Preferential Treatment may Hurt: Another Application of the All-Pay Auction*

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Abstract

In many contests a subset of contestants is granted preferential treatment which is presumably intended to be advantageous. Examples include affirmative action and biased procurement policies. In this paper, however, I show that some of the supposed beneficiaries may in fact become worse off when the favored group is diverse. The reason is that the other favored contestants become more aggressive, which may outweigh the advantage that is gained over contestants who do not receive preferential treatment. Likewise, a contestant may be made better off when a subset of his competitors is granted preferential treatment. The contest is modelled as an incomplete-information all-pay auction in which contestants have heterogenous and non-linear cost functions. Incomplete information is crucial for the results.

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1 Introduction

There are many examples of contests in which a subset of contestants receive "preferential treatment". On the labor market, affirmative action may influence which job applicant wins the prize, in this case the job. The same is true in the contest to win admission into university. Internal applicants are sometimes given preference over external applicants when a firm seeks to fill a senior position. In public procurement, domestic firms may be given preferential treatment over foreign firms, and so on.

In all these examples it is a *diverse* group of contestants who are favored. Affirmative action apply to individuals with different backgrounds, internal applicants for senior positions are likely to be heterogeneous, and domestic firms may have different technologies. Another feature of the examples is that the prize is not awarded based on the identities of the contestants alone, but also on the qualifications of the contestants in question. The investment in these qualifications – obtaining an education before applying for a job, preparing for the SAT, working hard to prove one's worth to the company, or building up expertise prior to seeking a procurement contract – may entail very significant costs. Importantly, the size of this investment is endogenous; it is likely to depend on the perceived strength of the competition and on whether the contestant is given preferential treatment. Since a given contestant may not have complete information regarding the skills, costs, or preferences of his rivals, asymmetric information may also play a role in determining the magnitude of a contestant's investments.

The objective of this paper is to study the consequences of preferential treatment in contests that are characterized by within-group diversity and incomplete information. I will show that the combination of these realistic features may, somewhat perversely, produce outcomes that are precisely the opposite of what intuition would suggests. The main results are:

1. If the group of contestants who are given preferential treatment is diverse, a subset of them may *participate less often*, *win less often*, and overall be *worse off* when preferential treatment is introduced.

2. If the group of contestants who are not given preferential treatment is diverse, a subset of them may *participate more often*, *win more often*, and overall be *better off* when preferential treatment is introduced.

The first outcome is unlikely to be what is intended with policies that give preferential treatment to select contestants. Thus, the current paper serves as a note of caution; rather than "leveling the playing field", preferential treatment may, in principle, increase the severity of the problems or inequalities it was intended to minimize.[1] The intention is not to dismiss preferential treatment in its many forms, but rather to challenge the common intuition in a formal model and invite more research into the complex interaction of heterogenous contestants. Thus, this paper is concerned with contests in general, rather than specifically with affirmative action per se. There are several theoretical papers that address the specific question of affirmative action on the labor market or in university admission. See De Fraja (2005), Moro and Norman (2003), or Fang and Norman (2006) for examples. The last two papers also reach surprising conclusions about the effects of affirmative action, but for very different reasons than those presented here.

The contest is modelled as a deterministic contest or, more formally, an all-pay auction, in which heterogenous contestants have private information about costs.[2] Thus, the paper is related to the literature on auctions with heterogenous participants. The papers by Lebrun (1999), Kirkegaard (2009a), and Parreiras and Rubinchik (2010) are among the few that consider specific auction formats with more than two heterogenous participants. However, due to the technical challenges, most theoretical papers that compare different auction formats or the consequences of changes to the auction design assume there are exactly two heterogeneous participants.[3] Clearly, since the purpose of the current paper is to consider a setting with two groups, at least one of which is diverse, a model with only two contestants is not adequate.

In this paper, I take a first step towards a more general analysis. One of the consequences

[1] Sowell (2004), for example, argues that there are other, behavioral reasons affirmative action may not be advantageous in the real world. Specifically, affirmative action may breed resentment, or it may trigger discrimination within the diverse group that is given preferential treatment. In contrast, the driving force in this paper is the strategic response by the contestants themselves to the changes in the rules of the game. While Sowell (2004) points out that there may be an incentive to lower investments, he considers this problematic only insofar as it leads to a population with lower qualifications. Sowell's (2004) main objective is to empirically evaluate and question the actual consequences of affirmative action. The current paper complements Sowell (2004) by pointing out that theoretical predictions – even in a simple model – do not necessarily support the received wisdom either. See Fryer and Loury (2005) for a brief discussion of "affirmative action and its mythology".

[2] The contest is deterministic in the sense that the contestant with the largest investment wins with probability one (in the absence of preferential treatment). This simplifying assumption facilitates the inclusion of private information in the model. Due to private information, a contestant is never sure of the actual investment of his rivals, and on whether a particular level of investment will be sufficient to win. Alternatively, the situation could be modelled as a tournament á la Lazaer and Rosen (1981) or as a Tullock contest, Tullock (1980). These contests are not deterministic, but nor are they conducive to the inclusion of private information.

[3] Recent prominent examples include Maskin and Riley's (2000) seminal comparison of first-price and second-price auctions, Hafalir and Krishna's (2008, 2009) analysis of such auctions with resale, Hörner and Sahuguet's (2007) analysis of jump bidding, and Goeree and Offerman's (2004) study of the "Amsterdam auction". In the context of all-pay auctions with private information, Clark and Riis (2000) and Kirkegaard (2009b) consider the revenue effects of various forms of preferential treatment.

is to cast into doubt the robustness of results that are based on models with just two participants, at least for all-pay auctions. For example, in a contest with just two contestants, the sole beneficiary of preferential treatment is unambiguously made better off. Concerning the analysis itself, a measure of traction is obtained by explicitly engineering the set-up of the problem to minimize the complications that arise from having several contestants and instead maximizing the use of insights from two-player contests. In the first part of the paper, the reaction to preferential treatment in a contest with two participants is used to infer the first result. In the second part, a model is constructed in which only two contestants are active at any given investment level, although this pair may depend on the actual level. The analysis is outlined next.

Consider a contest with a "strong", a "weak", and a "very weak" contestant, and assume the two weaker contestants are given preferential treatment. As a consequence, they have less to fear from the strong contestant. However, that does not necessarily mean that they will work less hard to obtain qualifications. In fact, the "weak" contestant may push his newfound advantage by investing more aggressively. From the point of view of the "very weak" contestant, one rival has become less of a threat, but the other more of a threat. I show, under mild assumptions, that the very weak contestant would be less likely to win the prize with a small investment when he and the other weak bidder are given preferential treatment. The second step is to show that there are cost structures for which a monotonic equilibrium exists in which the "very weak" contestant wins the prize with probability zero and earns zero payoff, but that such an equilibrium does not exist without preferential treatment. The cost function of the very weak bidder must have the right amount of "curvature", not too much and not too little. Moreover, the cost functions of the weak and very weak contestants may not be ordered (they may cross). These assumptions are discussed in Section 4.

An equilibrium is not characterized in the absence of preferential treatment; it is merely shown that there is no equilibrium where the very weak contestant is inactive. Siegel's (2010) paper nicely illustrates the technical difficulties. He characterizes equilibrium in certain *complete information* contests. However, in contests with just one prize, his characterization is generally valid only if there are exactly two contestants. For example, if there are three or more contestants with a common value of winning but non-ordered cost function, then it is generally not possible to characterize an equilibrium, even with complete information. Here, in contrast, I allow cost functions to be non-ordered and information to be incomplete.

To overcome these difficulties, a parameterized version of the model is examined in the second part of the paper. In this model, an equilibrium can be characterized, and its properties serve to further motivate the main result.

In the modified version of the model the strong contestant is not only the sole contestant to participate with probability one, he also wins more often than the other contestants combined. In an attempt to correct such an unbalanced outcome, weaker contestants may be given preferential treatment. Since the equilibrium can be characterized, it is possible to show that small doses of preferential treatment work more or less as expected for its beneficiaries. The weaker contestants participate more often and win more often. However, large interventions may fundamentally alter the equilibrium structure and leave the very weak contestant worse off.

As a counterpart to the main result mentioned previously, I also examine the use of preferential treatment in the modified model when the set of contestants who do not receive preferential treatment is diverse. Here, it is shown that a member of this group may participate and win more often once preferential treatment is introduced. In summary, the two results imply that a recipient of preferential treatment may be made worse off by it, while someone outside the favoured group may gain.[4]

This paper is not the first to document some arguably counterintuitive properties of contests. For instance, Che and Gale (1998) have shown that a cap or ceiling on investments may increase total investments, while Baye, Kovenock, and de Vries (1993) have shown that total investments may increase by excluding a subset of the contestants – specifically the strong contestants. However, what sets the current paper apart from these seminal papers is the pivotal role played by asymmetric information in generating new results.

Using recent results by Siegel (2009) for complete information contests, it is shown that the assumption of incomplete information is critical for the results; weaker contestants are never hurt by preferential treatment in a complete information contest. This complements a finding by Kirkegaard (2009b) that it may be profitable to handicap the weak contestant in a two-player contest when information is incomplete, but not when it is complete. Thus, the assumption of incomplete information adds an extra dimension and yields richer results.[5]

The general model is described in Section 2 and analyzed in Section 3. Section 4 contains a number of observations related to the main result. Section 5 develops and analyses a tractable version of the model. Section 6 concludes. All proofs are in the Appendix.

[4] The modified version of the model may also be interesting on purely theoretical grounds. Parreiras and Rubinchik (2010) have shown that an incomplete-information all-pay auction with more than two contestants may, in principle, have the property that (1) some contestant uses a discontinuous strategy (if he invests, he invests a lot), and/or (2) some contestant never invests enough to win with a probability close to one. It is precisely these features that makes it difficult to characterize equilibrium in general, or to prove uniqueness. The model presented here appears to be the first for which an equilibrium with both features can be explicitly characterized in a setting with incomplete information. See Siegel (2010) for a counterpart for contests with complete information.

[5] In contrast, the papers by Che and Gale (1998) and Baye, Kovenock, and de Vries (1993) assume valuations and abilities are common knowledge. In those papers, the commonly known heterogeneity among contestants yields a model that is sufficiently rich to produce their surprising results.

