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Biomanipulation of Food-Webs in Eutrophic Lakes

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***Abstract*— We consider an optimal control problem for a nonlinear 'nutrient-phytoplankton-zooplankton-fish' reaction-diffusion system, where the parameter for fish predation is treated as a multiplicative control variable. The model has implications for the biomanipulation of food-webs in eutrophic lakes to help improve water quality. We implement a semi-implicit (in time) finite element method with 'mass lumping', and present the results of numerical experiments in two space dimensions.**

***Index Terms*— chaos, optimal control, biomanipulation, predator-prey interaction, finite element method**

I. Introduction

A. Model equations

In this paper we study the following nutrient - phytoplankton - zooplankton - fish reaction-diffusion system

$$\begin{cases} \frac{\partial A}{\partial \tau} = D_A \Delta A + \widehat{r}\frac{n}{n+h_n}A - cA^2 - pZ\frac{A}{A+h_a}, \\ \frac{\partial Z}{\partial \tau} = D_Z \Delta Z + peZ\frac{A}{A+h_a} - \widehat{m}Z - F\frac{Z^2}{Z^2+h_Z^2}, \end{cases} \tag{1.1}$$

where A is phytoplankton biomass, Z is zooplankton biomass, and D_a and D_z are the diffusion coefficients of phytoplankton and zooplankton respectively, and n is the nutrient level of the system. F is the rate of zooplankton biomass consumed by fish per unit volume of water per day (average predation rate times the density of fish). It is important to note that in our formulation $F \equiv F(x,t)$, i.e. F is defined at every point in the lake and at every point in time. For definitions of the positive parameters c, e, $\widehat{m}$, $\widehat{r}$, p, h_a, h_z, and h_n see [33]. The symbol τ denotes time in days, and $\Delta = \sum_{i=1}^{d} \partial^2/\partial X_i^2$, is the usual Laplacian operator in $d = 2$ or 3 space dimensions. The grazing rate of zooplankton on phytoplankton is of a type II functional response, while the predation rate of fish on zooplankton is of type III [15]. In the absence of zooplankton the phytoplankton are assumed to grow logistically.

The nutrient - phytoplankton - zooplankton - fish model was originally formulated as a system of ordinary differential equations (ODEs) by Scheffer [33], and has since been spatially extended (see for example, [27], [24], [25], [28], [29], [38]).

The reaction-diffusion system models a simple food-web in lakes where planktivorous fish feed on zooplankton, and the zooplankton feed on phytoplankton (algae). The basic interactions in the model are illustrated in Figure 1. The model is 'minimal' in the sense that only a few important interactions are taken into account. For example, the positive effect that fish have on the nutrients of the system is omitted in this model (indicated by the dotted arrow in Figure 1). Nevertheless, such minimal predator-prey systems display a wide range of ecologically relevant behavior, for example, spiral waves [29], target waves [35], diffusion-induced instability [25], and chaos [31], [29]. See [26] for a historical overview of modeling plankton dynamics and pattern formation mechanisms.

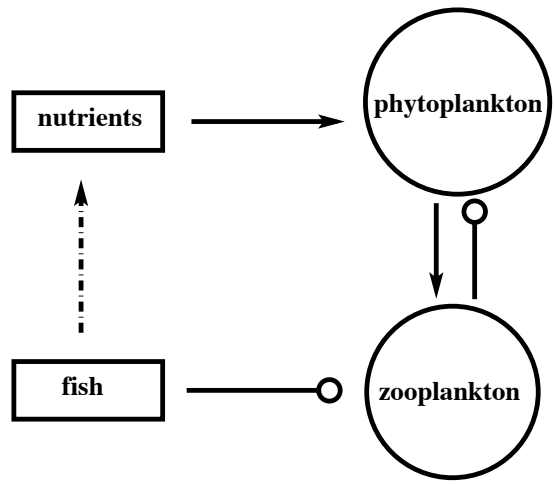

Fig. 1: Interactions incorporated in a nutrients-plankton-fish model. Arrows indicate positive effects, circles indicate negative effects (redrawn from [33])

It is simpler to work with equations that have been scaled to nondimensional form, thus after letting $N := n/(n+h_n)$ in (1.1) we define dimensionless phytoplankton densities, zooplankton densities, spatial coordinates, and time via

$$u = \frac{cA}{\widehat{r}N}, \quad v = \frac{cZ}{\widehat{r}eN}, \quad x_i = \frac{kX_i}{L}, \quad t = R_0\tau,$$

