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A dynamic Cournot–Nash game: a representation of a finitely repeated feedback game

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Abstract This paper studies market outcome equivalence of two dynamic production-capital investment games under uncertainty. One is played under complete information, while the other, feedback (FB) game, is played under incomplete information about the opponents' costs and market demand. The FB game structure may be observed in some newly initiated industries, in which a homogeneous good is exchanged via an auction mechanism. In that case, the FB game setting may predict the complete information equilibrium market outcomes.

Keywords Production-investment dynamic game · Complete/incomplete information · Communication · Demand uncertainty

Mathematics Subject Classification (2000) 91-08 · 91A25

1 Introduction

This paper outlines a two-period dynamic game. The objective is to study an industry structure in which economic agents (players or firms) noncooperatively find an optimal way to produce a homogeneous good and make capacity investments in production under uncertainty and incomplete information. Specifically we consider that demand for a product is uncertain and that the exact functional form of demand may not be known to the players (or players have no perfect foresight of it). Moreover each player only knows his own production capacity levels, and production and investment costs of several technologies, but not the

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rivals' information. In this setting we address the question of how equilibrium outcomes might be formed in the market. Suppose that an auctioneer procures the homogeneous good that is produced by firms. We assume multiple interactions between players and the auctioneer before optimal (production and capital investment) decisions are made. During each communication players simultaneously reveal their willingness to produce quantities to the auctioneer who then determines the price at the level of total quantity supplied. Firms solve their 'sub-problems' to determine their quantities and ultimately a set of 'artificial equilibrium' outcomes is obtained in each communication period. We call this setting a finitely repeated feedback (FB) game and explain the details of this game in the following section.

The rules and equilibrium concept of the FB game are different than the ones in the literature. For instance the structure of the FB game may recall 'communication equilibria' and/or 'rational learning equilibria', in which players learn how to play Nash equilibrium with or without complete information in finitely and infinitely repeated games. Forges (1986) proposed extensive form correlated equilibrium and communication equilibrium games in which players were allowed to observe private signals and to transmit inputs at every stage (of the event tree) to a correlation device. He showed that payoffs associated with these equilibrium concepts are equivalent to those of Nash equilibrium. In these equilibrium concepts, players communicate with each other during preplay or intraplay stages. Mertens et al. (1994) surveys the literature regarding how players learn to play Nash equilibrium in repeated games. Kalai and Lehrer (1993) study an infinitely repeated game with discounting in which players are subjectively rational: if a player's belief about a rival's strategies is compatible then, by means of Bayesian updating, players' strategies will converge to the Nash equilibrium outcome in the long run. Kalai and Lehrer criticize other models (by Selten 1991; Milgrom and Roberts 1991; Fudenberg and Levine 1993) since these models restrict the behavior of players to be myopic and bounded.

This paper also has equilibrium computational aspects. We show that FB game equilibrium outcomes converge to dynamic Cournot–Nash (CN) equilibrium outcomes under demand uncertainty. This correspondence may also be considered as computing CN outcomes algorithmically in a different space such that both spaces have different information structures and rules.

Finding and/or computing Nash equilibria has been studied since 1950s for both normal and extensive form games. Von Stengel (2002) surveys the exposition of linear methods used to find equilibrium for two-person games. For a survey of nonlinear methods for computing Nash equilibria for more than two player noncooperative games see McKelvey and McLennan (1996). The algorithms designed for solving Nash equilibrium outcomes mostly deal with small-scale optimization problems in game-theoretical settings. For example, Uryasev and Rubinstein (1994) consider a special class of numerical algorithms, the so-called relaxation algorithm, to compute Nash equilibrium points in noncooperative games. Belenkii et al. (1974), Basar (1987), and Li and Basar (1987) studied relaxation algorithms for deterministic games, where fixed-point theorems are used to check equilibrium convergence conditions. Basar (1987) and

Li and Basar (1987) proved convergence for a two-player static game via a contraction-mapping theorem. For linear-quadratic settings, it may be relatively easy to check the convergence conditions. However, for other nonlinear payoff functions with coupled constraints, it may be intractable to check these conditions. Uryasev and Rubinstein proposed a different approach to tackle the problem for nonlinear functions. They utilize "the residual terms" of the Nikaido-Isoda Hukuhane and Kazuo (1955) function. They show convergence of the algorithm via non-smooth weakly convex/concave Nikaido-Isoda functions. The usefulness of their methodology was discussed only for static games without constraints. Krawczyk and Uryasev (2000) study another algorithm to solve a multi-player, non-zero-sum dynamic game with coupled constraints. Krawczyk and Uryasev introduce an improvement to the relaxation algorithm by implementing the steepest-descent step-size control technique. They prove the convergence of their algorithm and test it on a several problems. They specifically apply their procedure to a river basin pollution problem with coupled environmental constraints and show that the algorithm demonstrates fast convergence for a wide range of parameters. Their algorithm minimizes a multivariate Nikaido-Isoda function by using a standard nonlinear programming routine at each level of iteration.

In this paper the FB game has two main features. First, from the computational point of view, it may be considered as an algorithm (game) to compute CN equilibrium outcomes. As an algorithm, but not from the point of view of the informational structure and rules, it resembles the work of Krawczyk and Uryasev (2000), although we use a much simpler algorithm. Second, it shows how a communication scheme under incomplete information leads to an equilibrium whose outcomes may be obtained via a dynamic game structure in which there is no communication and/or learning but players have full knowledge about rivals' strategies and full information about all parameters of the game.

The structure of the paper is as follows. In Sect. 2 we present the correspondence between one period FB and CN production games with symmetric and asymmetric players that have coupled constraints. Section 3 extends the procedure to two periods and considers capacity investment in production under demand uncertainty and presents an illustrative example. Section 4 concludes.

2 Model

In Sect. 2.1 we show how our approach is implemented for one-period production oligopoly (FB and CN) games. In Sect. 2.2 we study these games with capacity constraints.

2.1 One-period oligopoly games

2.1.1 Finitely repeated feedback game

Timing Let t_c denote a communication period. Let τ denote a production period.

Communication period: $t_c < \tau$.
Production period: $\tau = 0$.

