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A reaction-diffusion system of λ–ω type Part I: Mathematical analysis

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We study two coupled reaction-diffusion equations of the λ–ω type [11] in $d \leqslant 3$ space dimensions, on a convex bounded domain with a C^2 boundary. The equations are close to a supercritical Hopf bifurcation in the reaction kinetics and are model equations for oscillatory reaction-diffusion equations. Global existence, uniqueness and continuous dependence on initial data of strong and weak solutions are proved using the classical Faedo-Galerkin method of Lions [15] and compactness arguments. We also present a complete case study for the application of this method to systems of nonlinear reaction-diffusion equations.

1 Introduction

In this paper, we consider a reaction-diffusion system of the λ–ω type [11], with the following general form: find $\{u(\boldsymbol{x},t), v(\boldsymbol{x},t)\}$ such that

$$\frac{\partial u}{\partial t} = \Delta u + \lambda(r)\, u - \omega(r)\, v \qquad \text{in} \quad \Omega_T, \tag{1.1 a}$$

$$\frac{\partial v}{\partial t} = \Delta v + \omega(r)\, u + \lambda(r)\, v \qquad \text{in} \quad \Omega_T, \tag{1.1 b}$$

where $\Omega_T := \Omega \times (0,T)$, $T > 0$ and the 'amplitude' is given by

$$r := \sqrt{u^2 + v^2}, \tag{1.1 c}$$

with initial and boundary conditions

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \qquad v(\boldsymbol{x},0) = v_0(\boldsymbol{x}), \tag{1.1 d}$$

$$\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{\partial v}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \Sigma := \partial\Omega \times (0,T), \tag{1.1 e}$$

where $\boldsymbol{\nu}$ denotes the outward normal to $\partial\Omega$. Throughout Δ denotes $\sum_{i=1}^{d} \partial^2/\partial x_i^2$ and $\Omega \subset \mathbb{R}^d$ ($d \leqslant 3$) is an open, bounded, convex domain, with a boundary of class C^2. The Lebesgue measure is denoted $dx \equiv dx_1\, dx_2 \ldots dx_d$. The specific class of λ and ω functions

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we study are

$$\lambda(r) := \lambda_0 - \lambda_1 r^\rho, \qquad \omega(r) := \omega_0 + \omega_1 r^\rho, \tag{1.1f}$$

where λ_0, λ_1, $\rho > 0$ and ω_0, ω_1 are non-zero numbers; all parameters are assumed real and finite. We consider a more general class of nonlinear functions than was originally proposed in Kopell & Howard [11], with an arbitrary power of the amplitude instead of a quadratic power (cf. Sherratt [20, 21]). From an applications point of view (discussed below), the most important case is $\rho = 2$. For the purposes of later calculation, we express the λ–ω system in the following vector form:

$$\boldsymbol{u}_t = \Delta \boldsymbol{u} + B\boldsymbol{u} + |\boldsymbol{u}|^\rho A\boldsymbol{u}, \quad \text{in} \quad \Omega_T, \tag{1.2a}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{v}} = \boldsymbol{0} \quad \text{on} \quad \Sigma, \tag{1.2b}$$

$$\text{where} \quad B = \begin{bmatrix} \lambda_0 & -\omega_0 \\ \omega_0 & \lambda_0 \end{bmatrix}, \; A = \begin{bmatrix} -\lambda_1 & -\omega_1 \\ \omega_1 & -\lambda_1 \end{bmatrix}, \; \boldsymbol{u} = \begin{bmatrix} u \\ v \end{bmatrix}. \tag{1.2c}$$

Reaction-diffusion equations of the λ–ω type were first studied 30 years ago by Kopell & Howard [11], who were motivated by an attempt to describe the formation of patterns in the Belousov–Zhabotinskii reaction. The system is important and interesting for a number of reasons. First, systems of the λ–ω type display features common to many real biological and chemical models, although they are independent of any specific physical problem. Furthermore, the system is a model for general reaction-diffusion equations with a limit-cycle in the reaction kinetics ('oscillatory' reaction diffusion equations). From normal form theory and the Hopf Bifurcation Theorem [28], any system of two ODEs near a supercritical Hopf bifurcation will have the general λ–ω form (1.1 *a*)–(1.1 *b*), with the λ and ω functions defined as in (1.1*f*) with $\rho = 2$. This is relevant to general reaction-diffusion systems provided the diffusion coefficients are equal [11]. For a study of reaction-diffusion equations close to a subcritical Hopf bifurcation, see Ermentrout *et al.* [7].

For a review of the λ–ω system and early work, see Murray [16]. Recent interest in the λ–ω system is due to a series of papers by Sherratt [20]–[24] and Kay & Sherratt [10], who used a combination of analytical and numerical methods to investigate the dynamics of solutions in one dimension with, for example, locally exponentially decaying initial conditions.

In spite of the importance of this system, there are few existence results in the literature, and these concern specific (ansatz) solutions. Kopell & Howard [11] proved that the λ–ω system on the real line has a simple one-parameter family of 2π-periodic (in space and time) travelling-wave solutions, called 'periodic plane waves'. In two space dimensions the periodic plane waves correspond to spiral waves or concentric ring waves ('target patterns') (see Murray [16, pp. 343–356], and references therein). Existence proofs for these special types of solutions can be found in Cohen *et al.* [3], Greenberg [8] and Kopell & Howard [12]. See also Romero *et al.* [19], where nonclassical symmetries of a one-dimensional system of λ–ω type were studied, leading to several classes of exact solution.

We comment on the similarity of the λ–ω system to the Complex Ginzburg–Landau (CGL) equation (e.g. see Temam [27, p. 226], and references therein). The λ–ω system written in complex form with $\rho = 2$ is

$$c_t = \Delta c + (\lambda_0 + i\,\omega_0)c + (-\lambda_1 + i\,\omega_1)|c|^2 c,$$

where $c := u + i\,v$ and $r := |c| \equiv \sqrt{u^2 + v^2}$. If we apply the rotation $c \to c\exp(i\,\omega_0\,t)$ (effectively removing ω_0), followed by the re-scaling of dependent and independent variables $t \to (1/\lambda_0)t$, $x_i \to (1/\lambda_0^{1/2})x_i$, $c \to (\lambda_0/\lambda_1)^{1/2}c$, then we obtain the CGL equation

$$c_t = \Delta c + (1 + i\,\alpha)c - (1 + i\,\beta)|c|^2 c, \tag{1.3}$$

where $\alpha = 0$ and $\beta = -\omega_1/\lambda_1$. In general, for the CGL equation α is non-zero and the Laplacian has a complex coefficient, so if we split this equation into real and imaginary parts then the diffusion matrix is antisymmetric (in the λ–ω case, the off-diagonal terms in this matrix are zero). Thus our λ–ω model with $\rho = 2$ is a special case of the CGL equation. In the CGL equation, it is necessary that the α term is non-zero for the existence of unstable spatially homogeneous oscillations, a feature not possible in the λ–ω system [13, pp. 21, 140]. There is extensive literature on the regularity of the CGL equations (and a generalised CGL equation), and two key papers relevant to bounded domains are those by Doering *et al.* [6] and Levermore & Oliver [14]. For a study of subcritical, generalised one-dimensional CGL equations that focuses on 'coherent structure solutions', see van Saarloos & Hohenberg [27].

Of particular relevance here is a theorem of Temam [26, Theorem 5.1, p. 228] for a CGL equation in a form covering (1.3) with $\alpha = 0$, and therefore also applicable to the λ–ω system with $\rho = 2$. From this theorem we have the following results, which are consistent with the results obtained in this paper. Assume that $d = 1$ or 2, $\rho = 2$ and c, c_0 are complex-valued functions. Given initial data $c_0 \in L^2(\Omega)$, then there exists a unique weak solution of (1.1 *a*)–(1.1 *f*), where

$$c \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \qquad \forall\ T < \infty,$$

and the solution depends continuously upon the initial data in $L^2(\Omega)$. Given initial data $c_0 \in H^1(\Omega)$, then a unique strong solution exists where

$$c \in C([0,T]; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \qquad \forall\ T < \infty.$$

In this paper. we use the Faedo–Galerkin method of Lions [15] and compactness arguments to rigorously establish the well-posedness of (1.1 *a*)–(1.1 *f*) in the weak and strong solution contexts, for $d \leqslant 3$ space dimensions. We present a complete case study for the application of the Faedo–Galerkin method to systems of reaction-diffusion equations, and collect together a number of results that are often used implicitly in the literature. Our work also lays the foundation for a numerical analysis of the system (Part II of this work).