(cf. [28] and [29]), where R_0 is the characteristic (or typical) growth rate of phytoplankton, k is a factor related to the scale of expected patchy patterns, and L is the maximum diameter of the lake in the coordinate direction x_2, or x_3. Note that in the case of a square domain L is the side length. We also rescale the parameters via

$$r = \frac{R}{R_0}, \quad a = \frac{C_1 K}{C_2 R_0}, \quad b = \frac{K}{C_2}, \quad m = \frac{M}{R_0},$$
$$f = \frac{F}{C_3 R_0}, \quad g = \frac{K}{C_3 A}, \quad d_1 = \frac{k^2 D_A}{L^2 R_0}, \quad d_2 = \frac{k^2 D_Z}{L^2 R_0},$$

which leads to the dimensionless system

$$\begin{cases} \frac{\partial u}{\partial t} = d_1 \Delta u + ru(1-u) - \frac{auv}{1+bu}, \\ \frac{\partial v}{\partial t} = d_2 \Delta v + \frac{auv}{1+bu} - mv - f\frac{gv^2}{1+g^2v^2}. \end{cases} \tag{1.2}$$

Any 2-component reaction-diffusion system with reaction-kinetics close to a supercritical Hopf bifurcation, with equal

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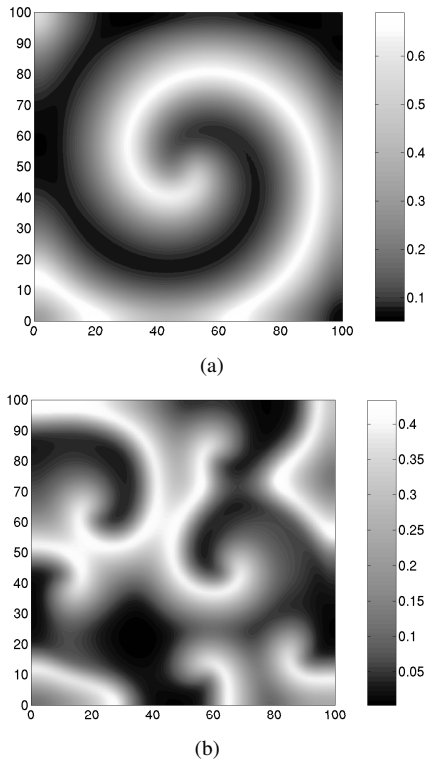

Fig. 2: Numerical solution v of (1.2) at time $T = 1000$ with $d_1 = d_2 = 0.05$, $f = 0$, $g = 10$, solved on a space-time grid of $101 \times 101 \times 12000$. A semi-implicit (in time) Galerkin finite element method with piecewise linear continuous basis functions was employed with homogeneous Neumann boundary conditions. In both cases initial data of the form $u_0 = A(x/100) + B$, $v_0 = C(100 - y)/100 + D$ was employed with the following parameter values: (a) $A = 0.02$, $B = 0.19$, $C = 0.02$, $D = 0.31$, $r = 1$, $a = b = 5$, $m = 0.5$; (b) $A = 0.2$, $B = 0.1$, $C = 0.2$, $D = 0.1$, $r = 1$, $a = b = 20$, $m = 0.8$.

diffusion coefficients, can be transformed into a generic reaction-diffusion system of 'λ-ω' form [34]. Thus as spiral wave solutions of λ-ω reaction-diffusion systems have been proved to exist [8], for appropriate parameter values and initial data we also expect spiral wave solutions to exist for system (1.1). In Figure 2 we present snapshots at $t = 1000$ for the uncontrolled system (1.2) representing spiral wave solutions that persist indefinitely (Figure 2(a)), or, after initialization, rapidly break-up into irregular patterns (Figure 2(b)). In both cases we checked that this behavior persists up-to $t = 10,000$. This behavior is important to our study as we apply controls that drive the system from the unstable regime to form regular patterns.