The communication period precedes the production period and is represented, without loss of generality, by integers. To not deal with negative numbers let $t = \mid t_c \mid$ such that $t = 1, 2, \ldots, T$, where $T < \infty$. Here t denotes iteration period.

Description of the FB game Consider an oligopoly with $n \geq 2$ profit-maximizing-firms who produce a homogeneous product so that each player faces the same price in the market. Each player has a differentiable cost function. In a given period of time each player is not sure how much to produce since market price and demand are both unknown. Each one starts with a quantity that he may be willing to produce. These quantities are submitted to the auctioneer, who procures the good and knows the market demand (or has a perfect forecast for the demand). We assume that communication between the auctioneer and firms is costless. Given these submitted quantities, the auctioneer calculates the price (that buyers are willing to pay for the total quantity supplied) and delivers it to the players. Then each player solves his own sub-problem that maximizes his 'regularized' payoff function for the quantities he wants to produce at this price level. After that, each player truthfully submits a quantity which is a convex combination of the previous and current quantities. The auctioneer uses these new quantities to calculate a new *candidate* market price. It is called a candidate price since the players may not agree to produce at that price. Again each firm gets this new price and solves his optimization problem. This process continues until the candidate market price and/or proposed quantities at iteration t is approximately equal to the price and the quantities at iteration $t-1$. We note that during this finitely repeated interaction between the players and the auctioneer, players do not observe the rivals' submitted quantity levels directly, nor does the candidate market price reflect the opponents' outputs directly since the exact functional form of the demand is unknown to the players. It is clear that iteration t output levels are a function of the past candidate market prices.

We call the above process the finitely repeated feedback (FB) game because prior to the actual production at a given period players interact finitely many times with the auctioneer and exchange information about the candidate market outcomes before they decide how much to actually produce.

Algorithmically the FB game is defined as follows.

Step 0. (Initialization) Each firm i ($i = 1, 2, \ldots, n$, where $n < N$) selects $\bar{q}_{i,0} \geq 0$ arbitrarily at the beginning and the auctioneer calculates a candidate market price $\bar{P}_0$ and submits this to the firms.

Step 1. (Solving sub-problems) Each firm solves the following sub-problem in every iteration:

$$\max_{q_{i,t} \geq 0} \bar{P}_{t-1} q_{i,t} - c(q_{i,t}) - f(q_{i,t}) \tag{1}$$

Let the solution sequence be $(q^*_{i,t})_i$ at iteration t. Note that here $\bar{P}_{t-1}$ is a fixed number, not a function of quantities.

Step 2. (Communication and keeping track of short-term memories) Each player computes the following: $\bar{q}_{i,t} = (1-\alpha)\bar{q}_{i,t-1}+\alpha q^*_{i,t}$, where t represents iteration, ($t = 1, 2, \ldots, T$, where $T < \infty$), and $\alpha \in (0, 2/(n+1))$. The auctioneer calculates the new price, $\bar{P}_t$ by using $\left(\bar{q}_{i,t}\right)_i$.

Step 3. (Stopping criteria) Let ξ_1, ξ_2 be sufficiently small positive real numbers. If $\left|q^*_{i,t} - q^*_{i,t-1}\right| < \xi_1$ and/or $\left|\bar{P}_t - \bar{P}_{t-1}\right| < \xi_2$ hold, then stop, otherwise repeat Step 1 and Step 2. $\diamond$

Definition 1 *The set of exchanged candidate prices and quantities* $(\bar{P}_t, \bar{q}_{i,t})_{i,t}$ *defined above is called artificial equilibria of the sub-problems defined in the expression* (1).

It is observed that, by mathematical induction, the formulation in Step 2 at iteration T becomes,

$$\bar{q}_{i,T} = (1-\alpha)^T \bar{q}_{i,0} + \alpha \sum_{t=0}^{T-1} (1-\alpha)^t q^*_{i,T-t} \tag{2}$$

We call the payoff function in (1) the ‘regularized’ profit function, since the price is constant and it has an additional term — the ‘regularization’ function $f(\cdot)$, which may be any polynomial function. (We note that the function $f(\cdot)$ is chosen to allow any differentiable cost functions including affine cost functions. If the cost function is affine, then the function $f(\cdot)$ entails the first order condition of (1) to be a function of q. Alternatively, if we exclude affine cost functions then we may discard the function $f(\cdot)$ in (1). However, for the sake of computational simplicity the function $f(\cdot)$ is chosen to be of the smallest order, $(o(f^{'}(\cdot)) = 1)$ so that even if the cost function is linear, the first order condition for (1) will be a function of q. Hence, without loss of generality let $f(\cdot) = \frac{1}{\gamma} q_i^2$, where $\gamma = 2$. Taking a convex combination of the quantities in Step 2 may be referred to as keeping track of short-term memories. Thus, each player uses all the information that he collects to reach optimal final decisions. We assume that each player uses the same weighting scheme (α), and it is constant. If n is large then each player puts little weight on the quantity q^*_i . In (1) it is not required to optimize and update α for each iteration, as opposed to the ‘steepest-descent-method’ (see Bertsekas 1999). With regard to the economic importance of α , if the number of players is large then players put more weights on their previously proposed quantities. This has two advantages. First, it is possible that the other players have started with some initial quantities that are far away from the optimal ones. Then, the price obtained from the auctioneer will reflect this fact and hence for a player the possible production quantity with this given price will be off. If the number of players is large, then that α choice will alleviate the initialization problem. Second, when the number of players is large they may need to collect more information (through price) about rivals’ strategies before they choose their optimal actions. Hence they cautiously put little weight on their current plans. As they get more information, the weights of very past actions do gradually decrease. However, if the number of players gets small then the interval of α enlarges, and hence players can put more value on their current strategies.

2.1.2 A Cournot–Nash (CN) game

Timing Before production takes place each firm has complete information about market demand and rivals' costs. Since this is a one-shot game no capital investment is considered.
Production period: $\tau = 0$.