The Faedo–Galerkin procedure is not the only method for proving existence and uniqueness results for the λ–ω system; for example, the Invariant Region method of Smoller [25] and Chueh *et al.* [2] is also applicable. A 'invariant region' region $\mathbb{D}$ is defined to be a closed subset of the phase space $\mathbb{R}^n$ with the property that if $u_0(x) \in \mathbb{D}$ for all $x \in \Omega$, then $u(x,t) \in \mathbb{D}$ for all $x \in \Omega$ and all $t > 0$ for which the solution exists.

To find an invariant region for the λ–ω system, first transform the ODE corresponding to the λ–ω system to polar form in phase space, via $r := \sqrt{u^2+v^2}$ and $\theta = \tan^{-1}(v/u)$, which yields $r_t = (\lambda_0 - \lambda_1 r^\rho)r$ and $\theta_t = \omega(r)$. We now argue as Smoller [26, Example 3, p. 210] and Chueh *et al.* [2, Example D]. Let B be any convex region containing the disk $u^2+v^2 = r_0^2$, where $r_0 := (\lambda_0/\lambda_1)^{1/\rho}$, then it is easy to see that the vector field corresponding to the reaction terms of the λ–ω system points into B, and thus by Corollary 14.8(b) in Smoller [25], we deduce that B is an invariant region for the λ–ω system. In particular, the region

$$B_\delta := \{(u,v)\,|\,u^2+v^2 \leqslant (\lambda_0/\lambda_1)^{2/\rho} + \delta\}, \quad \delta > 0,$$

is invariant. We claim that the ball B_0 is invariant for the λ–ω system, however we cannot apply Corollary 14.8(b) in Smoller [25] directly, since the vector field vanishes identically on ∂B_0. However, if $(u_0(x), v_0(x)) \in B_0$ for all x, then $(u_0(x), v_0(x)) \in B_\delta$ for all x and every $\delta > 0$. Thus, the solution $(u(x,t), v(x,t)) \in B_\delta$ for all x and every $\delta > 0$, which implies $(u(x,t), v(x,t)) \in B_0$ for all x and all $t > 0$ for which the solution exists. Thus, provided we have $L^\infty(\Omega)$ initial data and local existence of our solutions, then the solution asymptotically lies in B_0. That is, we have global existence in time of solutions to the full reaction diffusion system. In this paper, we are mainly interested in investigating solutions of the λ–ω system evolving from less regular initial data (e.g. data in $L^2(\Omega)$, or $H^1(\Omega)$). There is also the question of uniqueness to consider.

The paper is organised as follows. In § 2 the basic notation is laid out. § 3 deals with existence, uniqueness and regularity of weak solutions. Finally, in § 4 we consider existence, regularity and continuous dependence of strong solutions on the initial data. The overall approach we use via a 'composite' Galerkin approximation is a generalisation of that in Robinson [18] applied to a model reaction-diffusion equation with a polynomial nonlinearity.

2 Notation and preliminaries

In this study the dual space of a Banach space X is written X'. We use the usual Sobolev spaces $W^{m,p}(\Omega)$, $m \in \mathbb{N}$, $p \in [1,\infty]$, with associated norms and semi-norms given by

$$\|u\|_{m,p} := \left(\sum_{0\leqslant|\alpha|\leqslant m} \|D^\alpha u\|_{0,p}^p\right)^{1/p}, \qquad |u|_{m,p} := \left(\sum_{|\alpha|=m} \|D^\alpha u\|_{0,p}^p\right)^{1/p},$$

respectively, where D^α is the standard multi-index notation for the mixed (generalised) partial derivative of order $|\alpha|$ ($\alpha \in \mathbb{N} \cup \{0\}$). Another standard Banach space we use is $L^\infty(\Omega)$, with associated essential supremum norm

$$\|u\|_{0,\infty} \equiv \|u\|_{L^\infty(\Omega)} := \inf\{M : |u(x)| \leqslant M \quad \text{a.e. on } \Omega\}.$$

$W^{m,2}(\Omega)$ is denoted $H^m(\Omega)$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$, and additionally $W^{0,2}(\Omega) \equiv L^2(\Omega)$. The usual $L^2(\Omega)$ inner product over Ω with norm $\|\cdot\|_0$ is denoted by $(\cdot,\cdot)$, and $\langle\cdot,\cdot\rangle$ represents the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$. We denote the Euclidean norm by $|\cdot|$. To simplify notation, we define $H := L^2(\Omega)$ and $V := H^1(\Omega)$ so $V' = [H^1(\Omega)]'$ and the inner product on V is denoted $(\cdot,\cdot)_V$. We extend the Banach/Sobolev spaces to vector functions via $\boldsymbol{X} := \{X\}^2$, by requiring each component belong to the Banach/Sobolev space X.

We define function spaces depending on space and time [26, p. 45]. Let X be a Banach space and $p \in [1, \infty]$. Denote $L^p(0, T; X)$ to be the Banach space of all measurable functions $u : (0, T) \mapsto X$ such that $t \mapsto \|u(t)\|_X$ is in $L^p(0, T)$, with norm

$$\|u\|_{L^p(0,T;X)} := \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} \qquad \text{for} \quad 1 \leqslant p < \infty, \tag{2.1}$$

$$\|u\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{t \in (0,T)} \|u(t)\|_X \qquad \text{if} \quad p = \infty. \tag{2.2}$$

In addition, we write $L^p(\Omega_T) \equiv L^p(0, T; L^p(\Omega))$. In § 3 we shall also need to use $C_0^\infty(0, T; X)$, the space of infinitely differentiable functions $u : (0, T) \mapsto X$ with compact support in $(0, T)$ and density in $L^p(0, T; X)$.

We assume the standard Hilbert space setup [27, p. 55]

$$V \overset{c}{\hookrightarrow} H \equiv H' \hookrightarrow V', \tag{2.3}$$

where each space is dense in the previous one, '$\overset{c}{\hookrightarrow}$' denotes compact injection, '$\hookrightarrow$' denotes continuous injection, and $\equiv$ indicates the explicit identification of elements in H and H'.

For later calculations we need the following Sobolev interpolation result [1]: let $s \in [1, \infty]$, $m \geqslant 1$ and assume $v \in W^{m,s}(\Omega)$. Then there are constants C and $\mu = \frac{d}{m}\left(\frac{1}{s} - \frac{1}{r}\right)$ such that the inequality

$$\|v\|_{0,r} \leqslant C \|v\|_{0,s}^{1-\mu} \|v\|_{m,s}^{\mu} \quad \text{holds for} \quad r \in \begin{cases} [s, \infty] & \text{if } m - \frac{d}{s} > 0, \\ [s, \infty) & \text{if } m - \frac{d}{s} = 0, \\ [s, -\frac{d}{m-(d/s)}] & \text{if } m - \frac{d}{s} < 0. \end{cases} \tag{2.4}$$

To obtain the existence of weak solutions we make the following assumption on ρ:

(A 1) ρ is any finite, positive number if $d = 1, 2$ and $\rho \leqslant 4$ if $d = 3$.

To prove uniqueness of solutions and strong solution results we assume:

(A 2)
$$\rho \leqslant \begin{cases} 4 & \text{if } d = 1, \\ 2 & \text{if } d = 2, \\ 4/3 & \text{if } d = 3. \end{cases}$$

Note by the Sobolev Embedding Theorem that assumption (A 1) is sufficient for V to have continuous injection into $L^{\rho+2}(\Omega)$, while assumption (A 2) is sufficient for V to have continuous injection into $L^{2\rho+2}(\Omega)$.