We consider the above model in the context of eutrophication of lakes. Eutrophication is the process where excessive inputs of nutrients in lakes lead to high levels of phytoplankton (algae) and hence degraded water quality (see [5], [16] and the references therein). The most common approach to improving water quality in this situation is to either reduce the external nutrient loading, or, to enhance zooplankton by reducing planktivorous fish, thereby, reducing algal biomass. We focus on the latter approach ('top' down control) where the nutrient level in the system is determined by a parameter. In practice, planktivorous fish can be reduced by fish removal, or by piscivore* stocking. This manipulation of the food-web is called biomanipulation, and is an important approach for improving water quality in eutrophic lakes (e.g., [19], [21], [23], [32], [37], [41]). Evidence from the limnological literature supports the hypothesis that the most effective biomanipulation strategy for improving water quality is the partial removal of fish, and that there may be an optimum harvesting rate of planktivorous fish [9], [18], [39], [40]. This observation helped motivate the work in this paper, where we consider the optimal control of phytoplankton u and zooplankton v densities, where f is the distributed control. We aim to minimize the following quadratic cost functional

$$J(u,v,f) = \int_Q \tfrac{1}{2}\left(|u-\bar{u}|^2 + |v-\bar{v}|^2\right) + \tfrac{\alpha}{2}\left|\tfrac{\partial f}{\partial t}\right|^2 dxdt, \quad (1.3)$$

where $(\bar{u}, \bar{v})$ are the desired phytoplankton-zooplankton densities, and $Q = \Omega \times (0, T)$ is the space-time domain of interest. The first two terms in (1.3) measure, with respect to the L^2-norm in space and time, the difference between the given target densities $(\bar{u}, \bar{v})$ and the state densities (u, v). The last term in (1.3) reflects the fact that we want to avoid changing the control (adding or removing fish) too often. The constant α can be chosen to adjust the relative importance of this cost. We assume the state equations (1.2) are augmented with appropriate initial and boundary conditions. Note that as there is no forcing in the first equation of (1.2), the control of phytoplankton must result indirectly through the coupling with the second equation. Furthermore, the forcing in (1.2) enters not merely as an additive inhomogeneous source term, but in a multiplicative manner. Additionally, as the control f is a *rate*, it is strictly nonnegative.

We emphasize that although this work has implications for the control of eutrophication in lakes, the main focus of this work is on the rigorous mathematical analysis of the optimal control problem. We do not provide practical implementational details for the improvement of water quality in lakes, but we do provide a theoretical basis on which such a task would be based. We also remark that the mathematical problem and results from numerical simulations may provide insights into the field situation.

It is important to distinguish between the mathematical (optimal control) problem and the practical problem in the field. Mathematically, we assume that f can be manipulated at every point in space and time. However, from a practical point of view, we only have direct control of the net density of fish in the lake at any instant. For example, fish released into a lake may distribute themselves uniformly throughout the lake, or move in schools. Nevertheless, there is significant overlap between the mathematical problem and the field situation, for example, in the case where we wish to reduce the (net) algal growth. In addition to improving water quality we are also interested in the more fundamental aim of maintaining a stable equilibrium between the plankton, thus avoiding the extinction of one or more species.

*Species that feed on fish.

The aim of this paper is to undertake the mathematical analysis of the optimal control problem introduced above, namely, to minimize (1.3) subject to (1.2). We also provide some numerical results that illustrate the theoretical results. We remark that there are few optimal control studies in the literature for interacting species involving space and time (e.g. [1], [2], [3], [7], [11], [20], [22]) or, for reaction-diffusion equations applied in other contexts (e.g. [4], [6], [10], [17], [36]).

II. Mathematical preliminaries

A. Local analysis

We present some details of the local dynamics of the reaction-diffusion system for the state equations (1.2). This is important for deriving necessary conditions on the system parameters for the kinetics to possess biologically meaningful equilibria. Furthermore, this acts as a guide in the appropriate choice of parameters for the numerical simulation of optimal solutions.

The local dynamics can be analyzed by considering the nullclines ('zero-isoclines') of this system, which are the solution curves for

$$\begin{aligned} v &= \psi(u) := \tfrac{r}{a}(1-u)(1+bu), \\ u &= \phi(v) := \tfrac{g^2mv^2+fgv+m}{(a-bm)g^2v^2-bfgv+a-bm}, \end{aligned} \tag{2.4}$$

corresponding to the first and second equations of (1.2). It is easy to show that there are saddle points at $(0,0)$ and $(0,1)$, and a stationary point (u_s, v_s) (stable or unstable) corresponding to the coexistence of phytoplankton and zooplankton. Note that for positive u_s and v_s we must have

$$m < \tfrac{a}{1+b}, \tag{2.5}$$

which follows from the restriction $0 < u_s < 1$. The nullclines are illustrated in Figure 3 for a specific parameter set and increasing predation rate. With no fish present, or low fish predation rate, there is a limit cycle in the reaction kinetics surrounding the unstable stationary point, while at higher predation rates the system is dominated by a phytoplankton only state [28], [33]. In general there are no closed form expressions for u_s and v_s and consequently the analysis must be done numerically in each case. However, in the special case with no fish present, the analysis is straightforward and is presented in [14].