Description of the CN game Consider an oligopoly market with $n \geq 2$ firms such that each firm's cost function is convex and differentiable, and firms face a linear demand. Let $P(Q)$ be the inverse demand function which determines the price of output as a function of total production; this is a strictly decreasing function of the total quantity of production in the market. Specifically, let $P(Q) = D - Q$ be the inverse demand with a constant $D > 0$. Each player produces the same quality of the good and has no capacity constraint. Later we will consider capacity constraints. Assume that firms compete a la Cournot and know their rivals' costs. Under these assumptions, when each player maximizes his profit function, Cournot–Nash equilibrium outcomes will prevail in the market since the Cournot–Nash equilibrium already presupposes that firms have an accurate knowledge about the opponents' cost functions. Formally, each player i solves the following maximization problem.

$$\max_{q_i} \quad P(Q)q_i - c(q_i)$$

$$\text{s.t.} \quad \sum_i q_i = Q, \quad P + Q = D, \;\; q_i > 0$$

The first order necessary condition for player i yields $q_i = P - c_i^{'}$ for an interior solution.

A symmetric CN equilibrium with identical cost firms implies $D - (n+1)q - c^{'}(q) = 0$. Denote the quantity that solves this equality as q^c.

The following proposition relates the outcomes of FB and CN games given the information structures of the games as defined above.

Proposition 1 *For $n \geq 2$ player symmetric oligopoly, the equilibrium market outcomes of the finitely repeated feedback (FB) game converge to the equilibrium market outcomes of the CN game.*

Proof We want to show that $\bar{q}_{i,T} \to q_i^c$ for some $T < \infty$. Since we assume that players are symmetric we will drop the subscript i hereafter. Let $\bar{q}_0 = \varepsilon + q^c$, where ε is a real number, then $\bar{P}_0 = D - n(q^c + \varepsilon)$. Also let, without loss of generality, $\gamma = 2$ in (1). Solution of the expression (1) at the first iteration yields $q_1^* = \bar{P}_0 - c^{'}(q_1^*)$. Then a player obtains, $\bar{q}_1 = (1-\alpha)\bar{q}_0 + \alpha q_1^* = \alpha(D - (n+1)q^c - c^{'}(q^c)) + \varepsilon(1-(n+1)\alpha) + q^c - \alpha a_1 = \varepsilon(1-(n+1)\alpha) + q^c - \alpha a_1$, where

$a_1 = \left|c^{'}(q^c) - c^{'}(q_1^*)\right|$. The first equality comes from Step 2, the second equality is because of the appropriate substitutions, and the final equality is due to $D-(n+1)q^c - c^{'}(q^c) = 0$. The new price at the first iteration becomes $\bar{P}_1 = D - n\bar{q}_1$. Let $z = (n+1)\alpha$, then the above equation can be rewritten as $\bar{q}_1 = (1-z)\varepsilon + q^c - \alpha a_1$. Let $a_t = \left|c^{'}(q^c) - c^{'}(q_t^*)\right|, t = 1, \ldots, T$. Similarly, by following this process for the second iteration we obtain $\bar{q}_2 = (1-\alpha)\bar{q}_1 + \alpha q_2^* = \varepsilon(1-z)^2 + q^c - \alpha a_1(1-z) - \alpha a_2$. Assume that $\bar{q}_{T-1} = (1-z)^{T-1}\varepsilon + q^c - \alpha\sum_{t=1}^{T-1} a_t(1-z)^{T-t-1}$, then we need to show that $\bar{q}_T = (1-z)^T\varepsilon + q^c - \alpha\sum_{t=1}^{T} a_t(1-z)^{T-t}$. At the iteration T, $\bar{q}_T = (1-\alpha)\bar{q}_{T-1} + \alpha q_T^*$ holds, where $q_T^* = \bar{P}_{T-1} - c^{'}(q_T^*)$, $\bar{P}_{T-1} = D - n\bar{q}_{T-1}$. Then, $\bar{q}_T = (1-\alpha)\bar{q}_{T-1} + \alpha q_T^* = \alpha(D-(n+1)\bar{q}_{T-1} - c^{'}) + \bar{q}_{T-1} = \varepsilon(1-z)^{T-1}[1-\alpha(n+1)] + q^c + \alpha z\sum_{t=1}^{T-1} a_t(1-z)^{T-1-t} - \alpha a_T - \alpha\sum_{t=1}^{T-1} a_t(1-z)^{T-1-t} = (1-z)^T\varepsilon + q^c - \alpha\sum_{t=1}^{T} a_t(1-z)^{T-t}$.

Let $a = \min(a_t)_t$, then $\bar{q}_T \leq (1-z)^T\varepsilon + q^c - \alpha a\sum_{t=1}^{T}(1-z)^{T-t} = q^c + \varepsilon^{'}((1-z)^T - 1) + \varepsilon(1-z)^T$, where $\varepsilon^{'} = a/(n+1)$.

Choose $\xi = \varepsilon^{'}((1-z)^T - 1) + \varepsilon(1-z)^T$, which is a small number. Hence for some iteration $T < \infty$, and with the tolerance level ξ, $\bar{q}_T \to q^c$. □

In the above proof as it can be seen, $\bar{q}_T$ is a separable function of $\left|c^{'}(q^c) - c^{'}(q_T^*)\right|$, and since convergence of $c^{'}(q_T^*)$ to $c^{'}(q^c)$ is shown here, in the following proofs we will shortly write $D-(n+1)q - c^{'} = 0$.

Next we extend the result of the above proposition to an asymmetric duopoly in which players have different costs. First we write the following Lemma.

