tau \in [\beta_s, \alpha_w)$. The two auctions are revenue equivalent if $\tau \geq \alpha_w$ because in that case the winner is the same in the two auctions and $u_s^1(\beta_s) = u_s^2(\beta_s) = 0$.

Proposition 9 *Assume (i) $F_w \leq_{rh} F_s$, (ii) for all $v \in [\tau, \alpha_w]$, $f_w(v) \geq f_s(x)$ for all $x \in [v, r(v)]$, and (iii) $\tau \in [\beta_s, \alpha_w)$. Then, the FPA with reserve price τ generates strictly higher expected revenue than the SPA with the same reserve price.*

Proof. Because $\tau \geq \beta_s \geq \beta_w$, $u_i^k(\beta_i) = 0$ for $i = s, w$ and $k = 1, 2$. A bidder stays out of the auction if and only if his type is below τ. Modifying (3) yields

$$\begin{aligned} ER^k(\tau) &= \int_{\beta_w}^{\tau} \left(J_w(v) \times 0 + \int_{\tau}^{\alpha_s} J_s(x) dF_s(x) \right) dF_w(v) \\ &\quad + \int_{\tau}^{\alpha_w} \left(J_w(v) F_s(k(v)) + \int_{k(v)}^{\alpha_s} J_s(x) dF_s(x) \right) dF_w(v). \end{aligned}$$

Hence,

$$ER^1(\tau) - ER^2(\tau) = \int_{\tau}^{\alpha_w} D(v|k_1, k_2) dF_w(v),$$

which, as shown in the proof of Theorem 1, is positive since $\tau \geq \beta_s$. ■

[25] Bidders with type exceeding the reserve price will change their bidding strategy in the FPA. Thus, $k_1(v)$ depends on τ, although that dependence is suppressed here. The important point is that the reserve price does not change the property that $k_1(v) \in [v, r(v)]$ for all $v \in [\tau, \alpha_w]$. See Lebrun (1999).

[26] If the reserve price exceeds α_w then all the weak bidder's types are excluded. In this case, the two auctions are revenue equivalent (i.e., the revenue ranking is not strict).

If (F_1, F_2) do not satisfy the assumptions of the proposition then it is possible that a reserve price may reverse the ranking. For instance, in Lebrun's (1996) model the FPA dominates in the absence of a reserve price. However, in that model $D(v|k_1, k_2)$ is negative for v close to α_w. Thus, if the reserve price is large only types for which $D(v|k_1, k_2) \leq 0$ will remain active and the SPA will therefore be superior.

7.2 Other auction formats

The conclusion in Theorem 1 is made possible because the allocation in the SPA can be described precisely, while the possible allocations in the FPA can be narrowed down to a relatively small set. It is not necessary to know the exact allocation in the FPA.

Aside from the issue of how much rent is extracted from β_i types, Theorem 1 therefore really says that the SPA is a poor auction format if the objective is to generate high expected revenue. For instance, if $\beta_w = \beta_s$ and $u_w(\beta_w) = u_s(\beta_s) = 0$, any auction with $k(v) \in [v, r(v)]$ is more profitable than the SPA if (1) is satisfied. In other words, it is profitable to design an auction that favours the weak bidder moderately.

It has long been understood that *optimal* auctions typically favor the weak bidder; see e.g. McAfee and McMillan (1989).[27] Based on this property, Klemperer (1999) argues that "it is plausible that a first-price auction may be more profitable [...] than a second-price auction". However, this paper establishes a bound on the amount of favoritism that can safely be extended to the weak bidder. Specifically, any mechanism where the weak bidder wins more often than is efficient but less often than he would in a counterfactual symmetric auction against another weak bidder is more profitable than a SPA.

To illustrate, define a winner-pay auction to be an auction in which the winner pays a proportion γ of his own bid and $(1-\gamma)$ of the losing bid, and the loser does not pay, $\gamma \in [0, 1]$. The FPA corresponds to $\gamma = 1$, the SPA to $\gamma = 0$.

Proposition 10 *Assume (i) $F_w \leq_{rh} F_s$, (ii) condition (1) holds, and (iii) $\beta_s = \beta_w$. Then, the SPA yields strictly the lowest expected revenue of all winner-pay auctions.*

[27] In Maskin and Riley's (2000) one model where the SPA is superior to the FPA, the optimal auction would in fact discriminate *against* the weak bidder. In contrast, (1) implies that F_s dominates F_w in terms of the hazard rate. An optimal auction therefore favors the weak bidder.