Frequent use will be made of the following Grönwall lemma in differential form: let $E(s) \in W^{1,1}(0, t)$ and $Q(s)$, $P(s)$, $R(s) \in L^1(0, t)$, where all functions are non-negative. Then,

$$\frac{dE}{ds} + P(s) \leqslant R(s)E(s) + Q(s) \quad \text{a.e. in } [0, t] \tag{2.5}$$

implies

$$E(t) + \int_0^t P(\tau)\,d\tau \leqslant e^{\Lambda(t)}E(0) + e^{\Lambda(t)} \int_0^t Q(\tau)\,d\tau, \tag{2.6}$$

where $\Lambda(t) := \int_0^t R(\tau)\,d\tau$.

Throughout this paper, C will represent a generic bounded positive constant, possibly depending on T, Ω, u_0, and v_0, which may change from expression to expression.

3 Weak solutions

We introduce a weak formulation of the system (1.1 a)–(1.1 f):

(**P$_1$**) Find $u(\cdot,t), v(\cdot,t) \in V$ such that $u(\cdot,0) = u_0(\cdot)$, $v(\cdot,0) = v_0(\cdot)$ and for almost every $t \in (0,T)$

$$\left\langle \tfrac{\partial u}{\partial t}, \eta \right\rangle + (\nabla u, \nabla \eta) = (\lambda(r)\,u, \eta) - (\omega(r)\,v, \eta) \qquad \forall\, \eta \in V, \tag{3.1 a}$$

$$\left\langle \tfrac{\partial v}{\partial t}, \eta \right\rangle + (\nabla v, \nabla \eta) = (\omega(r)\,u, \eta) + (\lambda(r)\,v, \eta) \qquad \forall\, \eta \in V. \tag{3.1 b}$$

Note from the above notation (see § 2) that $\partial u(\cdot,t)/\partial t$, $\partial v(\cdot,t)/\partial t$ lie in V' for a.e. $t \in (0,T)$. The equivalent weak formulation corresponding to the system (1.2 a)–(1.2 c) written in vector form is:

(**P$_2$**) Find $\boldsymbol{u}(\cdot,t) \in \boldsymbol{V}$ such that $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot)$ and for almost every $t \in (0,T)$

$$\langle \boldsymbol{u}_t, \boldsymbol{\eta} \rangle + (\nabla \boldsymbol{u}, \nabla \boldsymbol{\eta}) = (B\boldsymbol{u}, \boldsymbol{\eta}) + (|\boldsymbol{u}|^\rho A\boldsymbol{u}, \boldsymbol{\eta}) \qquad \forall\, \boldsymbol{\eta} \in \boldsymbol{V}. \tag{3.2}$$

A finite-dimensional version of (P$_1$) will be used to prove existence of weak solutions, while a finite-dimensional version of (P$_2$) will be used to prove uniqueness and obtain strong solution results. It will be useful to note that for any vector $\boldsymbol{x} \in \mathbb{R}^2$ we have

$$(B\boldsymbol{x}) \cdot \boldsymbol{x} = \lambda_0 |\boldsymbol{x}|^2, \qquad (A\boldsymbol{x}) \cdot \boldsymbol{x} = -\lambda_1 |\boldsymbol{x}|^2. \tag{3.3}$$

Theorem 3.1 *Assume $\Omega \subset \mathbb{R}^d$ ($d \leqslant 3$) is an open, bounded, convex domain. Let* (A 1) *hold and assume that u_0, $v_0 \in H$, then the λ–ω system* (1.1 a)–(1.1 f) *possesses at least one weak solution $\{u, v\}$ satisfying*

$$u, v \in L^2(0,T;V) \cap L^{\rho+2}(\Omega_T) \cap C([0,T];H),$$

and the equations (1.1 a) *and* (1.1 b) *hold as equalities in $L^{\frac{\rho+2}{\rho+1}}(0,T;V')$. Furthermore, with assumption* (A 2) *the weak solution is unique and the map*

$$(u_0(\cdot), v_0(\cdot)) \mapsto (u(\cdot,t;u_0,v_0), v(\cdot,t;u_0,v_0)),$$

is continuous in H.

Proof We separate the proof into four parts showing: local existence of the Galerkin approximations; global existence of the Galerkin approximations; passage to the limit;

and uniqueness. For notational convenience in the proof we define the conjugate exponents

$$p := \rho + 2, \qquad q := \frac{\rho+2}{\rho+1} \in (1,2), \tag{3.4}$$

unless stated to the contrary.

Local existence of the approximations

With $L := -\Delta + I$, domain$(L) := \{\eta \in H^2(\Omega)\,|\,\partial\eta/\partial\boldsymbol{v} = 0 \text{ on } \partial\Omega\}$, L^{-1} is a symmetric, bounded, compact operator from H to H and thus the Hilbert–Schmidt Theorem applies (e.g., [17], p. 267). Consequently, from the spectral theory of such operators we introduce $\{z_i\}_{i=1}^{\infty}$ to be an orthogonal basis for V and an orthonormal basis for H, consisting of eigenfunctions for

$$-\Delta z_i + z_i = \mu_i z_i \ \text{ in } \Omega, \quad \frac{\partial z_i}{\partial \boldsymbol{v}} = 0 \quad \text{on } \partial\Omega, \tag{3.5}$$

where

$$1 = \mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_k \leqslant \cdots \quad \text{with } \lim_{i\to\infty}\mu_i = \infty$$

is an infinite set of corresponding eigenvalues. Note $(z_i, z_j)_V = \mu_i\delta_{ij}$ and $(z_i, z_j) = \delta_{ij}$. Now set $V^k := \text{span}\{z_i\}_{i=1}^{k} \subset V$ and seek a finite-dimensional weak form corresponding to (P_1):

Find $u^k(\cdot,t), v^k(\cdot,t) \in V^k$ such that $u^k(\cdot,0) = u_0^k(\cdot)$, $v^k(\cdot,0) = v_0^k(\cdot)$ and for almost every $t \in (0,T)$

$$\left(\frac{\partial u^k}{\partial t}, \chi^k\right) + \left(\nabla u^k, \nabla\chi^k\right) = \left(\lambda(r^k)\,u^k, \chi^k\right) - \left(\omega(r^k)\,v^k, \chi^k\right) \quad \forall\,\chi^k \in V^k, \tag{3.6 a}$$

$$\left(\frac{\partial v^k}{\partial t}, \chi^k\right) + \left(\nabla v^k, \nabla\chi^k\right) = \left(\omega(r^k)\,u^k, \chi^k\right) + \left(\lambda(r^k)\,v^k, \chi^k\right) \quad \forall\,\chi^k \in V^k, \tag{3.6 b}$$

$$\text{where} \quad r^k := \sqrt{(u^k)^2 + (v^k)^2}.$$

To derive later estimates note that the finite-dimensional weak form corresponding to (P_2) is:

Find $\boldsymbol{u}^k(\cdot,t) \in \boldsymbol{V}^k$ such that $\boldsymbol{u}^k(\cdot,0) = \boldsymbol{u}_0^k(\cdot)$ and for almost every $t \in (0,T)$

$$\left(\boldsymbol{u}_t^k, \boldsymbol{\chi}^k\right) + (\nabla\boldsymbol{u}^k, \nabla\boldsymbol{\chi}^k) = (B\boldsymbol{u}^k, \boldsymbol{\chi}^k) + (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k, \boldsymbol{\chi}^k) \quad \forall\boldsymbol{\chi}^k \in \boldsymbol{V}^k. \tag{3.7}$$

Now let

$$u^k(\cdot,t) = \sum_{i=1}^{k} a_{ik}(t) z_i(\cdot), \qquad v^k(\cdot,t) = \sum_{i=1}^{k} b_{ik}(t) z_i(\cdot) \tag{3.8}$$

and set $\chi^k = z_j$ for $j = 1,\ldots,k$, where the a_{ik} and b_{ik} are to be determined.