Lemma 1 *Let A_m and B_m be two series with an index $m \in \mathbb{N}$. Suppose that $A_m + B_m \to 0$. If $A_m - B_m \to 0$ then $A_m \to 0$ and $B_m \to 0$.*

Proof A straightforward proof for this lemma is as follows. For a given $\zeta > 0$ we want to show that $|A_m| < \zeta$. There exist indices $N_1, N_2 \in \mathbb{N}$ such that for $m > N_1$, $|A_m + B_m| < \zeta$ and for $m > N_2$, $|A_m - B_m| < \zeta$. Summation of $-\zeta < A_m + B_m < \zeta$ and $-\zeta < A_m - B_m < \zeta$ implies $|A_m| < \zeta$. Then $B_m \to 0$ holds since $A_m + B_m \to 0$. □

Proposition 2 *For asymmetric duopoly, the equilibrium market outcomes of the finitely repeated feedback (FB) game converge to the equilibrium market outcomes of the CN game.*

Proof Let $\bar{q}_{i,0} = \varepsilon_i + q_i^c$, for $i = 1, 2$. Then the initial price is $\bar{P}_0 = D - q_1^c - q_2^c - \varepsilon_1 - \varepsilon_2$. Let $\gamma = 2$ in (1). The first iteration solution of (2) yields $q_i^* = \bar{P}_0 - c_i^{'}$. Then by suitable substitutions we obtain,

$$\begin{aligned}\bar{q}_{1,1} &= (1-\alpha)\bar{q}_{1,0} + \alpha q_{1,1}^* \\ &= \alpha(D - 2q_1^c - q_2^c - c_1^{'}) + \varepsilon_1(1-\alpha) - \alpha(\varepsilon_1 + \varepsilon_2) + q_1^c \\ &= \varepsilon_1(1-\alpha) - \alpha(\varepsilon_1 + \varepsilon_2) + q_1^c,\end{aligned}$$

where the first subscript refers to a player, the second subscript refers to iteration and q_i^c denotes the CN equilibrium quantity for player i. Similarly the first iteration quantity for player 2 is,

$$\bar{q}_{2,1} = \varepsilon_2(1-\alpha) - \alpha(\varepsilon_1 + \varepsilon_2) + q_2^c.$$

Given these two quantities the price becomes $\bar{P}_1 = D - \bar{q}_{1,1} - \bar{q}_{2,1}$. Using a similar analogy, the second iteration quantities for both players turn out to be $\bar{q}_{1,2} = q_1^c + \varepsilon_1(5\alpha^2 - 4\alpha + 1) + \varepsilon_2(4\alpha^2 - 2\alpha)$, $\bar{q}_{2,2} = q_2^c + \varepsilon_2(5\alpha^2 - 4\alpha + 1) + \varepsilon_1(4\alpha^2 - 2\alpha)$, with the price $\bar{P}_2 = D - \bar{q}_{1,2} - \bar{q}_{2,2}$. Similarly one can obtain other iteration results.

Case 1: Assume $\varepsilon_1 = \varepsilon_2 = \varepsilon$. Then the proof is the same as the proof of Proposition 1.

Case 2: Assume $\varepsilon_1 \neq \varepsilon_2$. Let $\varepsilon_1 = \varepsilon_2 + k$ be for some $k \in \mathbb{R}$. Then, $\bar{q}_{1,1} = q_1^c + \varepsilon_1(1-\alpha) - \alpha(\varepsilon_1 + \varepsilon_2) = q_1^c + (\varepsilon_2 + k)(1-2\alpha) - \alpha\varepsilon_2 = q_1^c + (-1)(z-1)\varepsilon_2 + kA_1$, where $A_1 = 1 - 2\alpha$ is a residual. For the second iteration it is obtained that $\bar{q}_{1,2} = q_1^c + (-1)^2(z-1)^2\varepsilon_2 + kA_2$, where $A_2 = 5\alpha^2 - 4\alpha + 1$. Similarly, the first iteration for the second player leads to $\bar{q}_{2,1} = q_2^c + (-1)(z-1)\varepsilon_2 + kB_1$, where $B_1 = -\alpha$, for the second iteration $\bar{q}_{2,2} = q_2^c + (-1)^2(z-1)^2\varepsilon_2 + kB_2$, where $B_2 = 4\alpha^2 - 2\alpha$. Observe that $A_1 + B_1 = (-1)(z-1)$, $A_2 + B_2 = (-1)^2(z-1)^2$, $A_1 - B_1 = (-1)(\alpha - 1)$, and $A_2 - B_2 = (-1)^2(\alpha-1)^2$. By mathemarical induction it is easily shown that at the Tth iteration $A_T + B_T = (-1)^T(z-1)^T$ and $A_T - B_T = (-1)^T(\alpha-1)^T$. Now we show that $\bar{q}_{1,T} = q_1^c + (-1)^T(z-1)^T\varepsilon_2 + kA_T$ and $\bar{q}_{2,T} = q_2^c + (-1)^T(z-1)^T\varepsilon_2 + kB_T$. Assume that $\bar{q}_{1,T-1} = q_1^c + (-1)^{T-1}(z-1)^{T-1}\varepsilon_2 + kA_{T-1}$ and $\bar{q}_{2,T-1} = q_2^c + (-1)^{T-1}(z-1)^{T-1}\varepsilon_2 + kB_{T-1}$. Clearly $\bar{q}_{1,T} = (1-\alpha)\bar{q}_{1,T-1} + \alpha q_{1,T}^*$, where $q_{1,T}^* = \bar{P}_{T-1} - c_1'$ and $\bar{P}_{T-1} = D - \bar{q}_{1,T-1} - \bar{q}_{2,T-1}$. Substitution of these terms leads to $\bar{q}_{1,T} = q_1^c + (-1)^T(z-1)^T\varepsilon_2 + k\alpha(A_{T-1} + B_{T-1}) + kA_{T-1}$. Because of the result of the Lemma 1 and given that ε_2 is a constant and $|z-1| < 1$, then for some T, $\bar{q}_{1,T} \to q_1^c$. Using similar arguments as we used before we can also obtain that $\bar{q}_{2,T} \to q_2^c$ for the second player. □

2.2 One-period oligopoly games with capacity constraints

In what follows we keep the structure of the FB and CN games and add capacity constraints to the production stage. Specifically the optimization problem of player i in the CN game will be,

$$\begin{aligned} \max_{q_i} \quad & P(Q)q_i - c(q_i) \\ \text{s.t.} \quad & q_i \le K_i, \quad \sum_i q_i = Q, \;\; P+Q=D, \;\; q_i \ge 0 \end{aligned}$$

where K_i is the capacity constraint for player i.

The following proposition shows how the FB game may be used to compute the equilibrium outcomes in the capacity-constrained CN game.