B. Well-posedness of the state equations

We use results from semigroup theory and the abstract theoretical set-up of Morgan [30], which is based on the kinetics satisfying a Lyapunov-type condition, to infer the global existence and uniqueness of classical solutions of the state equations (1.2).

Before proving well-posedness of the equations we need to establish the formal setting and re-state the fish-plankton system with appropriate initial and boundary data. Let Ω be a bounded and open subset of $\mathbb{R}^d$, $d \le 3$, with a boundary

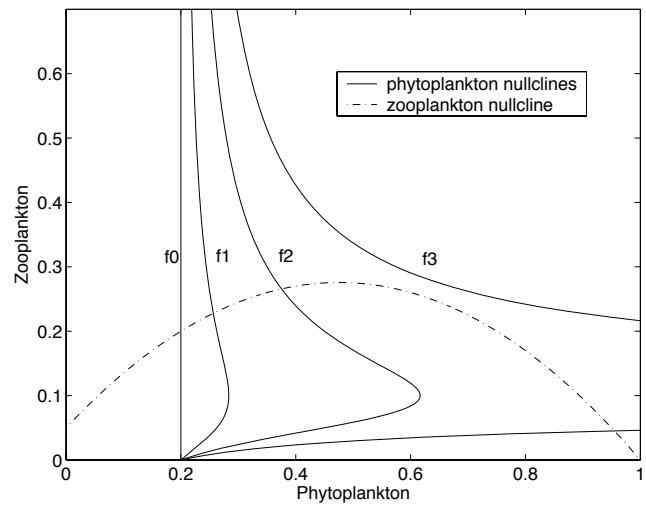

Fig. 3: Nullclines for the local kinetics of (1.2) with $r = 1$, $a = b = 20$, $m = 0.8$, $g = 10$, for fish-predation rates $f_0 = 0$ (limit cycle - no fish), $f_1 = 0.1$ (limit cycle), $f_2 = 0.25$ (limit cycle), and $f_3 = 0.4$ (phytoplankton dominance).

$\partial\Omega$ of class C^{2+s}, $s > 0$, i.e., $\partial\Omega$ is a $d-1$ dimensional $C^{2+\nu}$ manifold on which Ω lies locally on one side. The model problem is formulated as follows:

Find the phytoplankton $u(\mathbf{x},t)$ and zooplankton $v(\mathbf{x},t)$ densities such that

$$\tfrac{\partial u}{\partial t} = d_1\Delta u + ru(1-u) - \tfrac{auv}{1+bu} \quad \text{in } Q=\Omega\times(0,T), \tag{2.6a}$$

$$\tfrac{\partial v}{\partial t} = d_2\Delta v + \tfrac{auv}{1+bu} - m\,v - f\tfrac{gv^2}{1+g^2v^2} \quad \text{in } Q, \tag{2.6b}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad v(\mathbf{x},0) = v_0(\mathbf{x}), \quad \mathbf{x}\in\Omega \tag{2.6c}$$

$$\tfrac{\partial u}{\partial \boldsymbol{\nu}} = \tfrac{\partial v}{\partial \boldsymbol{\nu}} = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{2.6d}$$

where the parameters a, b, m, and g, are real and strictly positive, and $\boldsymbol{\nu}$ denotes the outward normal to $\partial\Omega$. We assume that the control $f \equiv f(\mathbf{x},t)$ is a Lipschitz continuous function on Q, which we denote by $f \in \mathrm{Lip}(Q)$, and that the initial data is bounded, i.e. $u_0(\mathbf{x}), v_0(\mathbf{x}) \in L^\infty(\Omega)$.