Let $P^k : H \mapsto V^k$ be the orthogonal projection from H onto V^k that satisfies $(P^k\eta, \chi^k) = (\eta, \chi^k)$ for all $\chi^k \in V^k$. For the work that follows we need the fact that the gradient operator satisfies the following symmetry condition:

Lemma 3.1 *For any $v \in V$ we have*

$$(\nabla(P^k v), \nabla\chi^k) = (\nabla v, \nabla\chi^k), \quad \forall\chi \in V^k.$$

Proof Let $\chi^k := \sum_{i=1}^k c_i z_i \in V^k$. From the weak form of the eigenvalue problem we have

$$(\nabla z_i, \nabla v) = (\mu_i - 1)(z_i, v), \quad \forall v \in V \tag{3.9}$$

and as v is also in H we have

$$P^k v = \sum_{j=1}^k (v, z_j) z_j \quad \text{so} \quad \nabla(P^k v) = \sum_{j=1}^k (v, z_j) \nabla z_j. \tag{3.10}$$

Thus from (3.9) and (3.10)

$$\begin{aligned}(\nabla(P^k v), \nabla z_i) &= \sum_{j=1}^k (v, z_j)(\nabla z_i, \nabla z_j) \\ &= \sum_{j=1}^k (v, z_j)(\mu_j - 1)\delta_{ij} \\ &= (v, z_i)(\mu_i - 1) \\ &= (\nabla z_i, \nabla v).\end{aligned}$$

Multiplying both sides by c_i and summing over $i = 1, \ldots, k$ gives the desired result. □

As a direct consequence of the above lemma, we have the following useful projection property:

$$\|\nabla(P^k v)\|_0 \leqslant \|\nabla v\|_0, \quad \forall v \in V. \tag{3.11}$$

For the initial approximations we take

$$u^k(\cdot, 0) := P^k u_0(\cdot), \qquad v^k(\cdot, 0) := P^k v_0(\cdot), \tag{3.12}$$

which satisfy

$$\{u_0^k, v_0^k\} \to \{u_0, v_0\} \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad k \to \infty. \tag{3.13}$$

The weak form (3.6 a)–(3.6 b) can be written as an equivalent initial value problem (IVP) for a system of $2k$ ordinary differential equations in a_{ik}, b_{ik}. We write this IVP in the following (equivalent) 'composite' form:

$$\frac{du^k}{dt} = \Delta u^k + P^k f(u^k, v^k), \quad u^k(\cdot, 0) = P^k u_0(\cdot), \tag{3.14 a}$$

$$\frac{dv^k}{dt} = \Delta v^k + P^k g(u^k, v^k), \quad v^k(\cdot, 0) = P^k v_0(\cdot), \tag{3.14 b}$$

where

$$f(u, v) := \lambda(r)\, u - \omega(r)\, v, \qquad g(u, v) := \omega(r)\, u + \lambda(r)\, v. \tag{3.15}$$

To show that f and g are locally Lipschitz functions with respect to u and v, consider the vector form of the λ–ω system in (1.2 a). Noting that $\boldsymbol{Bu}$ is linear, a calculation shows

$$\begin{aligned} ||\boldsymbol{u}_1|^{\rho} A\boldsymbol{u}_1 - |\boldsymbol{u}_2|^{\rho} A\boldsymbol{u}_2| &\leqslant \|A\|_2 \; ||\boldsymbol{u}_1|^{\rho}\boldsymbol{u}_1 - |\boldsymbol{u}_2|^{\rho}\boldsymbol{u}_2| \\ &\leqslant \|A\|_2 \; L(\boldsymbol{u}_1, \boldsymbol{u}_2) \; |\boldsymbol{u}_1 - \boldsymbol{u}_2|, \end{aligned} \tag{3.16}$$

where $\|A\|_2 = \sqrt{\lambda_1^2 + \omega_1^2}$ is the spectral norm of A and

$$L(\boldsymbol{u}_1, \boldsymbol{u}_2) = (\rho + 1) \max\{|\boldsymbol{u}_1|^{\rho}, |\boldsymbol{u}_2|^{\rho}\}.$$

From standard existence theory (Picard's Theorem) for systems of ordinary differential equations (since f and g are locally Lipschitz), the system has a unique solution $\{u^k, v^k\}$ on some finite time interval $(0, t_k)$, $t_k > 0$.

Global existence of the approximations

To prove global existence of the Galerkin approximations we derive an *a priori* estimate bounding u^k and v^k (independently of k) in various Banach spaces. From the uniform bounds we conclude $t_k = T$ (independent of k), implying global existence of the Galerkin approximations.

Estimate I: Choosing $\boldsymbol{\chi}^k = \boldsymbol{u}^k$ in (3.7) leads to

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} |\boldsymbol{u}^k|^2 \, dx + \int_{\Omega} |\nabla \boldsymbol{u}^k|^2 \, dx + \lambda_1 \int_{\Omega} |\boldsymbol{u}^k|^{\rho+2} \, dx = \lambda_0 \int_{\Omega} |\boldsymbol{u}^k|^2 \, dx, \tag{3.17}$$

and the application of the Grönwall lemma yields for a.e. $t \in (0, T)$

$$\|\boldsymbol{u}^k(t)\|_0^2 + 2\int_0^t \left(|\boldsymbol{u}^k(s)|_1^2 + \lambda_1 \|\boldsymbol{u}^k(s)\|_{0,\rho+2}^{\rho+2}\right) ds \leqslant \|\boldsymbol{u}^k(0)\|_0^2 \exp(2\lambda_0 t). \tag{3.18}$$

Recalling u_0^k, $v_0^k \in H$ we have

$$u^k, \; v^k \text{ are uniformly bounded in } L^{\infty}(0, T; H) \cap L^{\rho+2}(\Omega_T), \tag{3.19}$$

and noting the injection $L^{\infty} \hookrightarrow L^2$ and the semi-norm bound for V we have

$$u^k, \; v^k \text{ are uniformly bounded in } L^2(0, T; V). \tag{3.20}$$

Passage to the limit

Using classical compactness arguments (e.g. Dautray & Lions [4, Theorems 4, 5]), from the uniformly bounded sequences of functions $\{u^k\}_{k=1}^{\infty}$ and $\{v^k\}_{k=1}^{\infty}$ we extract convergent subsequences, still denoted $\{u^k\}$, $\{v^k\}$, such that

$$\{u^k, v^k\} \rightharpoonup \{u, v\} \quad \text{in} \quad L^p(\Omega_T) \cap L^2(0, T; V) \quad \text{as} \quad k \to \infty, \tag{3.21}$$

$$\{u^k, v^k\} \rightharpoonup^* \{u, v\} \quad \text{in} \quad L^{\infty}(0, T; H) \quad \text{as} \quad k \to \infty, \tag{3.22}$$

where '$\rightharpoonup$' and '$\rightharpoonup^*$' represent weak and weak-star convergence, respectively. We show passage to the limit of the terms in the first composite Galerkin approximation (3.14 *a*). The arguments will apply in a similar way to (3.14 *b*). Consider first the term $P^k f(u^k, v^k)$. It is easy to show

$$|f(u^k, v^k)| \leqslant C(|u^k| + |v^k| + |u^k|^{\rho+1} + |v^k|^{\rho+1}), \tag{3.23}$$

and therefore $f(u^k, v^k)$ is uniformly bounded in $L^q(\Omega_T)$, and so from weak compactness arguments there exists some $\chi \in L^q(\Omega_T)$ such that

$$f(u^k, v^k) \rightharpoonup \chi \quad \text{in } L^q(\Omega_T) \text{ as } k \to \infty. \tag{3.24}$$

We show that $P^k f$ also tends weakly to χ in $L^q(\Omega_T)$. Define $Q^k := I - P^k$, the projection orthogonal to P^k. Now recall from the proof of Lemma 3.1 that $(P^k v, \chi^k)_V = (v, \chi^k)_V$ for all $\chi^k \in V^k$, $v \in V$, which implies $\|P^k v - v\|_1 \leqslant \|\chi^k - v\|_1$ for all $\chi^k \in V^k$, $v \in V$. Thus as V^k is dense in V we have $P^k u \to u$ in V for $\forall u \in V$, i.e. $Q^k u \to 0$ in V as $k \to \infty$. From assumption (A 1) we have $V \hookrightarrow L^p(\Omega)$ and so $Q^k u \to 0$ in $L^p(\Omega)$, $\forall\, u \in L^p(\Omega)$. Consider an arbitrary $\phi \in L^p(\Omega_T)$, then using Hölder's inequality and the orthogonality of Q^k

$$\left| \int_0^T \left(P^k f(u^k, v^k) - \chi, \phi\right) dt \right| \leqslant \left| \int_0^T \left(f(u^k, v^k) - \chi, \phi\right) dt \right| + \left| \int_0^T \left(f(u^k, v^k), Q^k \phi\right) dt \right|$$
$$\leqslant \left| \int_0^T \left(f(u^k, v^k) - \chi, \phi\right) dt \right| + \int_0^T \|f(u^k, v^k)\|_{0,q} \|Q^k \phi\|_{0,p}\, dt \to 0 \quad \text{as } k \to \infty,$$

on noting the strong convergence of $Q^k \phi$ to 0 in $L^p(\Omega)$ and (3.24). Thus, we have

$$P^k f(u^k, v^k) \rightharpoonup \chi \quad \text{in } L^q(\Omega_T) \quad \text{as } k \to \infty. \tag{3.25}$$