Proposition 3 *For symmetric capacity-constrained oligopolies, the quantities and prices obtained from the FB game will converge to the equilibrium outcomes of the CN game.*

Proof We will prove this proposition for symmetric duopolies, an extension of it to many players is similar. The first order necessary condition of the maximization problem for a player for an interior solution satisfies $D-3q^c-c^{'}-\lambda^c=0$ with a complementarity condition $(K-q^c)\lambda^c=0$, where (q^c,λ^c) is the pair of CN equilibrium quantity and the shadow price of the inequality constraint. Next we implement the FB game as follows. Let each player initially submit production quantity $\bar{q}_0=\varepsilon+q^c$ to the auctioneer who sets the price as $\bar{P}_0=D-2(\varepsilon+q^c)$. Then each player solves (1), say for $\gamma=2$, and the first iteration solution yields $q_1^*=\bar{P}_0-c^{'}-\lambda_1^*$ and $\lambda_1^*(K-q_1^*)=0$. Assume that $\lambda_1^*=\lambda^c+\eta_1$ for some number η_1. Then a player computes $\bar{q}_1=(1-\alpha)\bar{q}_0+\alpha q_1^*=\varepsilon(1-3\alpha)+q^c+\eta_1(-\alpha)$, where $(\lambda^c+\eta_1)(K-q_1^*)=0$ holds. The second iteration of the algorithm yields $q_2^*=\bar{P}_1-c^{'}-\lambda_2^*$, where $\bar{P}_1=D-2\bar{q}_1$, and $\lambda_2^*=\lambda^c+\eta_2$. By suitable substitutions, $\bar{q}_2=(1-\alpha)\bar{q}_1+\alpha q_2^*=\varepsilon(1-3\alpha)^2+q^c+\eta_1(3\alpha^2-\alpha)+\eta_2(-\alpha)$, with $(\lambda^c+\eta_2)(K-q_2^*)=0$. By mathematical induction it is shown that at the iteration T, $\bar{q}_T=\varepsilon(1-z)^T+q^c-\alpha\sum_{j=1}^T(1-z)^{T-j}\eta_j$, where $z=3\alpha$, $(\lambda^c+\eta_j)(K-q_j^*)=0$ for all j , and q_j^* is the solution of (1) at iteration j. Next we show that the third term in $\bar{q}_T$ approaches to zero as T increases.

From the complementarity conditions we can solve for η_j for all j …For the first iteration, $0=(\lambda^c+\eta_1)(K-q_1^*)=(\lambda^c+\eta_1)(K-(\bar{P}_0-c^{'}-\lambda_1^*))=(\lambda^c+\eta_1)(K+2\varepsilon+\eta_1-q^c)=\eta_1(K+2\varepsilon+\eta_1-q^c+\lambda^c)+2\lambda^c\varepsilon=\eta_1(K-q_1^*+\lambda^c)+2\lambda^c\varepsilon \Rightarrow \eta_1=\frac{-2\lambda^c\varepsilon}{K-q_1^*+\lambda^c}$. Note that $|\eta_1|\le 2|\varepsilon|$. By using similar arguments as used above, for the second iteration we obtain,

$$|\eta_2|=\left|\frac{-\lambda^c[2\varepsilon(1-3\alpha)+2\eta_1(-\alpha)]}{K-q_1^*+\lambda^c}\right| \le 2|\varepsilon|(1-3\alpha)+2\alpha|\eta_1| \le 2|\varepsilon|(1-\alpha).$$

Again by mathematical induction it is easily shown that, $|\eta_j|\le 2|\varepsilon|(1-\alpha)^{j-1}$, $j=1,2,\ldots,T$.

By triangular inequality, $\left|\alpha\sum_{j=1}^T(1-z)^{T-j}\eta_j\right| \le \alpha\sum_{j=1}^T(1-z)^{T-j}|\eta_j| \le 2\alpha|\varepsilon|\sum_{j=1}^T(1-3\alpha)^{T-j}(1-\alpha)^{j-1} \le 2\alpha|\varepsilon|\sum_{j=1}^T(1-\alpha)^{T-1}=2\alpha|\varepsilon|T(1-\alpha)^{T-1}$.

Here $2\alpha\,|\varepsilon|\,T(1-\alpha)^{T-1} \to 0$ as T increases, since $\alpha < 1$ and ε is fixed. Thus, $\bar{q}_T = \varepsilon(1-z)^T + q^c - \alpha\sum_{j=1}^{T}(1-z)^{T-j}\eta_j \to q^c$. □

Next we generalize the above findings to two-period oligopolies that make investment in production capacity under demand uncertainty.

3 Two-period oligopoly games

3.1 A Cournot–Nash game (CN) under uncertainty

3.1.1 Timing

Capital investments from several technologies are here-and-now decisions made under demand uncertainty and will be available for use in period 1.
Production and investment period: $\tau = 0$.
Production period: $\tau = 1$.

3.1.2 Description of the CN game

Consider a class of dynamic oligopolistic games in which players make capital investment and production decisions under uncertainty about future demand for a homogeneous product. In this game, we represent the demand uncertainty by a finite number of scenarios on a tree. Each scenario from root to terminal node represents a possible realization of a random process. It is assumed that in the beginning of the game all players share a common characterization of the random process. Moreover, this is a complete information game such that each player knows its own possible strategies, production and investment cost functions, initial capacities, and its rivals' information. We employ *S-adapted open-loop Nash equilibrium* solution paradigm. Every firm is risk neutral and maximizes its expected payoff. Given the initial capacity levels, in period 0 each firm makes its initial production decisions to maximize current profit, and chooses its investment in production capacity. These investments will be available in use for production in period 1. However, demand scenarios for period 1 are stochastic and investment decisions must be made in period 0. After uncertainty is resolved in period 1, players make their production decisions. Any investment opportunities in period 1 will not be considered since the game ends at that stage.

Assume that each player accurately predicts the market with the discrete random walk process illustrated in Fig. 1, where $D > 0, \delta > 0$ and δ is the random outcome of the uncertainty. Also let the "up-state" probability be U and the "down-state" probability be $1 - U$.