2 A contest with preferential treatment

Consider a deterministic contest with n contestants and the following timing:

0. Contestants are informed about the rules of the contest and discover the value they place on winning the prize. These valuations are private information.

1. Contestants simultaneously invest effort or other resources into obtaining qualifications. The cost for contestant i of obtaining b units of qualifications is described by the twice continuously differentiable cost function $c_i(b)$, with $c_i(0) = 0$, $0 < c_i'(\cdot) < \infty$, and $c_i''(\cdot) \geq 0$, $i = 1, 2, ..., n$.[6]

2. The winner of the contest is the contestant with the highest *score*. Each contestant's score is a function (as specified in step 0) of his qualifications (b) and, possibly, his identity. Ties are broken with the toss of a fair coin.

The game described above is isomorphic to an all-pay auction with private information in which a set of bidders submit bids, but where the cost of bidding may be different from bidder to bidder, and where the winner is not necessarily the bidder who submitted the highest bid. The defining characteristic of the all-pay auction is that the cost of the bid is forfeited, whether or not the auction is won. The auction terminology is used in the remainder of the paper.

All bidders are, for now, assumed to be risk neutral, and to share the same value of not participating, which is normalized to zero. The consequences of risk aversion are discussed in Section 4.2. Each bidder has a privately known type, v, which captures how much he values winning the prize. Bidder i's type is distributed according to some strictly increasing and twice continuously differentiable distribution function, $F(v)$, with no mass points and support $[\underline{v}, \overline{v}]$, where $\overline{v} > \underline{v} > 0$, $i = 1, 2.., n$. Densities, denoted by f, are bounded above and below, away from zero. The following assumption is imposed.

Assumption A: The "average probability", $F(v)/v$, is strictly increasing in v.

Note that the "average probability", $F(v)/v$, is strictly increasing if "total probability", $F(v)$, is convex. For example, Assumption A is satisfied by the uniform distribution with support $[\underline{v}, \overline{v}]$ whenever $\underline{v} > 0$. The assumption that bidders are ex ante homogenous in

[6] The cost function may differ from contestant to contestant because the ability or access to obtain qualifications may differ, or because the same amount of training does not translate into the same perceived qualifications.

terms of their desire to win can be relaxed, but it serves to highlight that the important source of heterogeneity is differences in costs.[7]

Two different possibilities are considered for how scores are computed in stage two. The first possibility is that the auction is unbiased, in which case the score is simply identical to the bid. In the alternative specification, bidder 1 is handicapped. Formally, bidders other than bidder 1 obtain a score equal to their bid, as before, but bidder 1 must bid $h(s)$ to obtain a score of s. It is assumed that $h(s)$ is twice continuously differentiable, with $h(0) = 0$, $h(s) > s$ for all $s \in \mathbb{R}_{++}$, and $h'(s) > 0$ for all $s \in \mathbb{R}_+$.[8] The important point is that bidders $2, 3, ..., n$ are given the *same* kind of advantage, and, in particular, that there is no bias when the bids of bidders in this group are compared with each other. Nevertheless, the set of favoured bidders may be diverse, in the sense that they may have different cost functions.

Note that handicapping bidder 1 is equivalent to giving preferential treatment to all other bidders. Assume that bidders $2, 3, ..., n$ obtain a score of $h(b) > b$ for any $b \in \mathbb{R}_{++}$, whereas bidder 1 only scores b when he bids b. This amounts to a simple "change of variables" compared to the model formulated in terms of handicaps. In particular, to tie with a rival bidder who scores s, bidder 1 has to bid $h(s)$, exactly as before. The difference between the two models amounts to an inconsequential rescaling or renaming of scores. More generally, if bidder i scores $H_i(b)$ with a bid of b, then the function $h(b) \equiv H_1^{-1}(H_2(b))$ measures the bid bidder 1 would have to submit in order to tie with bidder 2 if the latter bids b. The assumption in this paper is that bidder 1 is disadvantaged; $H_2(b) > H_1(b)$ or $h(b) > b$ for all $b \in \mathbb{R}_{++}$.

It is assumed that F and c_i are common knowledge among bidders. It is not necessarily assumed that the regulator of the contest knows the primitives of the game. The focus of the paper is not on determining what the "optimal" intervention may be, but merely on describing the actual consequences of changes to the game. Thus, it is also assumed that the regulator cannot or will not manipulate the number of prizes (which is assumed to be one), or their value.

Let Γ_n denote the auction involving bidders $1, 2, ..., n$ in which scores coincide with bids. Similarly, let Γ_n^h denote the game in which bidders $2, .., n$ are given preferential treatment. In the following, it is useful to think of bidders as choosing scores rather than bids. The set of actions is then $\{out\} \cup \mathbb{R}_+$, meaning that each bidder can choose to either stay out

[7] All results hold if, for example, bidder 1's distribution first order stochastically dominates the distribution of bidder 2. See Amann and Leininger (1996) and Parreiras and Rubinchik (2009) for an analysis of all-pay auctions with two or more bidders, respectively, whose types are drawn from different distributions but where costs are linear.

[8] For example, if the handicap is linear, by bidding b bidder 1 would obtain a score of $s = b/h$, $h \in (1, \infty)$, where h is the handicap. Thus, bidder 1's problem is equivalent to deciding which score, s, to obtain, given that to obtain a score of s he must bid $hs \geq s$.

of the auction or to enter the auction and submit a non-negative score. When bidder 1 is handicapped, his cost of obtaining a score of s is $c_1^h(s) \equiv c_1(h(s))$.

It is assumed that the handicap increases bidder 1's marginal costs. This is the case if, for example, $h(s)$ is linear, convex, or, more generally, if $h'(s) > 1$.

Assumption B: The handicap increases bidder 1's marginal costs; $c_1^{h\prime}(s) > c_1'(s)$ for all $s \in \mathbb{R}_+$.

To demonstrate the main point, it is sufficient to consider a situation with *three* bidders. In the games considered here, bidder 1, the bidder who is potentially handicapped, is the "strong" bidder. Bidder 2 and bidder 3 are "weaker" bidders, although their weakness is manifested in different ways, as illustrated in Figure 1. Specifically, of all the bidders, bidder 2 is the one for whom small bids are the most expensive. On the other hand, bidder 3 is the bidder for whom large bids are the most expensive. Thus, for both bidders there are bids for which they would incur the highest cost of obtaining such bids. There are no bids with this property for bidder 1, the strong bidder. Bidder 2 and bidder 3 will be referred to as "weak" and "very weak", respectively, because in a closely contested auction bidder 3's advantage over bidder 2 at low bids is less likely to be relevant. Alternatively, following Siegel (2009), define bidder i's *reach*, r_i, as the highest bid that the bidder would be willing to submit (in the absence of a handicap) even if he was guaranteed to win, such that $\overline{v} - c_i(r_i) = 0$ or $r_i = c_i^{-1}(\overline{v})$. No bid above r_i can be rationalized by bidder i. In Siegel's (2009) complete information contests (where types are common knowledge), the ranking of bidders' reaches is an important measure of strength (see also Che and Gale (2006)). This ranking, $r_1 > r_2 > r_3$, provides another justification for the strong, weak, and very weak terminology.

Following the previous discussion, the relationship between the cost functions of bidder 1 and bidder 2 is formalized by the next assumption. While Assumption B signifies that the handicap is detrimental to bidder 1, it is assumed that it is not big enough to completely negate the advantage he has over bidder 2.

Assumption C: Bidder 1 has a cost advantage over bidder 2, even after he is handicapped; $c_2'(s) > c_1^{h\prime}(s)$ for all $s \in [0, r_2]$.

At this point, no formal assumptions regarding c_3 is imposed. The reason is that it will be a *result* of this paper that there are c_3 functions with the general properties depicted in Figure 1 for which bidder 3 is worse off with preferential treatment. Thus, Figure 1 serves as a "preview" of the main result.

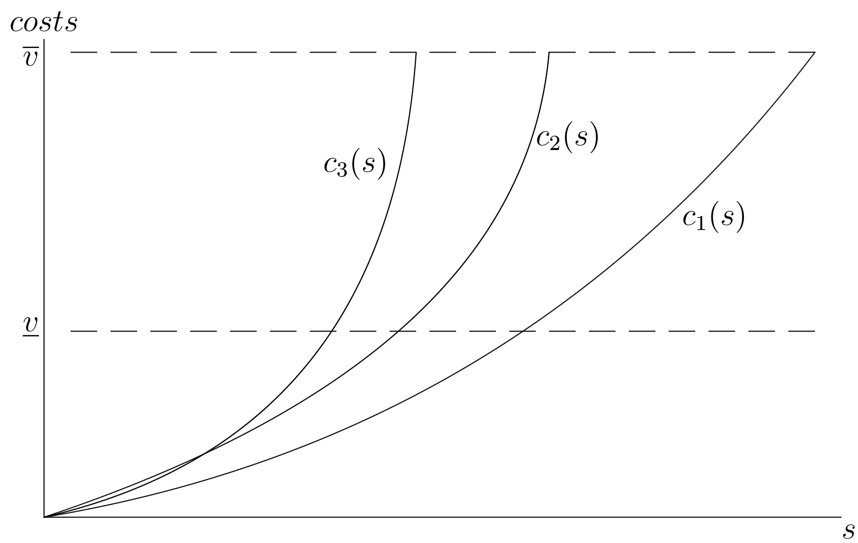

Figure 1: The strong, the weak, and the very weak.

3 Analysis

In this paper, I restrict attention to equilibria in increasing strategies. That is, each bidder has a cut-off type below which he scores zero or stays out of the auction, and above which he enters the auction with a score that is strictly increasing in his type. It follows from Athey's (2001) more general analysis that an equilibrium of this nature exists (see also Parreiras and Rubinchik (2010)).

The analysis is initiated by considering two games, Γ_2 and Γ_2^h. In these games, bidder 3 is ignored. Then, the larger games in which bidder 3 is potentially active, Γ_3 and Γ_3^h, are examined. Here, the central question is whether bidder 3 would select to be active if bidders 1 and 2 continue to play the increasing strategies from Γ_2 and Γ_2^h, respectively.