Proof. Consider $\gamma \in (0,1]$, i.e. an auction that is not a pure SPA. In this case, the two bidders must share the same maximal bid, $\bar{b}$. Let $\phi_i(b)$ denote bidder i's inverse bidding strategy, $i = s, w$, where $b \in [\beta_w, \bar{b}]$. Assume for the moment the bidding strategy is strictly increasing and differentiable. If bidder i has type v, his problem is

$$\max_b \int_{\beta_w}^{b} [v - (\gamma b + (1-\gamma)\, x)]\, dF_j(\phi_j(x)),$$

where $j \neq i$ denotes bidder i's rival. The first order condition is

$$\frac{f_j(\phi_j(b))}{F_j(\phi_j(b))} \phi_j'(b)) = \frac{\gamma}{v-b}.$$

In equilibrium, bidder i bids b if his type is $v = \phi_i(b)$. Substituting into the first order conditions produces the system of differential equations

$$\frac{f_w(\phi_w(b))}{F_w(\phi_w(b))} \phi_w'(b) = \frac{\gamma}{\phi_s(b)-b}, \quad \frac{f_s(\phi_s(b))}{F_s(\phi_s(b))} \phi_s'(b) = \frac{\gamma}{\phi_w(b)-b}.$$

The only difference from the FPA is that $\gamma \in (0,1]$ (the boundary conditions are the same). The proofs in Maskin and Riley (2000) can then be repeated to conclude that the auction has the same features as a FPA, $k_\gamma(v) \in [v, r(v)]$ for all $\gamma \in (0,1]$. Since bidders with type β_i earns zero rent for all $\gamma \in [0,1]$, Theorem 1 applies directly. ■

Not all auctions have the property that $k(v) \in [v, r(v)]$. The most prominent example is probably the all-pay auction for which $k(v) < v$ when v is small. The reason is that a weak bidder with a low type is deterred from bidding (which is a sunk cost in an all-pay auction) when facing a rival he perceives as strong. Thus, it is not possible to rank the SPA and the all-pay auction using the method developed in this paper.

7.3 An order statistics result: Contest architecture

Theorem 1 can be used to derive a new order statistics result, the implications of which are discussed momentarily. Let $E[M_{i,j}]$ denote the expected value of the second highest type from one draw from F_i and one draw from F_j, where $i, j \in \{s, w\}$ may or may not be identical. Note that $E[M_{i,j}]$ equals the expected revenue in a SPA with bidders i and j.

Proposition 11 *Assume condition (1) holds. Then,*

$$\frac{1}{2}E[M_{s,s}] + \frac{1}{2}E[M_{w,w}] \geq E[M_{s,w}].^{28} \tag{20}$$

Proof. Consider an auction with a strong and a weak bidder. Define a *rank-symmetric* mechanism as a mechanism in which $k(v) = r(v)$ and $u_i^k(\beta_i) = 0$, $i = s, w$. In such a mechanism, bidder i with type v wins with probability $F_i(v)$. From Theorem 1, the expected revenue in a rank-symmetric mechanism is strictly greater than expected revenue in a SPAs, where it equals $E[M_{s,w}]$ (the right hand side of (20)). To see this, let $b_* = \beta_w$ in Theorem 1 and replace $k_1(v)$ with $k(v) = r(v)$. Note that $b_* = \beta_w$ implies $u_s^k(\beta_s) = 0$ because the strong bidder with type β_s wins with probability zero.

In the rank-symmetric mechanism, bidder s wins with probability $F_s(v)$, just as he would in a SPA facing another strong bidder. In such a balanced SPA, the expected payment from each bidder would be $\frac{1}{2}E[M_{s,s}]$. In both mechanisms, the bidder earns zero rent if his type is β_s. By the Revenue Equivalence Theorem, the expected payment from bidder s in the rank-symmetric mechanism is therefore exactly $\frac{1}{2}E[M_{s,s}]$. The same argument proves that bidder w's expected payment in the rank-symmetric mechanism is precisely $\frac{1}{2}E[M_{w,w}]$. Thus, the expected revenue of the proposed mechanism is $\frac{1}{2}E[M_{s,s}] + \frac{1}{2}E[M_{w,w}]$ (the left hand side of (20)). ■

Consider a seller who has some limited control over the composition of a two-bidder auction. Proposition 11 implies that he would be better off flipping a coin between two symmetric SPAs – one with two strong bidders, the other with two weak bidders – than to settle for an asymmetric SPA with one weak and one strong bidder. This result is another manifestation of the fact that asymmetric SPAs are relatively unprofitable. It is better to gamble on symmetric auctions, even at the risk of ending up with one consisting of two weak bidders. By the Revenue Equivalence Theorem, this result holds for any efficient mechanism, but it may not hold for other auction formats. Cantillon (2008) also argues that bidder asymmetry is unprofitable, but her alternative symmetric auction is different.