Let $K_i^\tau = (K_{i,1}^\tau, K_{i,2}^\tau, \ldots, K_{i,m}^\tau)$, where $K_{i,k}^\tau$, $k = 1, 2, \ldots, m$, denotes the available capacity at time τ from technology k for firm i. Also let $c_i(q_i)$ be the total cost as a function of the vector of outputs for all technologies, and $F_i(I_i)$ be the total cost of investment for player i, where cost functions are differentiable.

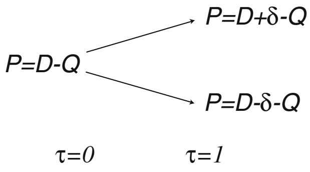

Fig. 1 Two-period demand scenarios

Then a player's maximization problem becomes,

$$\begin{aligned}
&\max \quad \Pi_i(q_{i,s}^{\tau}, I_i^{\tau}, K_{i,s}^{\tau}) \\
&\text{s.t.} \quad q_{i,s,k}^{\tau} - K_{i,s,k}^{\tau} \leq 0, \quad \tau = 0,1, \ \forall i,s,k \qquad (3.2)\\
&Q_s^{\tau} - \sum_{i,k} q_{i,s,k}^{\tau} = 0, \quad \tau = 0,1, \ \forall i,s,k \qquad (3.3)\\
&q_{i,s,k}^{\tau} \geq 0, \ K_{i,s,k}^{\tau} \geq 0, \quad \tau = 0,1, \ \forall i,s,k \qquad (3.4)\\
&I_{i,s,k}^{\tau} \geq 0, \quad \tau = 0, \ \forall i,s,k \qquad (3.5)\\
&I_{i,k}^{\tau} - E(I_{i,s,k}^{\tau}) = 0, \quad \tau = 0, \ \forall i,s,k \qquad (3.6)\\
&K_{i,s,k}^{\tau+1} - K_{i,s,k}^{\tau} - I_{i,s,k}^{\tau} = 0, \quad \tau = 0, \ \forall i,s,k \qquad (3.7)
\end{aligned}$$

where $s \in \{u,d\}, u$ and d denote for "up-state" and "down-state" scenarios, respectively. A player's expected profit function is:

$$\begin{aligned}
\Pi_i(\cdot) = {} & P_0 \sum_k q_{i,k}^0 - \sum_k c_{i,k}(q_{i,k}^0) - \sum_k F_{i,k}(I_{i,k}) \\
& + U\left[P_u^1 \sum_k q_{i,u,k}^1 - \sum_k c_{i,k}(q_{i,u,k}^1)\right] \\
& + (1-U)\left[P_d^1 \sum_k q_{i,d,k}^1 - \sum_k c_{i,k}(q_{i,d,k}^1)\right] \qquad (3.1)
\end{aligned}$$

where P_0 is the initial price, and P_u^1 and P_d^1 are prices in period 1 for the up and down states, respectively.

The objective function above maximizes the initial period profit plus the expected profit of the future. Because of (3.3), equilibrium conditions are imposed for each time period and scenario. The constraint (3.6) enforces the non-anticipativity (non-clairvoyance) condition, which implies that investment decisions must be implemented before the outcome of the random variable is observed. In the Eq. (3.6), E refers to "expectation operator". We note that this game setting is also studied in Genc et al. (2005).

3.2 Finitely repeated feedback (FB) game under uncertainty

3.2.1 Timing

Let t_c and $t_c^{'}$ denote communication periods. Let τ denote a production period. Again to avoid dealing with negative numbers let $t = |t_c|$ such that $t = 1, 2, \ldots, T$, where $T < \infty$, and t denotes iteration period.
Communication period: $t_c < 0$.
Production and investment period: $\tau = 0$.
Communication period: $0 < t_c^{'} |_{\tau=0} < 1$.
Production period: $\tau = 1$.

3.2.2 Description of the FB game under uncertainty

The description of the two-period FB game under demand uncertainty is in the same vein as the one-period FB game. The only difference is that capital investment takes place in the initial period. In period 0 each player chooses the capital investment quantity to meet the next period demand, which is uncertain. However, this chosen investment quantity is probably sub-optimal. In period 1 uncertainty is resolved and each player chooses production quantities similar to the process described in period 0, subject to total available capacity which is the summation of initial capacity plus the investment level chosen in period 0. Each player solves his sub-problem for each level of investment which results in artificial equilibrium quantities. To obtain globally optimal production quantities, each player iterates investment levels until the marginal cost of investment covers the summation of the shadow prices (opportunity cost) of production constraints. We assume whichever scenario occurs in period 1, players will stick to the strategies (production and investment decisions) that they have chosen in period 0. That is, planning and decisions for the future periods are made only during the initial period (period 0). That assumption corresponds to the open-loop Nash equilibrium concept used in the CN game under uncertainty.

To algorithmically solve the two-period CN game under uncertainty, we revise and add a few steps to the finitely repeated feedback game introduced in Sect. 2.1. Before period 0 each player follows Step 0 through Step 3 to obtain the initial-period artificial equilibrium points. They are irrelevant to period 0 investment and period 1 production decisions. At the same time each player i also chooses an investment level that he considers to make. For that given level of investment, for each scenario, each player follows procedures similar to those described in Step 0 through Step 3 to choose the quantities, but in Step 1 he solves

$$\begin{aligned} \max_{q_{i,t,s}} \quad & h[\bar{P}_{t-1,s} q_{i,t,s} - c(q_{i,t,s}) - \frac{1}{\gamma} q_{i,t,s}^2] \\ \text{s.t.} \quad & 0 \le q_{i,t,s} \le K_i + I_{i,0} \end{aligned}$$

where $h = \{U, 1-U\}$ denote scenario probabilities. Note that in the one-period game there is no scenario, hence $h = 1$.

We add the following step to have stopping criteria for the capital investment planning/iteration process.

Step 4. (Adding capacity) Iteration of investment levels continues until $\frac{F_{i.m}(I_{i,m})-F_{i.m'}(I_{i,m'})}{I_{i,m}-I_{i,m'}} = \sum_s \lambda_{i,m,s}$ is satisfied, where $I_{i,m} = I_{i,m'} + \mu$, μ is a small number, m, m'stand for iterations on investment I, and $F(\cdot)$ denotes the investment cost. The λ represents the shadow price of the production constraint. Step 4 means that capital investment will increase incrementally until the cost of doing so does not exceed the total benefit from the increase in production. We note that with different cost and demand structures it is possible to see cycles of excess demand and excess supply when the firms choose different capacity levels than they would choose in the equivalent CN game before convergence.