3.1 The strong and the weak bidder; Γ_2 and Γ_2^h

Consider the game Γ_2^h, in which bidder 3 is not present. Let $\varphi_i^h(s)$ denote bidder i's inverse bidding strategy among the set of types who participate, such that bidder i scores s if his type is φ_i^h, $i = 1, 2$. Bidder 1 with type v seeks to maximize $vF(\varphi_2^h(s)) - c_1^h(s)$, where $F(\varphi_2^h(s))$ is the probability that he outscores bidder 2 (and wins) with a score of s. Using the same arguments as in Amann and Leininger (1996, Lemmas 1-5), it can be shown that $F_1^h(s) \equiv F(\varphi_1^h(s))$ and $F_2^h(s) \equiv F(\varphi_2^h(s))$ have a common support of the form $[0, \overline{s}^h]$, where $\overline{s}^h$ is the common maximal score, and that they are continuous.[9] The latter implies that

[9]It is easy to see that the bidders must share a common maximal score. If this was not the case, one bidder could profitably deviate by lowering his score without affecting the probability that he wins.

there are no mass points, except possible at zero (see below). Since strategies are monotonic, they are differentiable almost everywhere.[10] Thus, any interior solution to bidder 1's problem must satisfy the first order condition,

$$v\frac{dF(\varphi_2^h(s))}{ds} - c_1^{h\prime}(s) = 0.$$

The first order condition for bidder 2 is obtained in similar fashion.

In equilibrium, bidder 1 scores s if his type is $v = \varphi_1^h(s)$ while bidder 2 scores s if his type is $\varphi_2^h(s)$. Substituting these into the first order conditions gives the following pair of conditions:

$$\frac{dF(\varphi_1^h(s))}{ds} = \frac{c_2'(s)}{\varphi_2^h(s)}, \quad \frac{dF(\varphi_2^h(s))}{ds} = \frac{c_1^{h\prime}(s)}{\varphi_1^h(s)}. \tag{1}$$

The assumption that types are strictly positive, or $\underline{v} > 0$, implies that the slopes of $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ are finite everywhere.

Recall that there is a common maximal score, $\overline{s}^h$, such that $\varphi_i^h(\overline{s}^h) = \overline{v}$, or $F(\varphi_i^h(\overline{s}^h)) = 1$, $i = 1, 2$. From any given guess on the value of $\overline{s}^h$, the two differential equations in (1) can be used to "shoot backwards", in order to evaluate the unique paths that φ_1^h and φ_2^h take as s is reduced.[11] Verifying whether the result is consistent with an equilibrium then helps to pinpoint the values of $\overline{s}^h$ that are equilibrium candidates.[12] To this end, note that it cannot be the case that both bidders stay out of the auction with strictly positive probability. The reason is that it would pay to enter the auction with a very small score, in order to win with a non-trivial probability.[13] Thus, it must hold that $F(\varphi_1^h(0)) = 0$ and $F(\varphi_2^h(0)) \geq 0$, or vice versa. In other words, one bidder enters with probability one with a minimum score of zero, while the other bidder may stay out with positive probability. By using this condition and the requirement of a common maximal score, it will be shown that there is a unique equilibrium in increasing strategies (see Proposition 1, below).

In the following, let φ_i denote the strategies and let $\overline{s}$ denote the maximum equilibrium

[10] Amann and Leininger (1996) assume that the cost function is identical (and linear) for the two bidders, but that types are draws from different distribution functions. However, their arguments also apply to the model in the current paper.

[11] The two differential equations in (1) can also be written as $d\varphi_1^h/ds = c_2'(s)/f(\varphi_1^h)\varphi_2^h$ and $d\varphi_2^h/ds = c_1^{h\prime}(s)/f(\varphi_2^h)\varphi_1^h$, respectively. The right hand sides are continuously differentiable in s, φ_1^h, and φ_2^h, which implies that the solution is unique given the boundary condition $\varphi_1^h(\overline{s}^h) = \varphi_2^h(\overline{s}^h) = \overline{v}$.

[12] Note that if φ_1^h and φ_2^h satisfy (1) then φ_i^h is increasing in s (the right hand side is strictly positive). The first derivative of bidder 1's payoff with respect to the score is $\left(v/\varphi_1^h(s) - 1\right) c_1^{h\prime}(s)$, which is positive when s is small (such that $\varphi_1^h(s) < v$) and negative when s is large (and $\varphi_1^h(s) > v$). Consequently, payoff is single peaked in s, and the first order conditions are sufficient if φ_1^h and φ_2^h satisfy (1).

[13] Likewise, no bidder scores or bids zero for a mass of types. Otherwise, the rival bidder with valuation $\underline{v}$ should not score zero, but rather marginally above zero, in order to dramatically increase the probability of winning (by ruling out the probability of a tie).

score when there is no handicap. The differential equations for this case are analogous to (1). Given Assumptions B and C, the strong bidder scores more aggressively than the weak bidder whether or not he is handicapped. Moreover, the weak bidder stays out of the auction with positive probability, whereas the strong bidder always participates. However, the strong bidder scores less aggressively when he is handicapped (although his bid may be higher). In response, the weak bidder becomes *more* aggressive, at least in the sense that he is now more likely to participate.

Proposition 1 (Equilibrium Properties) *There is a unique equilibrium in increasing strategies in Γ_2^h. In this equilibrium, the weak bidder stays out with strictly positive probability and is more likely to submit low bids than the strong bidder, $F(\varphi_1^h(s)) < F(\varphi_2^h(s))$ for all $s \in [0, \overline{s}^h)$, with $F(\varphi_1^h(0)) = 0 < F(\varphi_2^h(0))$.*[14] *The same properties hold for Γ_2.*

Proof. See the Appendix. ■

Proposition 2 (Comparative Statics) *The unique equilibrium in increasing strategies of Γ_2 compares with its counterpart in Γ_2^h as follows:*

1. *Scores are more compressed and the strong bidder scores less aggressively in Γ_2^h than in Γ_2: $\overline{s}^h < \overline{s}$, and $F(\varphi_1^h(s)) > F(\varphi_1(s))$ for all $s \in (0, \overline{s}^h]$.*

2. *The weak bidder participates more often in Γ_2^h than in Γ_2: $0 < F(\varphi_2^h(0)) < F(\varphi_2(0))$.*

Proof. See the Appendix. ■

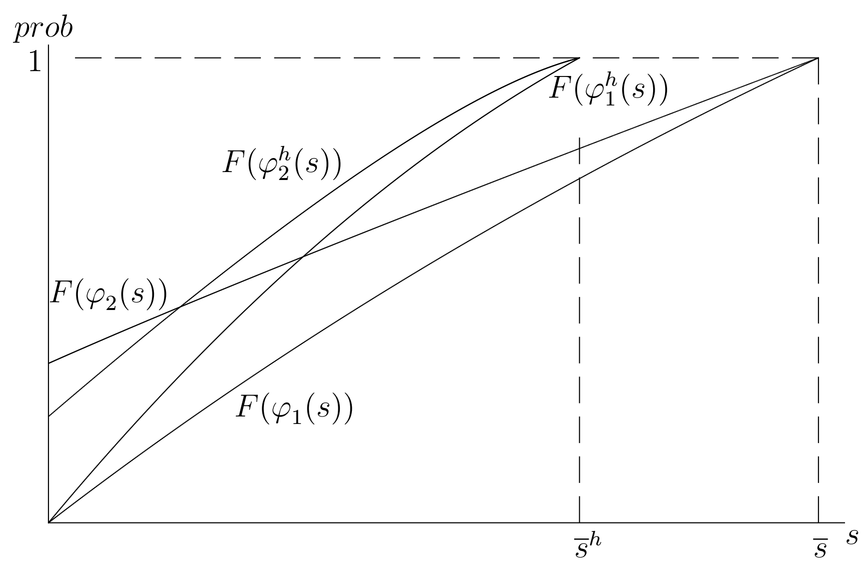

Figure 2: Individual distributions of scores.

[14] More precisely, the equilibrium is "essentially unique" because it does not matter whether bidder 2 with type $\varphi_2^h(0)$ scores zero or stays out.

Note that the strong bidder becomes less of a threat to the weak bidder when he is handicapped. For a fixed score, the weak bidder is more likely to win. Consequently, he is more likely to participate, and, if he participates, he is better off. Thus, depending on his type, v, he is either indifferent or strictly better off when he is given preferential treatment. Ex ante (before his type is known), he must therefore be strictly better off.

Corollary 1 *With just two bidders, bidder 2 is weakly better off regardless of his type if he is given preferential treatment, and strictly better of ex ante.*

3.2 The very weak bidder; Γ_3 and Γ_3^h

Consider now bidder 3, the very weak bidder. In particular, if bidder 1 and bidder 2 compete as described above, does bidder 3 have an incentive to become active in the auction? Figure 2 illustrates the response of bidders 1 and 2 to the handicap (assuming bidder 3 stays out), as described in Proposition 2. If bidder 3 enters with a small bid after preferential treatment is extended to the weak bidders, he is *more likely* to beat the strong bidder, but *less likely* to beat the weak bidder. Of course, bidder 3 is concerned with outscoring both bidders, the probability of which is

$$q_3^h(s) \equiv F(\varphi_1^h(s))F(\varphi_2^h(s)) \tag{2}$$

when he submits a score of s.

For small scores, both $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ are steeper than $F(\varphi_1(s))$ and $F(\varphi_2(s))$, which perhaps suggests a greater return to submitting a small score for bidder 3. However, this is counteracted by the fact that bidder 2 is more likely to participate. Given (1), the derivative of $q_3^h(s)$ is

$$q_3^{h\prime}(s) = \frac{F(\varphi_1^h(s))}{\varphi_1^h(s)} c_1^{h\prime}(s) + \frac{F(\varphi_2^h(s))}{\varphi_2^h(s)} c_2'(s) \,. \tag{3}$$

Since marginal costs are increasing, Assumption A imply that the right hand side is increasing in s. In other words, $q_3^h(s)$ is strictly convex. Since $F(\varphi_1^h(0)) = 0$ the derivative at $s = 0$ is

$$q_3^{h\prime}(0) = \frac{F(\varphi_2^h(0))}{\varphi_2^h(0)} c_2'(0) \,. \tag{4}$$

The main result follows from (4). In particular, $\varphi_2^h(0) < \varphi_2(0)$ (Proposition 2) and Assumption A together imply that $q_3^h(s)$ is *flatter* than $q_3(s) \equiv F(\varphi_1(s))F(\varphi_2(s))$ near $s = 0$, though each individual term is steeper. Since $q_3^h(0) = q_3(0) = 0$, $q_3^h(s)$ must be *below* $q_3(s)$ for small s; after bidder 3 is given preferential treatment (along with bidder 2), he faces a worse distribution of rival scores at the bottom.