Proposition 11 is directly relevant for "contest architecture". Moldovanu and Sela (2006) show that with symmetric bidders it is more profitable to stage one grand

[28] The assumption that $\beta_s \geq \beta_w$ and $f_w(v) \geq f_s(v)$ implies that $F_w(v) \geq F_s(v)$. The inequality in (20) is strict as long as the two distributions are not identical.

auction than to stage smaller, same-sized, auctions in which a fraction of the total prize is up for grabs in each. Order statistics play a dominant role in the analysis in that paper and a related paper on contests for status, Moldovanu et al (2007). The symmetry assumption makes it easier to apply known order statistics results.

Proposition 11 complements Moldovanu and Sela (2006). Suppose the contest designer is forced to stage smaller auctions (a grand auction may be unmanageable). Then, Proposition 11 means that it is better to stage two symmetric auctions, one with two weak bidders, the other with two strong bidders, than to stage two asymmetric (but efficient) auctions.

8 Larger auctions

It is straightforward to extend the revenue ranking to allow for more weak bidders.[29] However, allowing more strong bidders is considerably more difficult. As explained below, extending the ranking to this case is possible if the asymmetry between bidders is "large enough".

Let $m \geq 1$ and $n \geq 1$ denote the number of strong and weak bidders, respectively. With symmetric and monotonic strategies within each group, the auction winner must have the highest type within his group. Hence, (3) becomes

$$ER^k = \int_{\beta_w}^{\alpha_w} \left(J_w(v) F_s(k(v))^m + \int_{k(v)}^{\alpha_s} J_s(s) dF_s(x)^m \right) dF_w(v)^n - n u_w^k(\beta_w) - m u_s^k(\beta_s),$$

where the term in parenthesis is expected value of the winner's virtual valuation conditional on the *highest* type among the n weak players being equal to v. The counterpart to (7) is

$$D_m(v|k_1, k_2) = \int_{k_2(v)}^{k_1(v)} \left(J_w(v) - J_s(x) \right) dF_s(x)^m.$$

The method of proof in Maskin and Riley (2000) or in Kirkegaard (2009) can be used to prove that $k_1(v) \in [v, r(v)]$ holds in larger FPAs as well. A proof is omitted.

However, if there are several strong bidders who are much stronger than the weak bidder(s), the former may compete so hard among themselves that the latter would

[29] Using the same approach as in Maskin and Riley (2000), Amann and Qiao (2008) have shown that Maskin and Riley's (2000) results extends to the case with many weak bidders.

be content to submit relatively low bids, even with types close to α_w. Let $\bar{b}_i$ denote the bid submitted by a bidder of strength i, if his type is α_i, $i = s, w$. When $m = 1$, all bidders share the same maximal bid, $\bar{b}_w = \bar{b}_s$ or $k_1(\alpha_w) = \alpha_s$, but when $m > 1$ it is possible that $\bar{b}_w < \bar{b}_s$ or $k_1(\alpha_w) < \alpha_s$ (see Lebrun (2006)). Another difference is that when $m > 1$, a strong bidder with type β_s earns zero rent because he is certain to be outbid by another strong bidder.

8.1 One strong bidder, more weak bidders

Assume that $n \geq m = 1$. As before, $u_w^k(\beta_w) = 0$ in both the FPA and SPA. For the strong bidder, (6) is as before, but with $F_w(v)^n$ in place of $F_w(v)$. Hence, $ER^2 - ER^1$ takes the exact same form as (9), with $F_w(v)^n$ in place of $F_w(v)$. Thus, the proof of Theorem 1 applies to the situation with $n > 1$ weak bidders as well.

Proposition 12 *Assume that $F_w \leq_{rh} F_s$ and condition (1) holds. Then, the FPA generates strictly higher expected revenue than the SPA when $n \geq m = 1$.*

8.2 More strong bidders

The *strict* revenue ranking in Theorem 1 does not generalize to $m \geq 2$. To see this, consider the case with $C = \emptyset$. In both auctions, competition between bidders ensure that the winning bid must be at least β_s. Thus, the winner is the strong bidder with the highest type in both auctions. Moreover, $u_i^k(\beta_i) = 0$ in both auctions, $i = s, w$. Hence, the FPA and the SPA are revenue equivalent when $\beta_s \geq \alpha_w$.