The following proposition shows that under demand uncertainty the CN game outcomes are algorithmically approximated by the FB game.

Proposition 4 *For the two-stage production-investment symmetric oligopolies under uncertainty, the FB game artificial equilibrium outcomes will converge to the CN equilibrium outcomes.*

Proof This proof is for the case of a duopoly in which each firm has a single technology. The proof for the case of many players and multiple technologies is similar. When we form the Lagrange function for the problem described in the proposition, we will obtain the following first order necessary conditions.

At initial time period $D - 3q^c - c' - \lambda^c = 0$ holds with the complementarity condition $(K - q^c)\lambda^c = 0$, where the couple (q^c, λ^c) is the CN equilibrium quantity and the Lagrange multiplier of the inequality constraint. Hereafter, implementation of the algorithm is similar to the one in the proof of Proposition 3. At this period, a player chooses a capital investment level I_0, and let it be $I_0 = \sigma + I_c$, where σ is a real number.

At time 1 for both of the states $h[D - 3q_s^c - c'] - \lambda_s^c = 0$ holds with $(K - q_s^c)\lambda_s^c = 0$, where $s \in \{u, d\}$, and a state probability $h \in \{U, 1-U\}$. A player selects his production quantities initially as $\bar{q}_{0,u} = \varepsilon + q_u^c$ and $\bar{q}_{0,d} = \beta + q_d^c$, where $\varepsilon, \beta \in \mathbb{R}$ and $\bar{q}_{0,s} \geq 0$. Prices become $\bar{P}_{0,u} = D + \delta - 2(\varepsilon + q_u^c)$ and $\bar{P}_{0,d} = D - \delta - 2(\beta + q_d^c)$. When players solve sub-problems for the second step of the algorithm, each one obtains $U[\bar{P}_{0,u} - c_u' - q_{1,u}^*] - \lambda_{1,u}^* = 0$ and $(1 - U)[\bar{P}_{0,d} - c_d' - q_{1,d}^*] - \lambda_{1,d}^* = 0$ with $\lambda_{1,u}^*(K + I_0 - q_{1,u}^*) = 0$ and $\lambda_{1,d}^*(K + I_0 - q_{1,d}^*) = 0$. Let $\lambda_{1,u}^* = \lambda_u^c + \eta_1$ and $\lambda_{1,d}^* = \lambda_d^c + \zeta_1$, where $\eta_1, \zeta_1 \in \mathbb{R}$ such that $\lambda_{t,s}^* \geq 0$, $t = 1, 2, \ldots, T$. From here the process is similar to the one used before. From the first iteration, one obtains $\bar{q}_{1,u} = \varepsilon(1 - 3\alpha) + q_u^c + \eta_1(-\alpha/U)$ and $\bar{q}_{1,d} = \beta(1 - 3\alpha) + q_d^c + \zeta_1(-\alpha/(1 - U))$. By the second iteration, $\bar{q}_{2,u} = \varepsilon(1-3\alpha)^2 + q_u^c + \eta_1((3\alpha^2 - \alpha)/U) + \eta_2(-\alpha/U)$ and $\bar{q}_{2,d} =$

Table 1 Cost coefficients for each player and technology

	Technology 1		Technology 2	
	Prod. Coeff.	Inv. Coeff.	Prod. Coeff.	Inv. Coeff.
Player 1	(1,1)	(1,0)	(1,1)	(1,1)
Player 2	(4,2)	(2,0)	(2,1)	(1,2)
Player 3	(3,4)	(0,4)	(3,1)	(1,3)
Player 4	(4,4)	(4,0)	(3,1)	(1,4)

$\beta(1-3\alpha)^2 + q_d^c + \zeta_1((3\alpha^2-\alpha)/(1-U)) + \zeta_2(-\alpha/(1-U))$. Similar to the proof of Proposition 3, one can show that at iterations T and T', $\bar{q}_{T,u} = \varepsilon(1-z)^T + q_u^c - (\alpha/U)\sum_{j=1}^{T}(1-z)^{T-j}\eta_j$ and $\bar{q}_{T',d} = \beta(1-z)^{T'} + q_d^c - (\alpha/(1-U))\sum_{j=1}^{T'}(1-z)^{T'-j}\zeta_j$, where $z = 3\alpha$, with the complementarity conditions $(\lambda_u^c + \eta_T)(K + I_0 - q_{T,u}^*) = 0$ and $(\lambda_d^c + \zeta_{T'})(K + I_0 - q_{T',d}^*) = 0$. Let $o = \max\{T, T'\}$. Again, by using the proof of Proposition 3, it is shown that $|\eta_j| \leq 2|\varepsilon|(1-\alpha)^{j-1}$, $|\zeta_j| \leq 2|\beta|(1-\alpha)^{j-1}$ for $j = 1, 2, \ldots, o$, and $\left|(\alpha/U)\sum_{j=1}^{T}(1-z)^{T-j}\eta_j\right| \leq 2(\alpha/U)|\varepsilon| T(1-\alpha)^{T-1} \to 0$, $\left|(\alpha/(1-U))\sum_{j=1}^{T'}(1-z)^{T'-j}\zeta_j\right| \leq 2(\alpha/(1-U))|\beta| T'(1-\alpha)^{T'-1} \to 0$ as o increases. Thus, $\bar{q}_{o,u} \to q_u^c$ and $\bar{q}_{o,d} \to q_d^c$. That is, for each choice of I, $(q_s^c, \lambda_s^c)_s$ is obtained. Let I_m, $F_m(I_m)$ and $\lambda_{m,s}^c$ be iteration m investment level, its cost and the shadow prices of inequality constraints. Note that, here, the first iteration is because of the investment choice, and the second type of iteration is due to choosing optimal production quantities for each level of investment choice. Here for every investment iteration m, $\lambda_{T,s} \to \lambda_s^c$ for large T, since $\bar{q}_{T,s} \to q_s^c$. The Step 4 suggests that once $\frac{F_m(I_m) - F_{m'}(I_{m'})}{I_m - I_{m'}} = \sum_s \lambda_{m,s}$ holds for some $m > m'$ then I_m is the optimal investment level. But this is true since $F'_m(I_m) \approx \frac{F_m(I_m) - F_{m'}(I_{m'})}{I_m - I_{m'}}$ and the equilibrium first order condition yields that $F'(I) = \lambda_u^c + \lambda_d^c \to \lambda_{m,u} + \lambda_{m,d}$ for $I > 0$. Hence the result follows. □

Next, we present a simple example to illustrate how the FB game equilibrium outcomes converge to the CN game equilibrium outcomes.