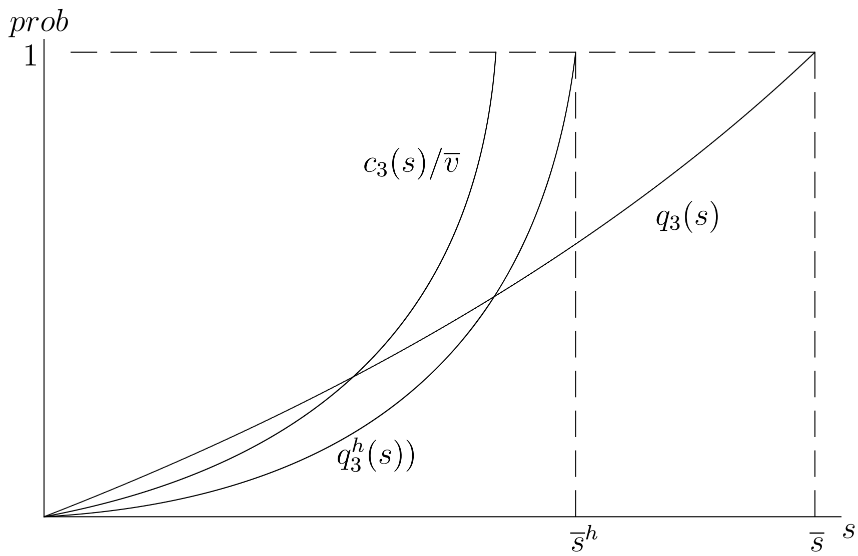

Figure 3: Distribution of the highest score and the cost function.

The incentive to enter the auction is strongest for bidder 3 if his type is $\overline{v}$, in which case he maximizes $\overline{v}q_3^h(s) - c_3(s)$, or, equivalently,

$$q_3^h(s) - \frac{c_3(s)}{\overline{v}}. \tag{5}$$

Thus, for bidder 3 to find entry profitable, there must be a score, $s > 0$, for which $q_3^h(s) \geq c_3(s)/\overline{v}$.

Figure 3 depicts $q_3^h(s)$ and $q_3(s)$. The important properties are: (i) $q_3^h(s)$ is flatter than $q_3(s)$ for low s, and (ii) $\overline{s}^h < \overline{s}$. Thus, $q_3^h(s)$ and $q_3(s)$ must cross. Now, with the cost function c_3 depicted in Figure 3, the very weak bidder should enter the auction with a strictly positive bid (and earn positive payoff) if there is no preferential treatment, but he should stay out if bidder 1 is handicapped. More precisely, it is not an equilibrium for bidder 3 to be inactive in the absence of preferential treatment, but there is an equilibrium in which he stays out after he and the other weak bidder is given preferential treatment.[15] Equilibrium is not characterized in the former case, but it follows from Athey (2001) that one exists, in increasing strategies. See Section 5 for an example.

Theorem 1 *There exists a strictly increasing and strictly convex cost function, c_3, for which there is no equilibrium in increasing strategies of Γ_3 where bidder 3 wins with probability zero, but for which there is an equilibrium in increasing strategies of Γ_3^h in which bidder 3 wins with probability zero.*

[15]The opposite is also possible. Specifically, there are other c_3 functions for which bidder 3 would be inactive without preferential treatment, but active with preferential treatment.

Proof. See the Appendix. ■

Consider now an arbitrary c_3 function for which Theorem 1 is valid. Note that multiple equilibria in increasing strategies has not been ruled out, but Theorem 1 proves that bidder 3 wins with strictly positive probability in *any* such equilibrium of Γ_3. However, this is not true for Γ_3^h. Even if there are multiple equilibria of Γ_3^h, the introduction of preferential treatment may afford bidder 1 and bidder 2 the opportunity to "collude" and effectively exclude bidder 3 in situations where that would have been impossible without preferential treatment.

Since bidder 3 wins with positive probability for a mass of types in Γ_3, he has types for which his payoff is positive.[16] In this case, bidder 3 would go from participating with positive probability, winning with positive probability, and having positive payoff, to participating with probability zero, winning with probability zero, and having zero payoff.

Corollary 2 *Bidder 3 may be worse off when he and bidder 2 are given preferential treatment.*

Sowell (2004) provides a plethora of examples from around the world in which preferential treatment was initially intended for a relatively small and well-defined group of individuals only, but where, over time, it grew to encompass a larger group of people. Even if preferential treatment as initially envisioned was beneficial to the former group, the results in this paper caution that adding more people to the list may not only "dilute" the advantage of the first group, as is intuitive, but may in fact *reverse* the effect of preferential treatment, and ultimately leave the initial beneficiaries worse off than before the advent of preferential treatment.

4 Discussion

The conditions under which Theorem 1 holds are discussed below.

4.1 Comparing bidders

Given Theorem 1, it is now possible to make more precise the sense in which bidder 3 is "very weak". For the cost function c_3 to satisfy Theorem 1, it is clearly necessary that a score of $\overline{s}^h$ is prohibitively expensive for bidder 3. On the other hand, it is not prohibitively expensive for bidders 1 and 2, since it is an equilibrium score. Hence, bidders 1 and 2 have

[16] It follows from Myerson's (1981) analysis that in any mechanism where a bidder wins with positive probability for a mass of types, these types must earn strictly positive payoff, except possibly for the lowest participating type.

an advantage over bidder 3 at high scores; their reaches are higher, $r_1 > r_1^h > r_2 > \overline{s}^h > r_3$, where r_1^h is bidder 1's reach when he is handicapped.

Next, consider low scores. Since bidder 3 is active when there is no handicap in place, there must be some profitable score if bidders 1 and 2 use their strategies from Γ_2. For instance, if bidder 3 profits by submitting a bid marginally above zero (as in the construct in the proof of Theorem 1, illustrated in Figure 3), then

$$0 < q_3'(0) - \frac{c_3'(0)}{\overline{v}} = \frac{F(\varphi_2(0))}{\varphi_2(0)} c_2'(0) - \frac{c_3'(0)}{\overline{v}} < \frac{c_2'(0)}{\overline{v}} - \frac{c_3'(0)}{\overline{v}},$$

where the first equality follows from (4) and the last inequality from Assumption A. In conclusion, $c_3'(0) < c_2'(0)$. Thus, bidder 3 has a cost advantage over bidder 2 for low scores, but bidder 2 has the advantage when scores are high. Figure 1 illustrates cost structures for which Theorem 1 is applicable.

Although it is tempting to make the mathematically expedient – and arguably more elegant – assumption that the bidders' cost functions can be ordered (that they do not cross), such an assumption may inadvertently cause the modeler to miss potentially important consequences of a policy intervention. Moreover, there is little reason to believe that real-world cost function can always be ranked in such a manner, and there may even be empirical evidence to suggest that cost functions cross in some contests.

For example, Fryer (2009) studies the relationship between academic achievement and social status among whites, blacks, and Hispanics in high-school. He finds that social status is increasing at a fast rate in grades for whites (Fryer (2009), Figure 1B). For blacks and Hispanics, the curve has an inverse U shape; it has a peak. The curve for blacks is relatively flat. For Hispanics, however, the curve is first increasing at a rate faster than the curve for blacks (but not as quickly as for whites), but it reaches its peak much sooner, after which it drops at a very fast rate.

Imagine now that obtaining higher grades involves two considerations on the costs side, incurring higher effort costs and experiencing an increase or decrease in one's social status. Assume the cost of effort is the same for all groups. Since whites experience a large increase in status from higher grades, their net costs are arguably lower than the overall costs of the other groups. Since the status of a Hispanic student at first rises faster in achievement than is the case for a black student, it could also be argued that blacks have the largest marginal costs at low achievement levels. However, because the curve drops so dramatically, and early, for Hispanic students, the overall costs of achieving high grades are very steep for Hispanics. The costs functions of whites, blacks, and Hispanics, may then resemble those of bidder 1, bidder 2, and bidder 3, respectively, in Figure 1.

4.2 Risk aversion

So far, bidders have been assumed to be risk neutral. Assume now that bidder 3 is risk averse, that w is his initial wealth, and that v measures the monetary value of winning the auction. Any bid or score can then be viewed as producing a lottery with an outcome of $w + v - c_3(s)$ with probability $q_3^h(s)$ and an outcome of $w - c_3(s)$ with probability $1 - q_3^h(s)$. Similarly, any point in Figure 3 can be viewed as representing a lottery, with a function of costs on the horizontal axis and the win probability on the vertical axis. The curves $q_3(s)$ and $q_3^h(s)$ can be thought of as feasible sets; a score of s results in a win probability of $q_3(s)$ and $q_3^h(s)$ without and with a handicap, respectively.

For the risk neutral case, the curve $q = c_3(s)/\overline{v}$ in Figure 3 captures an indifference curve; bidder 3 is indifferent between any combination of score (s) and win probability (q) on this curve. Any lottery to the north-west of this curve would generate higher expected utility than to not participate. Bidder 3 would "accept" such a lottery.

However, it is a standard result that the "acceptance set" diminishes as the agent becomes more risk averse (the indifference curve in Figure 3 shifts toward the north-west in the interior). Thus, the more risk averse bidder 3 is, the less likely any given score is to produce a lottery that bidder 3 would accept (compared to the risk-less alternative of not participating). In other words, he is less likely to participate in both Γ_3 and Γ_3^h. However, contingent on bidder 3 remaining active in Γ_3, the conclusion must be that handicapping bidder 1 is more likely to scare off bidder 3 the more risk averse he is; he was closer to giving up in Γ_3 and it takes less of a change to persuade him to stay out completely. The common assertion that risk aversion is diminishing in wealth then suggests that the intervention is more likely to deter bidder 3 the poorer he is. This is a thought-provoking conclusion since preferential treatment is often intended to help precisely those with limited resources.

See Section 5 for a version of the model in which bidder 3 has a binding resource constraint. There, preferential treatment may also hurt bidder 3, even when he is risk neutral.