Noting $\Delta u^k \in L^2(0, T; V')$ and $P^k f(u^k, v^k) \in L^q(\Omega_T)$ it follows from (3.14 *a*) that du^k/dt is uniformly bounded in $L^2(0, T; V') + L^q(\Omega_T)$ and from weak compactness arguments du^k/dt tends weakly to some $\dot{\eta}$ in this space. We now adapt an argument in Robinson [18, p. 203], to give $\dot{\eta} = du/dt$, i.e.

$$\frac{du^k}{dt} \rightharpoonup \frac{du}{dt} \quad \text{in } L^2(0, T; V') + L^q(\Omega_T) \quad \text{as } k \to \infty. \tag{3.26}$$

First, recall from (3.21) that $u^k \rightharpoonup u$ in the space $L^2(0, T; V) \cap L^p(\Omega_T)$, with dual space $L^2(0, T; V') + L^q(\Omega_T)$ (see Robinson [18, p. 224] or Temam [26, p. 93]). Furthermore, from the Sobolev Embedding Theorem and the fact that V is dense in H, we have the dense inclusion $V \hookrightarrow L^p(\Omega)$, thus $L^q(\Omega) \hookrightarrow V'$, and so $L^2(0, T; V') + L^q(\Omega_T) \subset L^q(0, T; V')$. Now consider an arbitrary $\phi(t) \in C_0^\infty(0, T; V) \subset L^p(0, T; V)$. Integrating by parts and using the weak convergence of u^k to u in $L^2(0, T; V') + L^q(\Omega_T)$ and hence $L^q(0, T; V')$ yields

$$\int_0^T \left(\frac{du^k}{dt}, \phi\right) dt = -\int_0^T \left(u^k, \frac{d\phi}{dt}\right) dt \to -\int_0^T \left(u, \frac{d\phi}{dt}\right) dt = \int_0^T \left(\frac{du}{dt}, \phi\right) dt,$$

after noting $d\phi/dt \in C_0^\infty(0,T;V)$. From the weak convergence of du^k/dt to $\dot\eta$ in $L^q(0,T;V')$, we also have

$$\int_0^T \left(\frac{du^k}{dt},\phi\right)dt \to \int_0^T (\dot\eta,\phi)\,dt \quad \text{as } k\to\infty,$$

and so by the uniqueness of weak limits we have $\dot\eta = du/dt$ as required.

Now as $u^k \rightharpoonup u$ in $L^2(0,T;V)$ we have (cf. Robinson [18, p. 204])

$$\Delta u^k \rightharpoonup \Delta u \quad \text{in } L^2(0,T;V') \subset L^q(0,T;V') \text{ as } k\to\infty. \tag{3.27}$$

Thus, we have the required passage to the limit of all terms in $L^q(0,T;V')$. To show $\chi \equiv f(u,v)$ in (3.24) we apply some classical theorems. From an application of the Lions–Aubin Theorem [15] with

$$W := \left\{\eta \mid \eta \in L^2(0,T;V), \quad \frac{d\eta}{dt} \in L^q(0,T;V')\right\},$$

we have $W \overset{c}{\hookrightarrow} L^2(\Omega_T)$ and as $u^k \in W$ we can extract a subsequence, still denoted u^k, such that $u^k \to u$ in $L^2(\Omega_T)$ (similarly for v^k), thus $u^k \to u$ ('pointwise') a.e. in Ω_T (similarly for v^k). As f is locally Lipschitz in Ω_T this implies by continuity that $f(u^k,v^k) \to f(u,v)$ ('pointwise') a.e. in Ω_T. The application of Lemma 1.3 of Lions [15] now gives

$$f(u^k,v^k) \rightharpoonup f(u,v) \quad \text{in } L^q(\Omega_T), \tag{3.28}$$

and due to the uniqueness of weak limits we deduce $\chi \equiv f(u,v)$, as required.

To obtain that $u \in C([0,T];H)$ we apply a modified version of another classical result (cf. Robinson [18, Exercise 8.2]), with $u \in L^2(0,T;V)\cap L^p(\Omega_T)$ and $du/dt \in L^2(0,T;V') + L^q(\Omega_T)$. The argument giving $u(\cdot,0) = u_0(\cdot)$ is also well known (see Robinson [18, p. 225] or Renardy & Rogers [17, p. 381]). This completes the existence part of the proof.

Uniqueness

To prove uniqueness of weak solutions we need the following lemma.

Lemma 3.2 *Assume ρ and ε are non-negative real numbers, ρ satisfies assumption* (A 1) *and C_1 is an arbitrary positive constant. Let $\eta \in L^{\rho+2}(\Omega)$ and $\psi \in V$ be functions defined on a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leqslant 3$), then there are constants $C_2(\varepsilon)$ and $\mu = d(\frac{1}{2} - \frac{1}{\rho+2})$, such that*

$$C_1 \int_\Omega |\eta|^\rho |\psi|^2\,dx \leqslant \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^{\frac{\rho}{1-\mu}}\right)\|\psi\|_0^2 + \frac{\mu}{\varepsilon}|\psi|_1^2, \quad \textit{where } 0<\mu<1. \tag{3.29}$$

Proof Observe using Hölder's inequality, followed by inequality (2.4) with $s = 2$, $m = 1$ and $r = \rho + 2$ that

$$C_1 \int_\Omega |\eta|^\rho \cdot |\psi|^2\,dx \leqslant C_1\|\eta\|_{0,\rho+2}^\rho \|\psi\|_{0,\rho+2}^2 \leqslant ab, \tag{3.30}$$

where $a := C\|\eta\|_{0,\rho+2}^{\rho}\|\psi\|_0^{2(1-\mu)}$, $b := \|\psi\|_1^{2\mu}$. An application of Young's inequality in the form

$$ab \leqslant \varepsilon^{m/n}\frac{a^m}{m} + \frac{1}{\varepsilon}\frac{b^n}{n}, \quad \frac{1}{m} + \frac{1}{n} = 1, \tag{3.31}$$

with $m := (1-\mu)^{-1}$, $n := \mu^{-1}$ ($\mu \neq 0, 1$) gives inequality (3.29). □

To prove uniqueness we assume there are two solutions $\boldsymbol{u} := (u_1, u_2)^T$, and $\boldsymbol{v} := (v_1, v_2)^T$ of the weak form (P_2), with initial conditions $\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0(\cdot)$ and $\boldsymbol{v}(\cdot, 0) = \boldsymbol{v}_0(\cdot)$ respectively. Setting $\boldsymbol{\eta} = \boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$, subtracting weak forms and using (3.3) leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{w}|^2\,dx + \int_\Omega |\nabla \boldsymbol{w}|^2\,dx = \lambda_0\int_\Omega |\boldsymbol{w}|^2\,dx + \int_\Omega \left(|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}\right)\cdot \boldsymbol{w}\,dx. \tag{3.32}$$