Theorem 2.1: Let $f \in \mathrm{Lip}(Q)$ and $u_0(\mathbf{x}), v_0(\mathbf{x}) \in L^\infty(\Omega)$. Then there exists a unique nonnegative classical solution of the fish-plankton system (2.6a)-(2.6d) for all $(\mathbf{x},t) \in \Omega\times[0,\infty)$. Furthermore, if $d_1 = d_2$ and the initial data is chosen in the positively invariant region $\Sigma \in [0,\infty)^2$ given by

$$\Sigma := \{(u,v) : u \le \ell, v - \tfrac{\ell}{m}(r+m) + u \le 0, u \ge 0, v \ge 0\},\ \ell > 1,$$

then $(u,v) \in \Sigma$ for all $(\mathbf{x},t) \in \Omega\times[0,\infty)$.

III. The optimal control problem

Let Ω be a bounded, open subset of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and let the set of all possible target densities $L^2_{\mathrm{loc}}(0,T;L^2(\Omega))$ be denoted by $\mathcal{T}_{ad}$. There are no particular requirements on the target densities $(\bar{u},\bar{v})$ other then the fact that the cost functional (1.3) must be bounded. The target densities need not to be solutions of (2.6a)-(2.6d).
Let $\mathcal{U}_{ad}$ be the set of admissible controls

$$\mathcal{U}_{ad} = \{f \in \mathrm{Lip}(Q);\ 0 \le f(x,t),\ \forall (x,t) \in Q\}.$$

Given $T > 0$, $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, and $(\bar{u}, \bar{v}) \in \mathcal{T}_{ad}$, then (u, v, f) is said to be an admissible element if $u, v \in L^2(0, T; H^1(\Omega))$, $f \in \mathcal{U}_{ad}$, the functional $J(u, v, f)$ is bounded and (u, v, f) satisfies (2.6a)-(2.6d). Let $\mathcal{A}_{ad}$ be the set of admissible states and controls. With this notation, the formulation of the optimal control problem is given by

$(\mathbf{P})$ Given $T > 0, u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ and $(\bar{u}, \bar{v}) \in \mathcal{T}_{ad}$, find $(u^*, v^*, f^*) \in \mathcal{A}_{ad}$ such that $J(u^*, v^*, f^*) \leq J(u, v, f)$ for all $(u, v, f) \in \mathcal{A}_{ad}$.

Theorem 3.1: Given $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ and $(\bar{u}, \bar{v}) \in \mathcal{T}_{ad}$, then there exists a solution (u^*, v^*, f^*) of the optimal control Problem (P).

A. First-order necessary conditions

We show that the optimal solution must satisfy the first-order necessary condition associated with the optimal control Problem (P).

We introduce the tangential (contingent) cone to $\mathcal{A}_{ad}$ at $(u, v, f) \in \mathcal{A}_{ad}$:

$$\operatorname{Tan} \mathcal{A}_{ad}(u, v, f) = \Big\{ (y, z, h) \mid y, z \in L^2(Q), h \in \operatorname{Tan} \mathcal{U}_{ad}(f)$$

and

$$\begin{aligned}
&\tfrac{\partial y}{\partial t} = d_1 \Delta y + ry(1-2u) - \tfrac{av}{(1+bu)^2} y - \tfrac{au}{1+bu} z \text{ in } Q,\\
&\tfrac{\partial z}{\partial t} = d_2 \Delta z + \tfrac{av}{(1+bu)^2} y + \Big(\tfrac{au}{1+bu} - m - \tfrac{2gvf}{(1+g^2v^2)^2} \Big) z\\
&\qquad - \tfrac{gv^2 h}{1+g^2v^2} \text{ in } Q,\\
&\tfrac{\partial y}{\partial \nu} = \tfrac{\partial z}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T),\\
&y(x, 0) = z(x, 0) = 0 \qquad \text{in } \Omega \Big\}.
\end{aligned} \tag{3.7}$$

Recall that if

$$J(u^*, v^*, f^*) = \inf_{(u,v,f) \in \mathcal{A}_{ad}} J(u, v, f)$$

and the functional $J(u, v, f)$ is Gâteaux differentiable, then necessarily

$$\partial J(u^*, v^*, f^*)(y, z, h) \geq 0, \quad \text{for all } (y, z, h) \in \operatorname{Tan} \mathcal{A}_{ad}(u^*, v^*, f^*), \tag{3.8}$$

where $\partial J(u^*, v^*, f^*)$ denotes the Gâteaux derivative of J at $(u^*, v^*, f^*) \in \mathcal{A}_{ad}$. Applying the optimum principle given by (3.8) it follows that

$$\int_Q \left[(u^* - \bar{u}) y + (v^* - \bar{v}) z + \alpha \frac{\partial f^*}{\partial t} \frac{\partial h}{\partial t} \right] dx dt \geq 0, \quad \forall (y, z, h) \in \operatorname{Tan} \mathcal{A}_{ad}(u^*, v^*, f^*). \tag{3.9}$$