4.3 Complete versus incomplete information

Siegel (2009) considers a very general class of contests that encompasses all-pay auctions with and without handicaps. While he allows bidders to be heterogenous in valuations and costs, it is assumed that information is complete. The implication of his analysis is that the bidder with the highest reach is the only bidder with strictly positive expected payoff. Thus, weak bidders earn zero payoff. Clearly, the two weak bidders cannot be worse off than this when they are given preferential treatment; preferential treatment cannot hurt the intended beneficiaries in a complete-information contest.[17] Consequently, the assumption of incomplete information imposed in this paper is as important as it is realistic.

5 Disjoint equilibria

In this Section a slightly modified but more tractable version of the model is presented. The advantage is that an equilibrium can be characterized in which all three bidders are active (prior to any intervention), such that the consequences of preferential treatment can be better illustrated.

The equilibrium described below has two interesting features: (1) bidder 1 is the only bidder to participate with probability one, and (2) bidder 1 wins more often than the other two bidders combined. Thus, if a policy maker has observed successive cohorts of bidders play the game over time, she will have seen successive incarnations of bidder 2 and bidder 3 appear disengaged or underrepresented (they are not always participating) and, partly for that reason, fairly unsuccessful. Although these observations may motivate preferential treatment of bidder 2 and bidder 3, Theorem 1 cautions that this may make matters worse, at least for bidder 3.

Theorem 1 is driven by the curvature of c_3. However, there is no need for c_1 and c_2 to be strictly convex. In this Section, it will therefore be assumed that

$$c_i(b) = \alpha_i b,\ i = 1, 2.$$

To introduce curvature into c_3, it is assumed from now on that

$$c_3(b) = \begin{cases} \alpha_3 b & \text{if } b \in [0, m] \\ \infty & \text{otherwise} \end{cases}, \tag{6}$$

where $m > 0$ can be thought of as a resource constraint[18]. Depending on the context, this could be a monetary constraint, a time constraint, or perhaps even a lack of access to higher education (when b represents schooling).

It is assumed that $\alpha_2 > \alpha_3 > \alpha_1 > 0$; bidder 3 would be more competitive than bidder 2 had it not been for the constraint. To fix ideas, assume that α_2 is so large compared to α_3 and α_1 that an equilibrium exists in which bidder 2 is inactive if m is sufficiently high. In such an

[17] More generally, a handicap reduces the reach of the handicapped bidder. The other bidders cannot be made worse off as a consequence. Thus, if bidder i is handicapped, the other two bidder are not hurt, $i = 1, 2, 3$.

[18] For "symmetry" in modeling, c_1 and c_2 could be allowed to take a similar form as c_3, but with a large and therefore irrelevant resource constraint.

equilibrium the interaction between bidder 1 and bidder 3 will yield a version of (3), which then implies that bidder 2's payoff of entering with a positive bid is convex in the bid, given that his bidding costs are linear. Thus, the optimal action is at a corner. The assumption that bidder 2 is inactive is therefore equivalent to the assumption that $\overline{v} - \alpha_2 \overline{b}_{-2} \leq 0$, where $\overline{b}_{-2}$ is the maximal bid in an auction without bidder 2 when m is so high that it does not play a role ($m > \overline{b}_{-2}$). As m is reduced and falls below $\overline{b}_{-2}$, bidder 3 is unable to bid as high as he would like. It is then possible that bidder 1 would bid m or marginally above m for a mass of types to guarantee a win. However, as m falls further, it must eventually be the case that bidder 2 would want to become active – with a high bid (due to the convexity of the objective function). This constitutes a challenge to bidder 1, who might then want to start bidding above m again, in order to deal with the new competitor.

The thought experiment in the previous paragraph suggest that an equilibrium with a particularly simple structure may exist when $\overline{v} - \alpha_2 m > 0$. In a *disjoint equilibrium*, bidder 1 and bidder 3 are the only bidders active at bids below m, whereas bidder 1 and bidder 2 are the only bidders active at bids above m. There is no overlap between the bids of bidder 2 and bidder 3; their equilibrium strategies are disjoint. Characterizing the equilibrium is then made easier, since only two bidders are active at any bid. However, it is possible that bidder 1 bids m for a mass of types in a disjoint equilibrium.[19,20]

Define $\overline{m} = \overline{v}/\alpha_2$ as the critical value of m where bidder 2 has an incentive to become active in the auction, $\overline{v} - \alpha_2 \overline{m} = 0$. For the reasons outlined above, assume that $m < \overline{m} < \overline{b}_{-2}$. Since $\overline{b}_{-2}$ depends on α_1 and α_3, and $\overline{m}$ depends on α_2, the latter inequality represents a joint restriction on the parameters $(\alpha_1, \alpha_2, \alpha_3)$. In the following, it is convenient to think of $(\alpha_1, \alpha_2, \alpha_3)$ as fixed. It will then be established that a disjoint equilibrium exists for some values of m (values close to, but below, $\overline{m}$).

Existence and uniqueness are discussed next, and the disjoint equilibrium is fully characterized. It is then verified that the equilibrium has the properties claimed in the beginning of this Section. The consequences of preferential treatment is then examined in this model. Since equilibrium can be characterized, it is also possible to consider the effects of preferential treatment on those that are handicapped.

[19] To ensure the existence of an equilibrium it is necessary to alter the tie-breaking rule to accommodate a mass of types taking the same action. In the following, it is assumed that bidder 2 wins any tie he is involved in, bidder 1 wins if he is in a tie with bidder 3 only, and bidder 3 loses any tie. It will remain the case that a tie occurs with probability zero, in equilibrium. It is the discontinuity in c_3 which may cause a mass of types to submit the same bid.

[20] Bidder 2 and bidder 3 must bid m with probability zero. Otherwise, bidder 1 would have an incentive to bid slightly above m (to experience a jump in his winning probability) when he is supposed to bid slightly below m. However, bidder 1 can bid m with positive probability, because it is impossible or prohibitively costly for bidder 3 to bid above m. Given the tie-breaking rule described in the previous footnote, there is therefore no incentive for bidder 3 to jump from a bid below m to a bid of m, or higher.

5.1 Characterizing disjoint equilibria

I begin by outlining how a disjoint equilibrium is characterized. Details are in the proof of Proposition 3, below.

Let $\widehat{v}_2$ denote bidder 2's lowest active type, i.e. the type that bids exactly m. Let $\widehat{v}_1$ denote bidder 1's highest type that bids m. Thus, if $v > \widehat{v}_i$, bidder i bids above m, $i = 1, 2$. Let $\widetilde{v}_1$ denote bidder 1's lowest type that bids m. In other words, bidder 1 bids below m if his type is below $\widetilde{v}_1$. He bids precisely m if his type is in the interval $[\widetilde{v}_1, \widehat{v}_1]$.

For bids above m, the interaction between bidder 1 and bidder 2 is as described in Section 3. However, since marginal costs are assumed constant, (1) becomes an autonomous system. The solution method proposed by Amann and Leininger (1996) can then be used to derive a "tying function", $k_2(v)$, where $k_2(v)$ is the type of bidder 2 who bids the same as bidder 1 with type v, $v \in [\widehat{v}_1, \overline{v}]$. A key step is to realize that the boundary condition is $k_2(\overline{v}) = \overline{v}$, since both bidders submit the maximum bid, $\overline{b}$, when their type is $\overline{v}$. Given the function $k_2(v)$, $\widehat{v}_1$ and $\widehat{v}_2 = k_2(\widehat{v}_1)$ are derived from the fact that bidder 2 with type $\widehat{v}_2$ must be indifferent between not participating and bidding m. Finally, bidding strategies for bidder i with type above $\widehat{v}_i$, $i = 1, 2$, can be derived by "integrating up" the first order conditions.

Likewise, for any guess concerning $\widetilde{v}_1$, a function $k_3(v|\widetilde{v}_1)$ can be derived that describes the type of bidder 3 who bids the same as bidder 1 with type v, $v \in [\underline{v}, \widetilde{v}_1]$, with $k_3(\widetilde{v}_1|\widetilde{v}_1) = \overline{v}$. It turns out that bidder 3 stays out if his type is low, $k_3(\underline{v}|\widetilde{v}_1) > \underline{v}$ for any $\widetilde{v}_1$. Since bidder 1 with type $\underline{v}$ and bidder 3 with type $k_3(\underline{v}|\widetilde{v}_1)$ bid zero, bidding strategies for higher types can once again be found be integrating up the first order condition. Then, $\widetilde{v}_1$ is pin-pointed by the observation that bidder 1 with type $\widetilde{v}_1$ and bidder 3 with type $\overline{v}$ bid exactly m. It is necessary to check that $\widetilde{v}_1 \leq \widehat{v}_1$, as a disjoint equilibrium does not exist if otherwise.

Since bidding strategies follow directly from the first order conditions once $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_2)$ is known, the unique triplet $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_2)$ completely characterizes the disjoint equilibrium. The following Proposition proves the existence and uniqueness of a disjoint equilibrium when m is not too small (an example will later demonstrate that disjoint equilibria do not always exist when m is small). Bidders 2 and 3 are relatively unsuccessful in such an equilibrium.

Proposition 3 *Assume that $\overline{m} < \overline{b}_{-2}$. Then, there exists some $\underline{m} < \overline{m}$, such that a unique disjoint equilibrium exists if $m \in (\underline{m}, \overline{m})$.*

Proof. See the Appendix. ■

Corollary 3 *In a disjoint equilibrium, bidder 1 is the only bidder to submit strictly positive bids with probability one. Ex ante, bidder 1 wins more often than bidders 2 and 3 combined.*

Proof. See the Appendix. ■

5.2 Preferential treatment of a diverse group

In light of the low participation- and success-rate of the weaker bidders, it may be tempting to intervene in the contest by handicapping the strong bidder. However, the point of Theorem 1 is that such an intervention may be harmful to bidder 3. The following example illustrates Theorem 1.

Example 1: Assume that $F(v) = v - 1$, $v \in [1, 2]$, and fix $\alpha_1 = 1$, $\alpha_2 = 2$. The parameters of bidder 3's cost function, (α_3, m), then determines the equilibrium. Assuming that m is sufficiently large, note that bidder 2 would stay out of the auction if $\bar{b}_{-2} \geq 1$, which can be shown to be satisfied if and only if α_3 falls below a critical value of approximately 1.677, such that the contest between bidder 1 and bidder 3 is not too uneven. Assuming that α_3 is below this critical value, bidder 2 will nevertheless be active in equilibrium if m falls below 1 and sufficiently constrains bidder 3's ability to compete.