Using (3.16), followed by Lemma 3.2 with $\varepsilon = 2$, $\eta \in \{\boldsymbol{u}, \boldsymbol{v}\}$, $\psi = \boldsymbol{w}$, we bound the last term in (3.32) via

$$\begin{aligned}\int_\Omega \left(|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}\right)\cdot \boldsymbol{w}\,dx &\leqslant (\rho+1)\sqrt{\lambda_1^2 + \omega_1^2}\int_\Omega \left(|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho\right)|\boldsymbol{w}|^2\,dx,\\ &\leqslant C\left[1 + \left(\|\boldsymbol{u}\|_{0,\rho+2}^{\nu} + \|\boldsymbol{v}\|_{0,\rho+2}^{\nu}\right)\right]\|\boldsymbol{w}\|_0^2 + \mu|\boldsymbol{w}|_1^2,\end{aligned} \tag{3.33}$$

where $\nu := \rho/(1-\mu)$. Noting $\mu < 1$, from (3.32) and (3.33) we have after kickback of $\mu|\boldsymbol{w}|_1^2$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_0^2 \leqslant C\left[1 + \left(\|\boldsymbol{u}\|_{0,\rho+2}^{\nu} + \|\boldsymbol{v}\|_{0,\rho+2}^{\nu}\right)\right]\|\boldsymbol{w}\|_0^2,$$

and multiplying through by 2 and applying a Grönwall lemma yields for a.e. $t \in (0, T)$

$$\|\boldsymbol{w}(t)\|_0^2 \leqslant \|\boldsymbol{w}(0)\|_0^2 \cdot \exp\left(2Ct + 2C\int_0^t \left(\|\boldsymbol{u}(s)\|_{0,\rho+2}^{\nu} + \|\boldsymbol{v}(s)\|_{0,\rho+2}^{\nu}\right)ds\right). \tag{3.34}$$

To use the regularity of solutions in $L^{\rho+2}(\Omega_T)$ we apply Hölder's inequality in time to the right-hand side of (3.34), requiring $\nu \leqslant \rho + 2$, which holds due to assumption (A 2). Thus we have $\|\boldsymbol{u} - \boldsymbol{v}\|_0^2 \leqslant C\|\boldsymbol{u}(0) - \boldsymbol{v}(0)\|_0^2$. If $\boldsymbol{u}_0 = \boldsymbol{v}_0$ we deduce uniqueness and if $\boldsymbol{u}_0 \neq \boldsymbol{v}_0$ we have continuous dependence in H. □

Remark The difficulty in the uniqueness proof is due to the nonlinear term $|\boldsymbol{u}|^\rho A\boldsymbol{u}$. An alternative approach would be to initially split this term into $\omega_1|\boldsymbol{u}|^\rho(-v, u)^T - \lambda_1|\boldsymbol{u}|^\rho\boldsymbol{u}$. The latter term involves the function $|\boldsymbol{u}|^\rho\boldsymbol{u}$ which is a monotonic function for $\rho > 0$ ([5], Lemma 4.4) and is encountered in a variety of physical contexts [15, 26]. Thus in proving uniqueness we are left with the former term to control, leading to effectively the same expression as the right-hand side of (3.33).

4 Strong solutions

Theorem 4.1 *Assume $\Omega \subset \mathbb{R}^d$ ($d \leqslant 3$) is an open, bounded, convex domain with a boundary $\partial\Omega$ of class C^2. Let* (A 2) *hold and assume that u_0, $v_0 \in V$, then the λ–ω system* (1.1 *a*)–

(1.1f) possesses a unique, strong solution $\{u,v\}$ satisfying

$$u, v \in L^2(0,T;H^2(\Omega)) \cap C([0,T];V)$$

and the equations (1.1 a) and (1.1 b) hold as equalities in $L^2(\Omega_T)$. Furthermore, the map

$$(u_0(\cdot), v_0(\cdot)) \mapsto (u(\cdot,t;u_0,v_0), v(\cdot,t;u_0,v_0)),$$

is continuous in V.

Proof To obtain existence and uniqueness of strong solutions we require additional regularity, which we obtain from further *a priori* estimates.

Existence

To prove strong solution results we need the following lemma (cf. Lemma 3.2):

Lemma 4.1 *Assume ρ and ε are non-negative real numbers, ρ satisfies assumption (A 1) and C_1 is an arbitrary positive constant. Let $\eta \in L^{\rho+2}(\Omega)$ and $\psi \in H^2(\Omega)$ be functions defined on a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leqslant 3$), then there are constants $C_2(\varepsilon)$ and $\mu = d(\frac{1}{2} - \frac{1}{\rho+2})$, such that*

$$C_1 \int_\Omega |\eta|^\rho |\nabla\phi|^2\,dx \leqslant \left(\frac{\mu}{\varepsilon} + C_2(\varepsilon)\|\eta\|_{0,\rho+2}^{\frac{\rho}{1-\mu}}\right)|\phi|_1^2 + \frac{2\mu}{\varepsilon}|\phi|_2^2, \quad \text{where } 0 < \mu < 1. \tag{4.1}$$

Proof In Lemma 3.2 take $\psi = \partial\phi/\partial x_i$ and summing both sides over $i = 1,\ldots,d$ leads to the desired result. □

Estimate II: Note from (3.5) and (3.8) that $-\Delta u^k = \sum_{j=1}^k (\mu_j - 1)a_{jk}z_j$, $-\Delta v^k = \sum_{j=1}^k (\mu_j - 1)b_{jk}z_j$, thus for fixed k we have $-\Delta \boldsymbol{u}^k \in \boldsymbol{V}^k$. Choosing $\boldsymbol{\chi}^k = -\Delta \boldsymbol{u}^k$ in (3.7) and integrating by parts leads to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \boldsymbol{u}^k|^2\,dx + \int_\Omega |\Delta \boldsymbol{u}^k|^2\,dx = -\int_\Omega (B\boldsymbol{u}^k)\cdot\Delta\boldsymbol{u}^k\,dx - \int_\Omega (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k)\cdot\Delta\boldsymbol{u}^k\,dx. \tag{4.2}$$

We deal with the last two terms in this inequality separately. Integrating by parts and recalling (3.3) yields

$$-\int_\Omega (B\boldsymbol{u}^k)\cdot\Delta\boldsymbol{u}^k\,dx = \int_\Omega (B\nabla\boldsymbol{u}^k)\cdot\nabla\boldsymbol{u}^k\,dx = \lambda_0\int_\Omega |\nabla\boldsymbol{u}^k|^2\,dx. \tag{4.3}$$

To control the remaining term first note the identity

$$\nabla\left(|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k\right) = |\boldsymbol{u}^k|^\rho\left(A\nabla\boldsymbol{u}^k\right) + A\boldsymbol{u}^k\left(\rho|\boldsymbol{u}^k|^{\rho-2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}^k\right). \tag{4.4}$$

Then integrating by parts, use of (4.4) and (3.3) again yields

$$
\begin{aligned}
&-\int_\Omega (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k)\cdot\Delta\boldsymbol{u}^k dx\\
&=\int_\Omega |\boldsymbol{u}^k|^\rho (A\nabla\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k)\,dx+\rho\int_\Omega |\boldsymbol{u}^k|^{\rho-2}(\nabla\boldsymbol{u}^k\cdot\boldsymbol{u}^k)(A\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k)\,dx\\
&\leqslant -\lambda_1\int_\Omega |\boldsymbol{u}^k|^\rho|\nabla\boldsymbol{u}^k|^2\,dx-\lambda_1\rho\int_\Omega |\boldsymbol{u}^k|^{\rho-2}(\boldsymbol{u}^k\cdot\nabla\boldsymbol{u}^k)^2\,dx+\rho|\omega_1|\int_\Omega|\boldsymbol{u}^k|^\rho\cdot|\nabla\boldsymbol{u}^k|^2\,dx. \quad (4.5)
\end{aligned}
$$

We apply Lemma 4.1 to the last term in (4.5) to give:

$$
\rho|\omega_1|\int_\Omega |\boldsymbol{u}^k|^\rho|\nabla\boldsymbol{u}^k|^2\,dx\leqslant C\left(1+\|\boldsymbol{u}^k\|^\nu_{0,\rho+2}\right)|\boldsymbol{u}^k|_1^2+\frac{2\mu}{\varepsilon}|\boldsymbol{u}^k|_2^2, \quad (4.6)
$$

where $\nu=\rho/(1-\mu)$. We apply some well-known elliptic regularity results for bounded, convex, open domains with a boundary of class C^2. From the eigenvalue equation (3.5) and Grisvard [9, Theorem 3.2.1.3], we have for fixed (finite) k that $z_i\in H^2(\Omega)$ $(i=1,\dots,k)$, and hence $u^k(\cdot,t),v^k(\cdot,t)\in H^2(\Omega)$ for a.e. $t\in(0,T)$. Thus, by Grisvard [9, Theorem 3.1.3.3], we have $\|\boldsymbol{u}^k\|_2\leqslant\widehat{C}\|\Delta\boldsymbol{u}^k\|_0$ for some positive constant $\widehat{C}$ and a.e. $t\in(0,T)$. Choosing $\varepsilon=4\mu\widehat{C}$ leads to

$$
\rho|\omega_1|\int_\Omega |\boldsymbol{u}^k|^\rho|\nabla\boldsymbol{u}^k|^2\,dx\leqslant C\left(1+\|\boldsymbol{u}^k\|^\nu_{0,\rho+2}\right)|\boldsymbol{u}^k|_1^2+\frac{1}{2}\|\Delta\boldsymbol{u}^k\|_0^2. \quad (4.7)
$$