Assume for the remainder of the section that $\beta_s < \alpha_w$. Assume, for now, that the asymmetry is so small that $\bar{b}_w = \bar{b}_s$, or $k_1(\alpha_w) = \alpha_s$. Since $J_w(\alpha_w) = \alpha_w$, evaluating D_m at $v = \alpha_w$ then yields

$$\int_{\alpha_w}^{\alpha_s} \left(J_w(\alpha_w) - J_s(x)\right) dF_s(x)^m = \alpha_w \left(1 - F_s(\alpha_w)^m\right) - \int_{\alpha_w}^{\alpha_s} J_s(x) dF_s(x)^m.$$

The last term on the right is equal to the expected value of an auction among m strong bidders with a reserve price of α_w. Clearly, such an auction would yield revenue in excess of α_w if it results in a sale, which occurs with probability $1 - F_s(\alpha_w)^m$. Hence, $D_m(\alpha_w|k_1, k_2) < 0$ when $m \geq 2$ (when $m = 1$, the auction with reserve price α_w

generates revenue of exactly α_w if the object is sold, implying that $D_1(\alpha_w|k_1,k_2)=0$). It follows that the approach in this paper and in Maskin and Riley (2000) will not work in general. Thus, the complication identified by Lebrun (1996) for the case with $\alpha_w=\alpha_s$ and $n=m=1$ is endemic to auctions with many strong bidders, even if $\alpha_w<\alpha_s$.

It is the assumption that all bidders share the same maximal bid that leads to the negative conclusion. In the following, I will demonstrate that a revenue ranking can sometimes be obtained if the asymmetry is "large", such that $\overline{b}_s>\overline{b}_w$.

8.2.1 Small overlap

Assume that α_w is "close" to β_s such that there is little overlap between the supports. As a starting point, if $\beta_s=\alpha_w$ then (i) $J_w(\alpha_w)=\alpha_w=\beta_s>J_s(\beta_s)$ and (ii) $\overline{b}_s>\overline{b}_w$ in a FPA (a strong bidder with type β_s bids $\beta_s=\alpha_w\geq\overline{b}_w$, and his strategy is strictly increasing). If α_w is "slightly above" β_s, it must remain the case that $\overline{b}_s>\overline{b}_w$, or $k_1(\alpha_w)<\alpha_s$, with $J_w(\alpha_w)>J_s(x)$ for all $x\in[\beta_s,k_1(\alpha_w)]$. Moreover, by continuity,

$$J_w(v)>J_s(x) \text{ for all } v\in[\beta_s,\alpha_w] \text{ and } x\in[\beta_s,k_1(\alpha_w)]\,, \tag{21}$$

when β_s and α_w are sufficiently close.[30] In the following, when the overlap is said to be "small", it should be taken to mean that (21) is satisfied.

In this case, the FPA yields higher expected revenue than the SPA because the weak bidders are winning more often against strong bidders with inferior marginal revenue. Recall that the two are revenue equivalent if there is no overlap.

Proposition 13 *Assume $F_w\leq_{rh}F_s$ and the overlap is small. Then, the FPA generates strictly higher expected revenue than the SPA when $m\geq 2$, $n\geq 1$.*

Proof. Both auctions ensure that $u_i^k(\beta_i)=0$, $i=s,w$. A weak bidder with type below β_s loses both auctions (competition between the strong bidders ensures that any serious bid must be at least β_s). By (i), a weak bidder with type $v\in(\beta_s,\alpha_w]$ wins more often in the FPA than in the SPA. By (ii) or (21), the winner's marginal revenue is no lower in the FPA, and may be higher. In other words, D_m is positive. This concludes the proof. ■

[30] F_s need not be more disperse then F_w. For instance, the former could have a smaller support than the latter. It is a general property that $J_i(\alpha_i)=\alpha_i$ and $J_i(\beta_i)<\beta_i$, $i=s,w$.

EXAMPLE 1, CONTINUED: Consider a many-bidder extension of Example 1, with $m \geq 2$. If F_s is shifted far to the right such that there is no overlap between supports, then the two auctions are revenue equivalent. The same is true if $a = 0$, in which case bidders are homogenous. Proposition 13 then states that the FPA is superior for large "interior" values of a. A comparison cannot be made for small values. Recall that Proposition 13 does not require F_s to be a "shifted" version of F_w.