Theorem 3.2: Let $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$. The mapping $(u, v) = (u(f), v(f))$ from $\mathcal{U}_{ad}$ to $L^2(0, T; H^1(\Omega))$, defined as the solution of (2.6a)-(2.6d), has a Gâteaux derivative $(D(u, v)/Df) \cdot h$ in every direction $h \in \operatorname{Tan} \mathcal{U}_{ad}(f)$. Furthermore, $(y(h), z(h)) = (D(u, v)/Df) \cdot h$ is the classical solution of the problem (3.7).

The Gâteaux derivative gives useful information about the sensitivity of the system (2.6a)-(2.6d) at a particular point (u, v) in a particular direction h, but complete information requires the solution of (3.7) for every direction h. Fortunately, in order to minimize the functional we need only an integral over all these directions which can be obtained by solving a single adjoint equation.

Theorem 3.3: Let $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ and $(\bar{u}, \bar{v}) \in \mathcal{T}_{ad}$ be given. If (u^*, v^*, f^*) is an optimal solution for (P), then we have

$$\int_Q \left(\frac{g v^{*2} k}{1 + g^2 v^{*2}} (f - f^*) + \alpha \frac{\partial f^*}{\partial t} \left(\frac{\partial f}{\partial t} - \frac{\partial f^*}{\partial t} \right) \right) dx dt \geq 0, \quad \forall f \in \mathcal{U}_{ad}, \tag{3.10}$$

where (p, k) is the unique classical solution of the adjoint equation

$$\begin{aligned}
&\tfrac{\partial p}{\partial t} + d_1 \Delta p + rp(1-2u^*) - a \tfrac{v^*}{(1+bu^*)^2} p + a \tfrac{v^*}{(1+bu^*)^2} k\\
&\qquad = u^* - \bar{u}, \text{ in } Q,\\
&\tfrac{\partial k}{\partial t} + d_2 \Delta k + a \tfrac{u^*}{1+bu^*} k - mk - 2 \tfrac{g f^* v^*}{(1+g^2 v^{*2})^2} k - a \tfrac{u^*}{1+bu^*} p\\
&\qquad = v^* - \bar{v} \text{ in } Q,\\
&\tfrac{\partial p}{\partial \nu} = \tfrac{\partial k}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T),\\
&p(x, T) = k(x, T) = 0 \qquad \text{in } \Omega.
\end{aligned}$$

IV. Numerical results

To illustrate the theoretical results of the optimal control problem in the previous section we present results of numerical experiments in two space dimensions. The state equations and adjoint equations were solved using a 'lumped mass', semi-implicit (in time) Galerkin finite element method with piecewise linear continuous basis functions. We showed previously that this approach was highly effective in solving the forward in time equations of a similar predator-prey system [12].

The control was updated using a variable-step gradient algorithm based on the fully-discrete optimality condition (see (4.15) below). At each iteration of the gradient algorithm the method requires the sequential solution of the discrete state and adjoint equations (see (4.12) and (4.14) below). In practice one cannot solve these systems simultaneously. The discrete state equations are solved by marching forward in time starting from an initial condition, while the discrete adjoint equations are solved by marching backward in time (from T) starting from a terminal condition. For further details see [13].

We employed a (uniform) right-angled triangulation Ω^h of the square $\Omega = [0, L] \times [0, L]$, with space steps h, and numerically solved the optimal control problem up-to time T with uniform time steps Δt. We introduce S^h, the standard finite element space

$$S^h := \{ v \in C(\overline{\Omega}) \ : \ v|_\tau \quad \text{is linear } \forall \tau \in \Omega^h \} \subset H^1(\Omega).$$