From (4.2), (4.3), (4.5), (4.7), a kickback of $\frac{1}{2}\|\Delta\boldsymbol{u}^k\|_0^2$, multiplying through by 2, and the Grönwall lemma we have for a.e. $t\in(0,T)$

$$
\begin{aligned}
|\boldsymbol{u}^k(t)|_1^2&+\int_0^t\Bigg(\|\Delta\boldsymbol{u}^k(s)\|_0^2+2\lambda_1\int_\Omega|\boldsymbol{u}^k(s)|^\rho|\nabla\boldsymbol{u}^k(s)|^2\,dx\\
&+2\lambda_1\rho\int_\Omega|\boldsymbol{u}^k(s)|^{\rho-2}(\boldsymbol{u}^k(s)\cdot\nabla\boldsymbol{u}^k(s))^2\,dx\Bigg)\,ds\\
&\leqslant|\boldsymbol{u}_0^k|_1^2\exp\left(2Ct+2C\int_0^t\|\boldsymbol{u}^k(s)\|^\nu_{0,\rho+2}\,ds\right). \quad (4.8)
\end{aligned}
$$

We apply Hölder's inequality in time on the right-hand side of (4.8) to use the fact that solutions lie in $L^{\rho+2}(\Omega_T)$. As in the uniqueness proof this requires $\nu\leqslant\rho+2$, which holds due to assumption (A 2). The boundedness of the term $|\boldsymbol{u}_0^k|_1^2\equiv|P^k\boldsymbol{u}_0|_1^2$ follows from the projection property (3.11) and the assumption that the initial data is in V. Then noting bound (3.19) we deduce

$$
u^k,\ v^k\quad\text{are uniformly bounded in } L^\infty(0,T;V). \quad (4.9)
$$

Furthermore, as $\boldsymbol{u}(\cdot,t),\Delta\boldsymbol{u}(\cdot,t)\in\boldsymbol{L}^2(\Omega)$ for a.e. $t\in(0,T)$, elliptic regularity theory (see Grisvard [9, Theorem 3.2.1.3, and Remark 3.2.1.4]) gives $\boldsymbol{u}(\cdot,t)\in\boldsymbol{H}^2(\Omega)$ for a.e. $t\in(0,T)$, thus

$$u^k,\ v^k \quad \text{are uniformly bounded in } L^2(0,T;H^2(\Omega)). \tag{4.10}$$

Remark From (4.5) if $\rho \leqslant \lambda_1/|\omega_1|$, then we can apply the Grönwall lemma to deduce the above uniform bounds without assumption (A 2) (but for the existence of solutions we still need assumption (A 1)). We make a further estimate on $d\boldsymbol{u}/dt$.

Estimate III: Set $\boldsymbol{\chi}^k = \boldsymbol{u}_t^k$ in (3.7), then direct calculation yields

$$\begin{aligned}
&\int_\Omega |\boldsymbol{u}_t^k|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \boldsymbol{u}^k|^2\,dx \\
&= \int_\Omega (B\boldsymbol{u}^k)\cdot \boldsymbol{u}_t^k\,dx + \int_\Omega (|\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k)\cdot \boldsymbol{u}_t^k\,dx \\
&= \frac{\lambda_0}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{u}^k|^2\,dx - \frac{\lambda_1}{2}\int_\Omega |\boldsymbol{u}^k|^\rho \frac{\partial |\boldsymbol{u}^k|^2}{\partial t}\,dx + \int_\Omega (\omega_0+\omega_1|\boldsymbol{u}^k|^\rho)\big(u^k v_t^k - v^k u_t^k\big)\,dx.
\end{aligned} \tag{4.11}$$

We apply a simple Young's inequality to the last term in (4.11) to give

$$\begin{aligned}
\int_\Omega (\omega_0+\omega_1|\boldsymbol{u}^k|^\rho)\big(u^k v_t^k - v^k u_t^k\big)\,dx &\equiv \int_\Omega (\omega_0+\omega_1|\boldsymbol{u}^k|^\rho)\,\boldsymbol{u}^k\cdot \begin{pmatrix} v_t^k \\ -u_t^k \end{pmatrix} dx \\
&\leqslant \int_\Omega (|\omega_0| + |\omega_1\|\boldsymbol{u}^k|^\rho)|\boldsymbol{u}^k|\big|\boldsymbol{u}_t^k\big|\;dx \\
&\leqslant \omega_0^2\|\boldsymbol{u}^k\|_0^2 + \omega_1^2\|\boldsymbol{u}^k\|_{0,2\rho+2}^{2\rho+2} + \tfrac{1}{2}\big\|\boldsymbol{u}_t^k\big\|_0^2.
\end{aligned} \tag{4.12}$$

Then after noting

$$|\boldsymbol{u}^k|^\rho \frac{\partial}{\partial t}|\boldsymbol{u}^k|^2 = \frac{2}{(\rho+2)}\frac{\partial}{\partial t}|\boldsymbol{u}^k|^{\rho+2},$$

on combining (4.11) and (4.12), a kickback of $\frac{1}{2}\|\boldsymbol{u}_t^k\|_0^2$, and multiplying through by 2, we have

$$\|\boldsymbol{u}_t^k\|_0^2 + \frac{d}{dt}\|\nabla \boldsymbol{u}^k\|_0^2 + \frac{2\lambda_1}{(\rho+2)}\frac{d}{dt}\|\boldsymbol{u}^k\|_{0,\rho+2}^{\rho+2} \leqslant \lambda_0\frac{d}{dt}\|\boldsymbol{u}^k\|_0^2 + 2\omega_0^2\|\boldsymbol{u}^k\|_0^2 + 2\omega_1^2\|\boldsymbol{u}^k\|_{0,2\rho+2}^{2\rho+2}. \tag{4.13}$$

Integrating both sides of (4.13) over $(0,t)$ yields for a.e. $t \in (0,T)$

$$\begin{aligned}
&\int_0^t \big\|\boldsymbol{u}_t^k(s)\big\|_0^2\,ds + |\boldsymbol{u}^k(t)|_1^2 + \frac{2\lambda_1}{(\rho+2)}\|\boldsymbol{u}^k(t)\|_{0,\rho+2}^{\rho+2} + \lambda_0\|\boldsymbol{u}_0^k\|_0^2 \\
&\qquad \leqslant \lambda_0\|\boldsymbol{u}^k(t)\|_0^2 + 2\omega_0^2\int_0^t \|\boldsymbol{u}^k(s)\|_0^2\,ds + 2\omega_1^2\int_0^t \|\boldsymbol{u}^k(s)\|_{0,2\rho+2}^{2\rho+2}\,ds \\
&\qquad\quad + \big|\boldsymbol{u}_0^k\big|_1^2 + \frac{2\lambda_1}{(\rho+2)}\big\|\boldsymbol{u}_0^k\big\|_{0,\rho+2}^{\rho+2}.
\end{aligned} \tag{4.14}$$

From the uniform bounds in Estimates I and II, assumption (A 2), the continuous injections $V \hookrightarrow L^{\rho+2}(\Omega)$, $L^\infty(0,T;V) \hookrightarrow L^{2\rho+2}(\Omega_T)$ and the projection property (3.11), we deduce that the right-hand side of (4.14) is bounded by a positive constant. Thus we have

$$\frac{\partial u^k}{\partial t},\ \frac{\partial v^k}{\partial t} \quad \text{are uniformly bounded in } L^2(\Omega_T). \tag{4.15}$$

By extracting the appropriate subsequences from (4.9), (4.10), and (4.15) we deduce

$$u,\ v \in L^\infty(0,T;V), \quad u,\ v \in L^2(0,T;H^2(\Omega)), \quad \frac{\partial u}{\partial t},\ \frac{\partial v}{\partial t} \in L^2(\Omega_T). \tag{4.16}$$

Application of Corollary 7.3 in Robinson [18] gives

$$u,\ v \in C([0,T];V).$$

To show the differential equation holds as an equality in $L^2(\Omega_T)$ we can repeat the passage to the limit argument with $p = q = 2$, without any additional complications.