Figure 4 describes the parameters of bidder 3's cost function, assuming that $m \in [0, 1]$ and $\alpha_3 \in [\alpha_1, \alpha_2] = [1, 2]$. A disjoint equilibrium as defined above exists if and only if (α_3, m) is to the north-west of the steep, unbroken, curve in Figure 4. On the curve, $\widetilde{v}_1 = \widehat{v}_1$; above it, $\widetilde{v}_1 < \widehat{v}_1$. Since the curve is increasing, a disjoint equilibrium will not necessarily exist if m is small, as claimed earlier. Indeed, if (α_3, m) is to the south-east of the flat, unbroken, curve, there is an equilibrium in which bidder 3 does not participate in the auction.[21,22]

Assume next that bidder 1 is handicapped. The dashed lines summarizes bidder 3's entry decision for different handicaps. The flat lines applies if $h(s) = 2s$ (a very severe handicap, for which $c_1^h(s) = c_2(s)$) and the steeper line applies if $h(s) = \frac{5}{4}s$ (a moderate handicap). If (α_3, m) is to the south-east of these lines, then there is an equilibrium in which bidder 3 never participates. Thus, Figure 4 confirms Theorem 1; there is an overlap of regions where all bidders are initially active (in a disjoint equilibrium) but where bidder 3 drops out once he and bidder 2 is given "preferential treatment".[23] This occurs when both α_3 and m are relatively low, which is consistent with the earlier assertion that c_3 needs to have sufficient "curvature" for Theorem 1 to hold. For fixed α_3, preferential treatment is therefore more likely to be harmful to bidder 3 the more disadvantaged he is initially (the lower m is). It should also be pointed out that bidder 3 may not participate even when α_3 is smaller than $c_1^h(0)$. That is, to participate it is not sufficient to have a cost advantage over all other bidders at low bids. ■

[21] The details of the example are omitted, but are available upon request.

[22] When (α_3, m) is in the region between the two curves, it is conjectured that all three bidders are active for a set of bids below m.

[23] There is also a region where the opposite conclusion applies. If α_3 is high and m is in a medium range, there are situations where bidder 3 is inactive before preferential treatment is introduced, but not after.

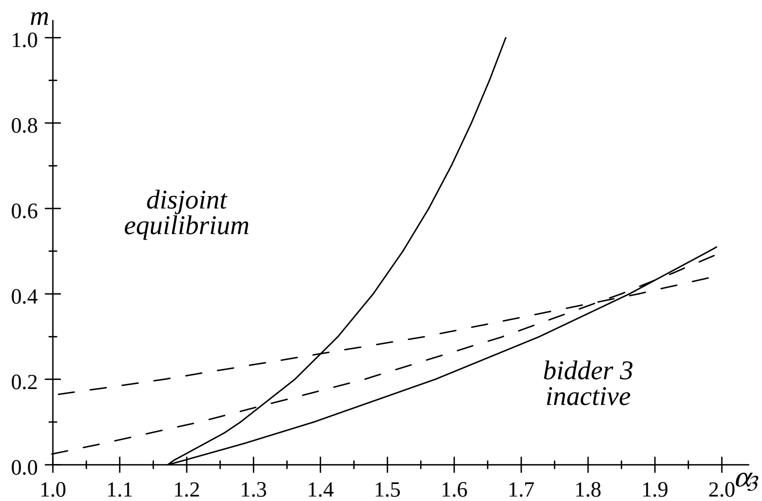

Figure 4: Disjoint equilibria and Theorem 1.

Assume that a disjoint equilibrium exists. One of the advantages of being able to characterize the equilibrium is that it is possible to examine the consequences of handicaps that are so small that a disjoint equilibrium still exists, meaning that bidder 3 is still active. The impact of a small handicap is described below, under the assumption that $h(s)$ is linear.

Proposition 4 *Assume that $h(s) = \tau s$, $\tau \geq 1$, and that a disjoint equilibrium exists when $\tau = 1$ (no handicap). For any handicap $\tau' \in (1, \frac{\alpha_3}{\alpha_1})$ for which a disjoint equilibrium exists, it holds that bidder 2 and bidder 3 participate more often, and win more often.*

Proof. See the Appendix. ■

Proposition 4 implies that *small* handicaps are successful on two dimensions; they entice the two weaker bidders to participate more often, and lead them to win more often as well. In this model, then, preferential treatment can be disadvantageous to its recipients only when it is large enough to alter the structure of the equilibrium.

5.3 Handicapping a diverse group

The analysis thus far has assumed that the set of bidders receiving preferential treatment is diverse, but that this is not the case for the set of bidders who are handicapped (since there is only one such bidder). In the following, I will switch the focus to the handicapped group, and ask whether diversity within that group matters.

Assume now that just one bidder receives preferential treatment. It is easy to check that if bidder 1 *or* bidder 3 is given preferential treatment (as captured by a decrease in α_1 or a decrease in α_3 and an increase in m, respectively) then the remaining bidders are adversely affected. The consequences of giving preferential treatment to bidder 2 are more surprising.

Proposition 5 *Assume that bidder 2 and only bidder 2 is given preferential treatment; with a bid of b he outscores a rival who bids γb, where $\gamma > 1$. Assume a disjoint equilibrium exists before and after preferential treatment. When preferential treatment is introduced, bidder 1 wins less often, bidder 2 participates and wins more often, and bidder 3 also participates and wins more often.*

Proof. See the Appendix. ■

The intuition behind Proposition 5 is as follows. Bidder 2's advantage leads him to participate more often. Thus, bidders 1 and 3 are less likely to win with a low bid, other things being equal. Consequently, they bid more cautiously. This opens up a gap between the small bids, those submitted by bidder 1 with type below $\widetilde{v}_1$, and the high bids, those submitted by bidder 1 with type above $\widetilde{v}_1$. Therefore, if bidder 1 was supposed to bid m, he can lower his bid without lowering his chance of winning. However, once bidder 1 starts submitting low bids more often, bidder 3 must respond by bidding more aggressively. Thus, if his type is close to $\overline{v}$, bidder 3 will now outscore more of bidder 1's types than before. This effect "trickles down" and affects the entire system or relationship between bidder 1 and bidder 3. The implication is that bidder 3 beats bidder 1 more often and participates more often ($k_3(v|\widetilde{v}_1)$ decreases). The following counterpart to Corollary 2 is immediate.

Corollary 4 *Bidder 3 may be made better off when he and bidder 1 are handicapped.*

Proof. See the Appendix. ■

6 Conclusion

This paper considered a contest with a number of realistic features: There are more than two contestants, contestants are heterogenous, and information is incomplete. In this environment, preferential treatment may have surprising and most likely unintended consequences. Specifically, when a diverse group of contestants are given preferential treatment compared to the remaining contestants, a subset of the intended beneficiaries may become worse off. The reason is that the dynamics within the "favored" group changes. In particular, the stronger of the favored contestants may become more aggressive. From the point of view of the weaker of the favored contestants, this effect may outweigh the advantage that is gained over contestants who do not receive preferential treatment. Similarly, a contestant who is not given preferential treatment may in fact be made better off by its introduction.

The possibility that preferential treatment may be disadvantageous was demonstrated in a setting with very specific cost structures, in which multiple equilibria were not ruled out.

However, since the result is a negative result, the main point remains valid: Jumping to the conclusion that preferential treatment is unambiguously beneficial to the weaker contestants is not justified. The theory of contests with more than two heterogeneous contestants is underdeveloped, and more research is needed to better assess the consequences of manipulating or regulating the contest.

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Appendix: Proofs

Proof of Proposition 1. Consider the game Γ_2^h. The system described by (1) is monotonic in $\overline{s}^h$. If $\overline{s}^h$ is reduced to $\widetilde{s}^h < \overline{s}^h$, it must be the case that $\varphi_i^h(\widetilde{s}^h) = \overline{v}$ after the change, but that $\varphi_i(\widetilde{s}^h) < \overline{v}$ before the change. Given (1), $F(\varphi_j^h(s))$ must therefore be strictly flatter than before at $s = \widetilde{s}^h$. Continuing this argument as s is reduced implies that $F(\varphi_1^h(\cdot))$ and $F(\varphi_2^h(\cdot))$ shift up when $\overline{s}^h$ is reduced. Thus, there is precisely one value for which $\overline{s}^h$ and the resulting unique paths of $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ satisfy the equilibrium requirement that $F(\varphi_1^h(0)) = 0$ and $F(\varphi_2^h(0)) \geq 0$ or vice versa.

Next, note that if $F(\varphi_1^h(s)) = F(\varphi_2^h(s))$ then $\varphi_1^h(s) = \varphi_2^h(s)$, in which case Assumption C and (1) together imply that $F(\varphi_1^h(s))$ is strictly steeper than $F(\varphi_2^h(s))$. Thus, $F(\varphi_1^h(s))$ and $F(\varphi_2^h(s))$ coincide at most once. In fact, this occurs at $\overline{s}^h$, since $F(\varphi_1^h(\overline{s}^h)) = F(\varphi_2^h(\overline{s}^h)) = 1$. Since $F(\varphi_1^h(s))$ is strictly steeper than $F(\varphi_2^h(s))$ at $\overline{s}^h$, the implication is that $F(\varphi_1^h(s)) < F(\varphi_2^h(s))$ for all $s \in [0, \overline{s}^h)$, as claimed. These arguments also apply to Γ_2. ■

Proof of Proposition 2. Assume that $\overline{s}^h = \overline{s}$. In this case $\varphi_i^h(\overline{s}) = \varphi_i(\overline{s}) = \overline{v}$, $i = 1, 2$, and it follows from Assumption B that $F(\varphi_2^h(\cdot))$ is strictly steeper than $F(\varphi_2(\cdot))$ at $\overline{s}$. Hence, $\varphi_2^h < \varphi_2$ immediately to the left of $\overline{s}$, which implies that $F(\varphi_1^h(\cdot))$ is strictly steeper than $F(\varphi_1(\cdot))$, and which in turn implies that $\varphi_1^h < \varphi_1$. These arguments repeat themselves as s is reduced even further, and it follows that $F(\varphi_1^h(\cdot))$ becomes zero for some $s > 0$. As discussed earlier, this contradicts that $\overline{s}^h$ and φ_1^h, φ_2^h form an equilibrium. It is easily seen that $\overline{s}^h > \overline{s}$ would lead to a similar contradiction. Thus, $\overline{s}^h < \overline{s}$, and $F(\varphi_1^h(\overline{s}^h)) = 1 > F(\varphi_1(\overline{s}^h))$.