Example 1 Consider a four-player market in which each player has two available technologies for production of a homogeneous good. Assume affine marginal production and investment cost functions. For each player $i = 1, 2, 3, 4$ and for each technology $k = 1, 2$, the production marginal cost function follows $c'_{i,k} = a_{i,k} + b_{i,k}q_{i,k}$, the investment marginal cost function follows $F'_{i,k} = g_{i,k} + h_{i,k}I_{i,k}$, where the cost parameters g, h, a, b are provided in Table 1.

Let each player have 7 units of initial production capacity from each technology. Let the demand scenarios in Fig. 1 be $D = 90$, $\delta = 10$, and $U = 0.5$.

Implementation of the FB game under uncertainty is depicted below. For the sake of brevity we only report player 1's production quantities from technology 1 for periods 0 and 1, and all players' expected profit levels versus iterations

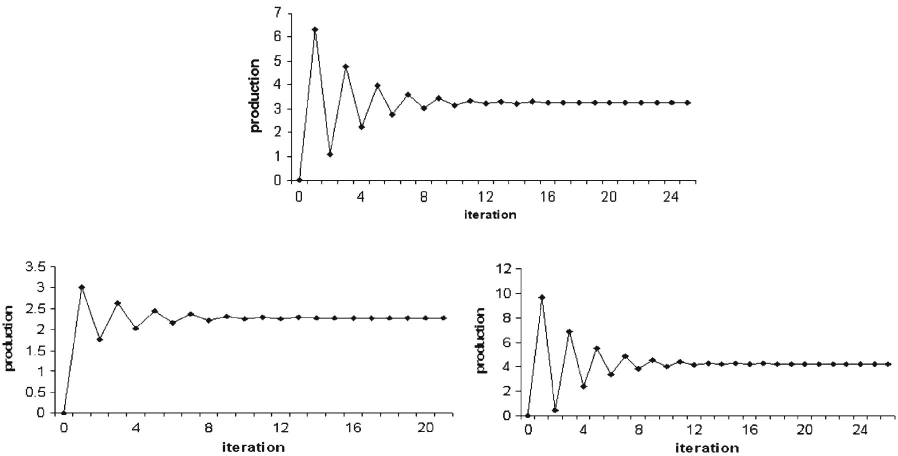

Fig. 2 Convergence of Player 1's production quantities from tech 1 for time=0 and time=1 down and up states, respectively

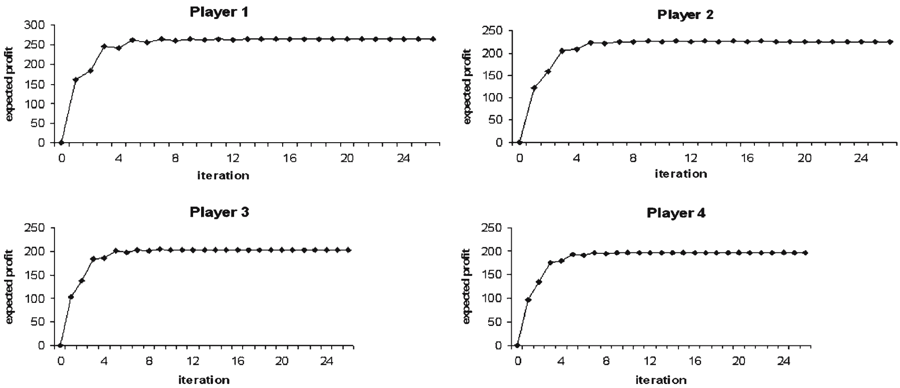

Fig. 3 Convergence of players' total expected profits

(the number of information exchanges with the auctioneer). In Fig. 2 after several oscillations the quantities converge to the CN equilibrium quantity levels. In Fig. 3 we plot expected profits at the optimal investment levels. As can be observed the FB game artificial equilibrium outcomes lead to concave-looking expected profit function. Indeed this should be expected since the cost functions are quadratic and demand is linear.

After several information (i.e., price and quantity) exchanges between players and the auctioneer, players produce and invest actual amounts. The number of iterations varies for each state and time period. However, in this example, the algorithm converges relatively quickly after several iterations. When we make several runs with different parameterizations we observe that initialization of the quantities in Step 0 of the FB game plays little role in determining whether the CN game equilibrium outcomes are reached. Nevertheless, it may increase the number of total iterations. This may stem from the fact that the short-term memory process (i.e., convex combinations of the quantities) picks up the local optimum in each iteration, hence the initialization levels are not very relevant for reaching the CN equilibrium quantities.

4 Concluding remarks

In this paper we show market outcome equivalence of two different dynamic games under demand uncertainty with different information structures and market rules. The FB game setting may be observed in a market where a new product is introduced and exchanged in an auction format such that firms do not have full information about their opponents and are unable to predict the market demand accurately. If the good is exchanged via an auction mechanism under uncertainty, the FB game setting will predict optimal market outcomes. We have shown that information exchanges between the auctioneer and the players lead to Cournot–Nash equilibrium outcomes, if the firms play according to the FB game-setting. Alternatively, the FB game structure may be considered as an algorithm that solves a large-scale CN game by decomposing the individual maximization problem into sub-problems. However, there are some limitations to the FB game solution method. For example, the algorithm may not converge to equilibrium or it may be inefficient from the computational-time point of view if the functional form of demand is complicated. In future research we will examine a generalization of the algorithm for different market structures.

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