Remark We could have obtained the above results more directly by extracting the kickback term $\frac{1}{2}\|\boldsymbol{u}_t^k\|_0^2$ from $(B\boldsymbol{u}^k + |\boldsymbol{u}^k|^\rho A\boldsymbol{u}^k, \boldsymbol{u}_t^k)$ with the aid of a simple Young's inequality, but the slightly longer approach shows there is nothing to be gained in keeping additional positive terms.

To complete the proof we still need to show continuous dependence of the strong solutions on the initial data in V.

4.1 Continuous dependence

Assume $\boldsymbol{u}$ and $\boldsymbol{v}$ satisfy the weak form (P_2), with initial conditions $\boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0(\cdot)$ and $\boldsymbol{v}(\cdot,0) = \boldsymbol{v}_0(\cdot)$, respectively, and $\boldsymbol{u}_0 \neq \boldsymbol{v}_0$. Setting $\boldsymbol{\eta} = -\Delta\boldsymbol{w} + \boldsymbol{w}$, $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{v}$ and subtracting weak forms leads to after integrating by parts and noting (3.3)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (|\boldsymbol{w}|^2 + |\nabla\boldsymbol{w}|^2)\,dx + \int_\Omega |\Delta\boldsymbol{w}|^2\,dx + \int_\Omega |\nabla\boldsymbol{w}|^2\,dx \\ = \lambda_0\int_\Omega (|\boldsymbol{w}|^2 + |\nabla\boldsymbol{w}|^2)\,dx + \int_\Omega (|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v})\cdot(-\Delta\boldsymbol{w} + \boldsymbol{w})\,dx.\end{aligned} \tag{4.17}$$

We split the last term in (4.17) and consider each term separately.

Noting (3.16), Hölder's inequality, and the continuous injection of V into $L^{\rho+2}(\Omega)$ yields

$$\begin{aligned}\int_\Omega (|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v})\cdot\boldsymbol{w}\,dx &\leqslant (\rho+1)\sqrt{\lambda_1^2 + \omega_1^2}\int_\Omega (|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho)|\boldsymbol{w}|^2\,dx \\ &\leqslant C\big(\|\boldsymbol{u}\|_{0,\rho+2}^\rho + \|\boldsymbol{v}\|_{0,\rho+2}^\rho\big)\|\boldsymbol{w}\|_{0,\rho+2}^2 \\ &\leqslant C\big(\|\boldsymbol{u}\|_{0,\rho+2}^\rho + \|\boldsymbol{v}\|_{0,\rho+2}^\rho\big)\|\boldsymbol{w}\|_1^2.\end{aligned} \tag{4.18}$$

In addition, from (3.16) and a simple Young's inequality we have

$$\int_\Omega \big(|\boldsymbol{u}|^\rho A\boldsymbol{u} - |\boldsymbol{v}|^\rho A\boldsymbol{v}\big)\cdot \Delta \boldsymbol{w}\,dx \leqslant (\rho+1)\sqrt{\lambda_1^2+\omega_1^2}\int_\Omega (|\boldsymbol{u}|^\rho + |\boldsymbol{v}|^\rho)|\boldsymbol{w}||\Delta\boldsymbol{w}|\,dx$$
$$\leqslant C\int_\Omega (|\boldsymbol{u}|^{2\rho}+|\boldsymbol{v}|^{2\rho})|\boldsymbol{w}|^2\,dx + \frac{1}{2}\int_\Omega |\Delta\boldsymbol{w}|^2\,dx. \tag{4.19}$$

From Hölder's inequality and the continuous injection of V into $L^{2\rho+2}(\Omega)$ we have

$$\int_\Omega (|\boldsymbol{u}|^{2\rho}+|\boldsymbol{v}|^{2\rho})|\boldsymbol{w}|^2\,dx \leqslant \big(\|\boldsymbol{u}\|_{0,2\rho+2}^{2\rho} + \|\boldsymbol{v}\|_{0,2\rho+2}^{2\rho}\big)\|\boldsymbol{w}\|_{0,2\rho+2}^2$$
$$\leqslant C\big(\|\boldsymbol{u}\|_{0,2\rho+2}^{2\rho} + \|\boldsymbol{v}\|_{0,2\rho+2}^{2\rho}\big)\|\boldsymbol{w}\|_1^2. \tag{4.20}$$

Combining (4.17)–(4.20) leads to after kickback of $\frac{1}{2}\int_\Omega |\Delta\boldsymbol{w}|^2 dx$

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|_1^2 \leqslant C\big(1 + \|\boldsymbol{u}\|_{0,\rho+2}^{\rho} + \|\boldsymbol{v}\|_{0,\rho+2}^{\rho} + \|\boldsymbol{u}\|_{0,2\rho+2}^{2\rho} + \|\boldsymbol{v}\|_{0,2\rho+2}^{2\rho}\big)\|\boldsymbol{w}\|_1^2. \tag{4.21}$$

Multiplying through by 2 and using the Grönwall lemma gives for a.e. $t \in (0,T)$

$$\|\boldsymbol{w}(t)\|_1^2 \leqslant \|\boldsymbol{w}(0)\|_1^2 \exp\left(2Ct + 2C\int_0^t (\|\boldsymbol{u}(s)\|_{0,\rho+2}^{\rho} + \|\boldsymbol{v}(s)\|_{0,\rho+2}^{\rho} + \|\boldsymbol{u}(s)\|_{0,2\rho+2}^{2\rho} + \|\boldsymbol{v}(s)\|_{0,2\rho+2}^{2\rho})\,ds\right).$$

Using Hölder's inequality in time and recalling that solutions belong to $L^{2\rho+2}(\Omega_T)$ leads to $\|\boldsymbol{u}-\boldsymbol{v}\|_1^2 \leqslant C\|\boldsymbol{u}_0-\boldsymbol{v}_0\|_1^2$. As $\boldsymbol{u}_0 \neq \boldsymbol{v}_0$ this gives continuous dependence in $\boldsymbol{V}$. This completes the proof of Theorem 4.1. □

Summary and discussion

We studied the weak and strong solutions of a generalised λ–ω reaction-diffusion system in $d \leqslant 3$ space dimensions. With minor adjustments of the proofs the results are applicable to the homogeneous Dirichlet and periodic boundary conditions as well. Provided the initial data is square integrable, we proved global existence, uniqueness and continuous dependence on initial data of the weak solution, subject to restrictions on the parameter ρ. Furthermore, if the initial data is in $H^1(\Omega)$, then there is a unique global strong solution depending continuously on the initial data, subject to additional restrictions on the parameter ρ. When $\rho = 2$, $d = 3$, we were unable to prove uniqueness of weak solutions, or global regularity results except in the special case (see Estimate II) when $\rho \leqslant \lambda_1/|\omega_1|$. Results in one and two space dimensions cover the important case when $\rho = 2$. Part II of this work covers the numerical analysis of the λ–ω system.

The main difficulty in this work was the lack of $L^{2\rho+2}(\Omega_T)$ regularity of solutions, which forced us to severely restrict the admissible values of ρ via (A 2). Bearing in mind the results obtainable by the Invariant Region method of Smoller (see the discussion in the

Introduction) this would appear to be a limitation of the Faedo–Galerkin method and the fact that we took the initial data in $L^2(\Omega)$ or $H^1(\Omega)$.

There is still additional work to be done, for example: extending results to cover the $\rho = 2$, $d = 3$ case; proving the continuous dependence of solutions on the system parameters; and investigating how the solution dynamics depend on the data (initial and boundary conditions).

We leave these questions for future study.

This paper significantly contributes to the mathematical analysis of reaction-diffusion systems with a supercritical Hopf bifurcation in the reaction kinetics and paves the way for subsequent numerical work.

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