Let $\{x_i\}_{i=0}^J$ be the set of nodes of the triangulation. We introduce $\pi^h : C(\overline{\Omega}) \mapsto S^h$, the Lagrange interpolation operator, such that $\pi^h v(x_j) = v(x_j)$ for all $j = 0, \ldots J$. In order to formulate our finite element approximation of the reaction-diffusion system we define a discrete L^2 inner product on $C(\overline{\Omega})$ given by $(u, v)^h := \int_\Omega \pi^h(u(x)v(x))\, dx$,

which approximates the usual L^2 inner product (u, v). Given $\mathbf{f}_h \in \mathrm{Lip}(\Omega)$ and $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $(\mathbf{u}_h, \mathbf{v}_h)$ is a solution of the fully discrete, semi-implicit (in time) fish-plankton system if $u_h^{(n)}, v_h^{(n)} \in S^h$ satisfies the system

$$\begin{aligned}
&\tfrac{1}{\Delta t}\left(u_h^{(n)} - u_h^{(n-1)}, \chi_h\right)^h + d_1\left(\nabla u_h^{(n)}, \nabla \chi_h\right) \qquad (4.12)\\
&\quad = \left(r u_h^{(n)}(1-|u_h^{(n-1)}|) - \tfrac{a u_h^{(n-1)} v_h^{(n)}}{1+b|u_h^{(n-1)}|}, \chi_h\right)^h,\\
&\tfrac{1}{\Delta t}\left(v_h^{(n)} - v_h^{(n-1)}, \chi_h\right)^h + d_2\left(\nabla v_h^{(n)}, \nabla \chi_h\right)\\
&\quad = \left(\tfrac{a u_h^{(n-1)} v_h^{(n)}}{1+b|u_h^{(n-1)}|} - m v_h^{(n)} - \tfrac{g f_h^{(n)} v_h^{(n-1)2}}{1+g^2 v_h^{(n-1)2}}, \chi_h\right)^h,
\end{aligned}$$

for all $\chi_h \in S^h$, $n = 1, \ldots, N$, with initial densities $u_h^{(0)} = \pi^h u_0(x)$, $v_h^{(0)} = \pi^h v_0(x)$. The discrete cost functional used in the optimal control problem is given by

$$\begin{aligned}
\mathcal{J}_h^N(\mathbf{u}_h, \mathbf{v}_h, \mathbf{f}_h) = &\frac{\Delta t}{2}\sum_{n=1}^{N}\left(\|u_h^{(n)} - \bar{u}^{(n)}\|^2 + \|v_h^{(n)} - \bar{v}^{(n)}\|^2\right)\\
&+ \frac{\alpha}{2\Delta t}\sum_{n=1}^{N}\|f_h^{(n)} - f_h^{(n-1)}\|^2. \qquad (4.13)
\end{aligned}$$

Thus we can now formulate the fully discrete optimal control problem as:

$(\mathbf{P}^{h,\Delta t})$ Given $\Delta t = T/N$, $h = L/J$, $u_0, v_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ and $(\bar{u}, \bar{v}) \in \mathcal{T}_{ad}$, find $(\mathbf{u}_h, \mathbf{v}_h, \mathbf{f}_h) \in S^h \times S^h \times S^h$ such that (4.12) is satisfied for $n = 1, 2, \ldots, N$ and the cost functional (4.13) is minimized.

To complete the fully discrete optimality system we also need the following fully discrete adjoint system: The adjoint functions $p_h^{(n)}, k_h^{(n)} \in S^h$ satisfy

$$\begin{aligned}
&-\frac{1}{\Delta t}\left(p_h^{(n)} - p_h^{(n-1)}, \chi_h\right)^h + d_1\left(\nabla p_h^{(n-1)}, \nabla \chi_h\right) \qquad (4.14)\\
&= \Bigg(r(1 - 2|u_h^{(n)}|)p_h^{(n-1)} - a\tfrac{v_h^{(n)}}{(1+b|u_h^{(n)}|^2)^2}p_h^{(n-1)}\\
&\quad + a\tfrac{v_h^{(n)}}{1+b|u_h^{(n)}|^2}k_h^{(n-1)} - u_h^{(n)} - \bar{u}^{(n)}, \chi_h\Bigg)^h\\
&-\frac{1}{\Delta t}\left(k_h^{(n)} - k_h^{(n-1)}, \chi_h\right)^h + d_2\left(\nabla k_h^{(n-1)}, \nabla \chi_h\right)\\
&= \Bigg(a\tfrac{u_h^{(n)}}{1+b|u_h^{(n)}|^2}p_h^{(n-1)} + a\tfrac{u_h^{(n)}}{1+b|u_h^{(n)}|^2}k_h^{(n-1)} - m k_h^{(n-1)}\\
&\quad - g f_h^{(n)}\tfrac{2v_h^{(n)}}{1+g^2 v_h^{(n)2}} - v_h^{(n)} - \bar{v}^{(n)}, \chi_h\Bigg)^h,
\end{aligned}$$