Moving to the left, consider the first $s \in (0, \overline{s}^h)$ to the left of $\overline{s}^h$, if it exists, for which $F(\varphi_1^h(\cdot))$ and $F(\varphi_1(\cdot))$ coincide, or $\varphi_1^h = \varphi_1$. Since $F(\varphi_1^h(\cdot)) > F(\varphi_1(\cdot))$ to the right of this point, by definition, $F(\varphi_1^h(\cdot))$ must be at least as steep as $F(\varphi_1(\cdot))$, which implies that $\varphi_2^h \leq \varphi_2$ or $F(\varphi_2^h) \leq F(\varphi_2)$. As $\varphi_1^h = \varphi_1$, it follows from Assumption B and (1) that $F(\varphi_2^h(\cdot))$ is strictly steeper than $F(\varphi_2(\cdot))$ at this point, which means that $\varphi_2^h < \varphi_2$ just to the left of this score. As before, this leads to the conclusion that $F(\varphi_1^h(\cdot))$ is steeper than $F(\varphi_1(\cdot))$ to the left of this point. Once again, this can be ruled out, because $F(\varphi_1^h(\cdot))$ becomes zero for some $s > 0$. Thus, by contradiction, $F(\varphi_1^h(s)) > F(\varphi_1(s))$ for all $s \in (0, \overline{s}^h]$.

Assumption B and $\varphi_1^h(0) = \varphi_1(0) = \underline{v}$ imply that $F(\varphi_2^h(\cdot))$ is strictly steeper than $F(\varphi_2(\cdot))$ at $s = 0$. Assume now that $F(\varphi_2^h(0)) \geq F(\varphi_2(0))$ or $\varphi_2^h(0) \geq \varphi_2(0)$. Then, $\varphi_2^h > \varphi_2$ for small, strictly positive s, implying that $F(\varphi_1^h(\cdot))$ is strictly flatter than $F(\varphi_1(\cdot))$. Consequently, $\varphi_1^h < \varphi_1$ for small s. However, this contradicts the property that bidder 1 is less aggressive when he is handicapped. ■

Proof of Theorem 1. To begin, note that q_3^h as defined in (2) is exogenous to the games Γ_3 and Γ_3^h, since it depends only on the strategies φ_1^h and φ_2^h from the game Γ_2^h. For similar reasons, $q_3(s) \equiv F(\varphi_1(s))F(\varphi_2(s))$ is also exogenous to the games Γ_3 and Γ_3^h. In particular, neither depends on bidder 3's characteristics, since he is assumed absent from Γ_2 and Γ_2^h.

So, assume bidder 3's cost function takes the form $c_3(s) = \alpha \overline{v} q_3^h(s)$, with $1 < \alpha < q_3'(0)/q_3^{h\prime}(0)$. This is a strictly increasing and strictly convex function. Since $\alpha > 1$, (5) reveals that there is no incentive for bidder 3 to become active in Γ_3^h when bidders 1 and 2 follow the increasing strategies from Γ_2^h. Given bidder 3 stays out with probability one, there is no incentive for bidders 1 and 2 to deviate from the increasing strategies in Γ_2^h. Thus, it is an equilibrium for bidder 3 to always stay out, and for bidders 1 and 2 to continue using the increasing strategies from Γ_2^h. Clearly, bidder 3 wins the auction with probability zero.

Consider now the game without a handicap, Γ_3. If bidder 3 wins with probability zero, no strictly positive score or bid can be rationalized. Thus, bidder 3 must either stay out, or bid zero. The arguments leading to Proposition 1 still apply, and bidder 1 and bidder 2's increasing strategies from Γ_2 are the unique pair of candidates for their increasing equilibrium strategies. However, (5) and $\alpha < q_3'(0)/q_3^{h\prime}(0)$ imply that bidder 3 should deviate; there is no equilibrium in increasing strategies of Γ_3 where bidder 3 wins with probability 0. ■

Proof of Proposition 3. The proof is in three steps.

Step 1 (Constructing an equilibrium candidate): A simple application of the method proposed by Amann and Leininger (1996) reveals that $k_2(v)$ is implicitly given by

$$\alpha_2 \int_{k_2(v)}^{\overline{v}} \frac{f(x)}{x} dx = \alpha_1 \int_{v}^{\overline{v}} \frac{f(x)}{x} dx. \tag{7}$$

See Amann and Leininger (1996) for details. Note that $k_2(v)$ is strictly increasing and $k_2(\overline{v}) = \overline{v}$. Note also that once $\widehat{v}_1$ is derived, bidding strategies are trivially characterized by the first order conditions. For instance, bidder 2's first order condition can be written as

$$\frac{d\varphi_1(b)}{db} = \frac{\alpha_2}{\varphi_2(b) f(\varphi_1(s))}.$$

Since $\varphi_2(b) = k_2(\varphi_1(b))$, inverting and letting $v = \varphi_1(b)$ yields $b_1'(v) = k_2(v)f(v)/\alpha_2$, where b_1 refers to bidder 1's bidding strategy. Therefore,

$$b_1(v) = m + \int_{\widehat{v}_1}^{v} \frac{k_2(x)}{\alpha_2} f(x) dx,\ v \in [\widehat{v}_1, \overline{v}]. \tag{8}$$

Bidder 2's bid can be derived in a similar manner, or directly from $b_2(v) = b_1(k^{-1}(v))$.

To find $\widehat{v}_1$ it is useful to reconsider bidder 2's problem. In a disjoint equilibrium, bidder 2 is supposed to stay out if his type is below $\widehat{v}_2$, and then jump to a bid of m if his type is $\widehat{v}_2$. Consequently, bidder 2 must be indifferent between the two actions if his type is $\widehat{v}_2$, or

$$\widehat{v}_2 F(\widehat{v}_1) - \alpha_2 m = 0, \tag{9}$$

since a bid of m wins the auction if bidder 1's type is below $\widehat{v}_1$ (given the tie-breaking rule). For the first order conditions to be satisfied it must hold that $\widehat{v}_2 = k_2(\widehat{v}_1)$, or

$$k_2(\widehat{v}_1) F(\widehat{v}_1) - \alpha_2 m = 0, \tag{10}$$

which has a unique solution. Thus, $\widehat{v}_1$ and $\widehat{v}_2$ have been identified, as have strategies for bidder i with type $v \in [\widehat{v}_i, \overline{v}]$, $i = 1, 2$.

Consider now bids below m. For bidder 1, such a bid wins only if it beats bidder 3 and bidder 2 stays out. Thus, bidder 1's expected payoff from a bid of b is

$$vF(\varphi_3(b))F(k_2(\widehat{v}_1)) - \alpha_1 b, \tag{11}$$

if his type is v. A similar expression holds for bidder 3. Although the first order conditions now contain a $F(\widehat{v}_2)$ term, this will cancel out when Amann and Leininger's (1996) method is used to derive the tying function $k_3(v|\widetilde{v}_1)$, which is implicitly defined by

$$\alpha_3 \int_{k_3(v|\widetilde{v}_1)}^{\overline{v}} \frac{f(x)}{x} dx = \alpha_1 \int_{v}^{\widetilde{v}_1} \frac{f(x)}{x} dx. \tag{12}$$

Note that $\varphi_3(0) = k_3(\underline{v}|\widetilde{v}_1) > 0$ since $\alpha_3 > \alpha_1$ and $\overline{v} \geq \widetilde{v}_1$, meaning that bidder 3 stays out of the auction with positive probability. Repeating a previous argument then produces

$$b_3(k_3(v|\widetilde{v}_1)|\widetilde{v}_1, \widehat{v}_1) = b_1(v|\widetilde{v}_1, \widehat{v}_1) = F(k_2(\widehat{v}_1)) \int_{\underline{v}}^{v} \frac{k_3(x|\widetilde{v}_1)}{\alpha_3} f(x) dx, \; v \in [0, \widetilde{v}_1]. \tag{13}$$

Note that $b_3(\overline{v}|\overline{v}, \overline{v}) = b_1(\overline{v}|\overline{v}, \overline{v}) = \overline{b}_{-2}$.

It remains to determine $\widetilde{v}_1$. In a disjoint equilibrium, bidder 3's highest bid is m, or

$$b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1) = b_1(\widetilde{v}_1|\widetilde{v}_1, \widehat{v}_1) = m.$$

Bidder 3's maximal bid can also be written as

$$b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1) = F(k_2(\widehat{v}_1)) \int_{k_3(\underline{v}|\widetilde{v}_1)}^{\overline{v}} \frac{k_3^{-1}(x|\widetilde{v}_1)}{\alpha_1} f(x) dx. \tag{14}$$

Since k_3 shifts down when $\widetilde{v}_1$ increases, $b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1)$ is increasing in $\widetilde{v}_1$. Thus, there is at most one value of $\widetilde{v}_1$ for which $b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1) = m$. However, it is not obvious that a solution exists for all m. In particular, small values of $\widehat{v}_1$ imply that the right hand side of (14) is small as well. Moreover, the equilibrium construct assumes that $\widetilde{v}_1$ is bounded above by $\widehat{v}_1$, or $\widetilde{v}_1 \leq \widehat{v}_1$. Thus, the existence of a disjoint equilibrium is not guaranteed.