8.2.2 Large stretches

Assume the asymmetry between bidders is so large that $\bar{b}_s > \bar{b}_w$. Define $\overline{\alpha}_s \equiv k_1(\alpha_w)$ as the highest strong type that competes with the weak bidders. A strong bidder outbids the weak bidders with probability one if his type exceeds $\overline{\alpha}_s$, $\overline{\alpha}_s < \alpha_s$.

Consider the consequences of "stretching" the strong bidder's distribution, transforming F_s with support $[\beta_s, \alpha_s]$ to F_s^λ with support $\left[\beta_s, \alpha_s^\lambda\right]$, $\alpha_s^\lambda > \alpha_s$, such that $F_s^\lambda = \lambda F_s$ on the subinterval $v \in [\beta_s, \alpha_s]$, with $\lambda \in (0,1)$. More concisely, F_s is a truncation of F_s^λ. Importantly, F_s and F_s^λ have the same reverse hazard rate on $[\beta_s, \alpha_s]$ and therefore on $[\beta_s, \overline{\alpha}_s]$. Thus, if F_s dominates F_w in terms of the reverse hazard rate, so does F_s^λ. Likewise, the system of first order conditions from the bidders' maximization problems is unchanged at bids below $\bar{b}_w$. This can be seen by examining the systems in Maskin and Riley (2000) or Lebrun (2006). The implication is that weak bidders regardless of type and strong bidders with type below $\overline{\alpha}_s$ use the exact same strategy in either case. Consequently, k_1 is the same in both environments.

For types in $[\beta_s, \overline{\alpha}_s]$, the strong bidders' marginal revenue is

$$J_s^\lambda(v) = v - \frac{1 - F_s^\lambda(v)}{f_s^\lambda(v)} = v - \frac{\frac{1}{\lambda} - F_s(v)}{f_s(v)}$$

as a function of λ. The important property is that J_s^λ decreases without bound as F_s is stretched more and more (that is, as λ decreases and goes to zero). Thus,

$$J_w(v) \geq J_s^\lambda(x) \text{ for all } v \in [\beta_w, \alpha_w] \text{ and } x \in [\beta_s, \overline{\alpha}_s] \tag{22}$$

when F_s is stretched sufficiently much. In the following, when F_s is said to be stretched "a lot", it should be taken to mean that (22) is satisfied.

Proposition 14 *Assume $F_w \leq_{rh} F_s$ and F_s is stretched a lot. Then, the FPA generates strictly higher expected revenue than the SPA when $m \geq 2$, $n \geq 1$.*

Proof. The proof is identical to the proof of Proposition 13. ■

EXAMPLE 3, CONTINUED: Proposition 14 applies directly if F_w is a truncation of F_s, in which case $F_s(v) = \lambda F_w(v)$ on $v \in [\beta_w, \alpha_w]$. As with Example 1, the two auctions are revenue equivalent if the bidders are homogenous, or $\lambda = 1$. A comparison cannot be made if λ is close to one, or the asymmetry is small. By Proposition 14, however, the FPA is superior when λ is close to zero. Note that Proposition 14 does not require F_s and F_w to be related in any way other than through reverse hazard rate dominance (it does not imply one is a truncation of the other), nor does it require log-concavity.

9 Conclusion

This paper identifies the most general conditions to date under which the FPA is more profitable than the SPA when bidders are heterogenous. It is argued that the sufficient conditions have natural and appealing economic interpretations. Thus, the paper complements mounting evidence that a seller who is unsure of bidders' beliefs, preferences, and opportunities is better off using a FPA rather than a SPA. For example, Holt (1980) finds that the FPA is more profitable than the SPA if bidders are homogeneous but risk averse. Che and Gale (1998, 2006) prove that this is also the case if bidders are financially constrained. Hafalir and Krishna (2008) show the same ranking holds if resale is possible after the auction, even if bidders are potentially heterogenous. The obvious and not inconsiderable qualifier is Milgrom and Weber's (1982) well-known result that the SPA is better when values are affiliated.

Mechanism design methods are used to simplify the analysis. Specifically, a simple reformulation of Myerson's (1981) expression for expected revenue allows the conclusion that the FPA dominates if the strong bidder's distribution dominates the weak bidder's distribution in terms of the reverse hazard rate and, in addition, the former is flatter and more disperse than the latter. The central role played by the dispersive order complements recent findings by Mares and Swinkels (2010a, 2010b) in other asymmetric auction settings. Thus, the dispersive order may prove to be as useful for auction design as the usual stochastic orders of strength. The dispersive order and a related order of spread, the star order, appear naturally in settings with multi-dimensional types where valuations are constructed by summing or multiplying independent random variables.

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