STEP 2 (EXISTENCE): Recall that any $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_2)$ triplet completely characterizes an equilibrium candidate. To summarize, $\widehat{v}_2$ is derived from $\widehat{v}_2 = k_2(\widehat{v}_1)$, while the pair $(\widetilde{v}_1, \widehat{v}_1)$ is derived from the pair of equations

$$\frac{k_2(\widehat{v}_1)F(\widehat{v}_1)}{\alpha_2} = m \tag{15}$$

$$b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1) = m. \tag{16}$$

To form a credible equilibrium candidate, it is required that $\widetilde{v}_1 \leq \widehat{v}_1 < \overline{v}$. Note that if $m = \overline{m}$ then (15) is satisfied at $\widehat{v}_1 = \overline{v}$ and if $m < \overline{m}$ it is satisfied at some $\widehat{v}_1 < \overline{v}$. Second, $\overline{b}_{-2} = b_3(\overline{v}|\overline{v}, \overline{v}) > \overline{m}$, by assumption. Since $b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1)$ is strictly increasing in $\widetilde{v}_1$ and is zero at $\widetilde{v}_1 = \underline{v}$, there is some $\widetilde{v}_1 \in (\underline{v}, \overline{v})$ for which $b_3(\overline{v}|\widetilde{v}_1, \overline{v}) = \overline{m}$, satisfying (16). Thus, both conditions are satisfied at $m = \overline{m}$, by some $\widetilde{v}_1 < \widehat{v}_1 = \overline{v}$. It is also the case that $k_2(\widehat{v}_1)F(\widehat{v}_1)$ and $b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1)$ are continuous and increasing in $\widehat{v}_1$ and $\widetilde{v}_1$. Thus, by continuity and monotonicity, all conditions remain satisfied if m is reduced slightly.

Finally, to prove that $(\widetilde{v}_1, \widehat{v}_1, \widehat{v}_2)$ characterizes a disjoint equilibrium, profitable deviations must be ruled out. First, local or small deviations can be ruled out for bidder 2 and bidder 3, as well as for bidder 1 with types outside the interval $[\widetilde{v}_1, \widehat{v}_1]$. The reason is that strategies are derived from first order conditions (and these conditions are sufficient, as explained in an earlier footnote). Large deviations must then be ruled out. Bidder 3 is unable to jump to bids in excess of m, and the tie-breaking rule ensures he has no incentive to jump to m, given the first order condition specifies what the optimal bid in the range $[0, m]$ is. Bidder 2 has no incentive to pick bids in the range $(0, m)$, due to the convexity of his payoff function in that range. The tie-breaking rule guarantees that bidder 2 wins if he is in a tie at bid m, so he has no incentive to jump from such a bid when his equilibrium strategy dictates that he bids m. Since the single-crossing condition is satisfied and type $\widehat{v}_2$ earns zero payoff, there is no incentive to change the entry decision for types below or above $\widehat{v}_2$. Consider now bidder 1. For him, the probability of a tie at a bid of m is zero, so there is no jump in payoff from bidding marginally higher than m. The first order conditions then ensure that bidder 1 with type outside $[\widetilde{v}_1, \widehat{v}_1]$ is playing a best response. The single-crossing condition then implies that types between $\widetilde{v}_1$ and $\widehat{v}_1$ must be maximizing by bidding exactly m, as required. In summary, there is no incentive to deviate.

Step 3 (Uniqueness): Assuming existence, $\widehat{v}_1$ is unique because the left hand side of (15) is strictly increasing in $\widehat{v}_1$. It follows that $\widehat{v}_2 = k_2(\widehat{v}_1)$ is unique as well. Finally, for this (and any other) fixed value of $\widehat{v}_1$, the left hand side of (16) is strictly increasing in $\widetilde{v}_1$, which means that $\widetilde{v}_1$ is unique as well. ■

Proof of Corollary 3. At least one bidder must submit strictly positive bids with probability one, or there would be an incentive to submit a small bid for someone who is supposed to bid zero or stay out. By construction, bidder 2 stays out bids with probability $F(\widehat{v}_2) > 0$ and bidder 3 with probability $F(k_3(\underline{v}|\widetilde{v}_1)) > 0$. Hence, bidder one is the only bidder to submit strictly positive bids with probability one.

For the second part, it is easily verified that (i) $k_3(v|\widetilde{v}_1) > v$, $v \in [\underline{v}, \widetilde{v}_1)$, (ii) $k_3(v|\widetilde{v}_1)$ is decreasing in $\widetilde{v}_1$, and (iii) $k_2(v) > k_3(v|\overline{v}) > v$, $v \in [\underline{v}, \overline{v})$. Thus, bidder 1 wins with probability $F(k_2(v)) > F(v)$ when his type is $v \in [\widehat{v}_1, \overline{v})$, and probability$F(k_2(\widehat{v}_1)) > F(\widehat{v}_1) > F(v)$ when his type is $v \in [\widetilde{v}_1, \widehat{v}_1)$. When $v \in [\underline{v}, \widetilde{v}_1)$, his winning probability is $F(\widehat{v}_2)F(k_3(v|\widetilde{v}_1))$, which is no smaller than $F(k_2(\widehat{v}_1))F(k_3(v|\widehat{v}_1))$. Viewing the last expression as a function of $\widehat{v}_1$, observation (ii) and Assumption A can be used to show that it is single-peaked in $\widehat{v}_1$, for a fixed v. Thus, the expression is minimized when $\widehat{v}_1 = \overline{v}$ or $\widehat{v}_1 = v$. At $\widehat{v}_1 = \overline{v}$, $F(k_2(\widehat{v}_1))F(k_3(v|\widehat{v}_1)) = F(k_3(v|\overline{v})) > F(v)$, while at $\widehat{v}_1 = v$, $F(k_2(\widehat{v}_1))F(k_3(v|\widehat{v}_1)) = F(k_2(v)) > F(v)$. In summary, it has been shown that bidder 1's winning probability strictly exceeds $F(v)$ for all $v \in [\underline{v}, \overline{v})$. Hence, his ex ante winning probability is strictly larger than

$$\int_{\underline{v}}^{\overline{v}} F(v)f(v)dv = \frac{1}{2},$$

which proves the last part of the Corollary. ■

Proof of Proposition 4. Imposing a linear handicap is equivalent to changing α_1 to $\alpha_1' = \tau\alpha_1$. Starting "at the top", (7) reveals that $k_2(v)$ decreases; bidder 2 is emboldened by bidder 1's handicap, and wins more often for any type that participates. Combining (7) and (9) implies that $\widehat{v}_2$ decreases but $\widehat{v}_1$ increases. Since $k_2(v)$ and $\widehat{v}_2$ decrease, the part of the Proposition dealing with bidder 2 follows. The increase in α_1 also leads $k_3(v|\widetilde{v}_1)$ to decrease, for fixed $\widetilde{v}_1$. Coupled with the decrease in $\widehat{v}_2$, (13) implies that $b_1(\widetilde{v}_1|\widetilde{v}_1, \widehat{v}_1)$ declines, for fixed $\widetilde{v}_1$. Since $b_1(\widetilde{v}_1|\widetilde{v}_1, \widehat{v}_1) = b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1)$, the latter must also have decreased. To maintain $b_3(\overline{v}|\widetilde{v}_1, \widehat{v}_1) = m$, it is therefore necessary that $\widetilde{v}_1$ increases. The increase in $\widetilde{v}_1$ and α_1' then means that k_3 decreases (bidder 3 is more successful against bidder 1, but less successful against bidder 2). In particular, $k_3(\underline{v})$ must decrease, thereby proving that bidder 3 participates more often. The assumption that $\tau' < \frac{\alpha_3}{\alpha_1}$ ensures that $k_3(\underline{v}) > \underline{v}$ (bidder 3 does not participate with probability one, but bidder 1 does). Suppressing the dependence on $\widetilde{v}_1$, bidder 3's ex ante winning probability is initially

$$w_3 = \int_{k_3(\underline{v})}^{\overline{v}} F(k_2(\widehat{v}_1))F\left(k_3^{-1}(x)\right)f(x)dx.$$

Substituting $z = k_3^{-1}(x)$ and noting, from (12), that $k_3'(z) = \frac{\alpha_1}{\alpha_3}\frac{k_3(z)}{z}\frac{f(z)}{f(k_3(z))}$ yields

$$w_3 = \int_{\underline{v}}^{\widetilde{v}_1} F(k_2(\widehat{v}_1))F\left(z\right)f(k_3(z))k_3'(z)dz = \int_{\underline{v}}^{\widetilde{v}_1} F(k_2(\widehat{v}_1))\frac{k_3(z)}{\alpha_3}f(z)\left(\alpha_1\frac{F\left(z\right)}{z}\right)dz.$$

Note that it is only the term in parenthesis that separates w_3 from the constant m, by (13). Let $G(z) = \frac{F(z)}{z}$; by Assumption A, $G'(z) > 0$. Integration by parts produces

$$w_3 = \alpha_1\int_{\underline{v}}^{\widetilde{v}_1}\left(m - \int_{\underline{v}}^{z} F(k_2(\widehat{v}_1))\frac{k_3(x)}{\alpha_3}f(x)dx\right)G'(z)dz.$$

The term in parenthesis is positive, by (13). Recall that since $\widetilde{v}_1$ and α_1 increase, $k_3(x)$ decreases. Moreover, $\widehat{v}_2 = k(\widehat{v}_1)$ decreases when bidder 1 is handicapped. Thus, the term in the parentheses is positive and increases when bidder 1 is handicapped. Since the multiplicative first term (α_1) on the right hand side increases and the upper bound on the integration ($\widetilde{v}_1$) increases as well, w_3 must increase. ■

Proof of Proposition 5. The advantage to bidder 2 is equivalent to lowering α_2 to α_2/γ. Such a decrease in α_2 lowers $k_2(v)$. Since the last term on the left hand side of (10) decreases, the first term must do the same. Thus, it cannot be the case that $\widehat{v}_2 = k_2(\widehat{v}_1)$ increases, since that would necessitate an increase in $F(\widehat{v}_1)$, and therefore an increase in $\widehat{v}_2F(\widehat{v}_1)$. In conclusion, $\widehat{v}_2$ decreases; bidder 2 participates more often. Since $k_2(v)$ has been lowered, bidder 2 also wins more often.

Since $k_2(\widehat{v}_1)$ decreases, $\widetilde{v}_1$ must increase. Thus, $k_3(v|\widetilde{v}_1)$ decreases, implying that bidder 3 participates more often ($k_3(\underline{v}|\widetilde{v}_1)$ decreases). An argument similar to that in Proposition 4 can be used to prove that bidder 3 wins more often as well. Thus, bidder 1 wins less often. ■

Proof of Corollary 4. Since bidder 3 participates more often in the disjoint equilibria in Proposition 5, he has a set of types who earn positive payoff after bidder 2 is given preferential treatment but who earned zero payoff (and did not participate) before. ■