for all $\chi_h \in S^h$, $n = 1, 2, \ldots, N$, with the terminal conditions $p_h^{(N)} = k_h^{(N)} = 0$. The fully discrete optimality condition is

$$\begin{aligned}
0 \le \Delta t \sum_{n=1}^{N-1} &\Bigg(\tilde{f}_h^{(n)}, g\tfrac{v_h^{(n)2}}{1+g^2 v_h^{(n)2}}k_h^{(n-1)} \qquad (4.15)\\
&- \alpha\tfrac{f_h^{(n+1)} - 2f_h^{(n)} + f_h^{(n-1)}}{\Delta t}\Bigg) - \tfrac{\alpha}{\Delta t}\left(\tilde{f}_h^{(0)}, f_h^{(1)} - f_h^{(0)}\right)\\
&+ \left(\tilde{f}_h^{(n)}, \Delta t g\tfrac{v_h^{(N)2}}{1+g^2 v_h^{(N)2}}k_h^{(n-1)} + \alpha\tfrac{f_h^{(N)} - f_h^{(N-1)}}{\Delta t}\right),
\end{aligned}$$

for all $\tilde{\mathbf{f}}_h \in \mathrm{Tan}\,\mathcal{U}_{ad}(\mathbf{f}_h)$.

In the numerical experiments we chose the domain $\Omega = [0, 100] \times [0, 100]$ and initial conditions of the optimal control problem to be the freely evolving ($f = 0$) system (1.2) at time $t = 1000$, with parameter values and initial data corresponding to Figure 2(b) (see caption). The target functions are one-armed Archimedian spirals with period of rotation equal to 20, and are given by

$$\begin{aligned}
\bar{u}(R, \theta, t) &= 0.495\cos(\theta + R/12 + \pi t/10) + 0.495,\\
\bar{v}(R, \theta, t) &= 0.21\sin(\theta + R/12 + \pi t/10) + 0.22,
\end{aligned}$$

where $R := \sqrt{(x-50)^2 + (y-50)^2}$ and $\theta := \arctan[(y - 50)/(x - 50)]$. The numerical results for the optimal control

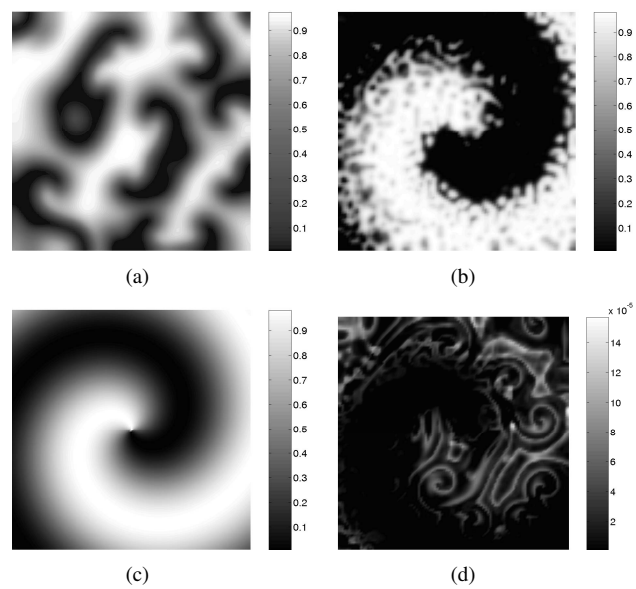

Fig. 4: Uncontrolled (a), optimally controlled (b), target (c), and the control f (d) for phytoplankton densities u at time $T = 100$ for (1.2). Parameter values: $d_1 = d_2 = 0.05$, $r = 1$, $a = b = 20$, $m = 0.8$, $g = 10$, $\alpha = 10^{-5}$. For details of initial data see text.

of phytoplankton densities u are shown in Figure 4 at time $T = 100$. A plot of the reduction in cost functional with increasing iteration count suggests the near optimality of the system (see [13]). The results show that at time $T = 100$ the controlled system is close to the desired state, and that we were successful in driving the system from a disordered state to an ordered one.

V. Conclusions

The mathematical formulation, analysis, and numerical solution of an optimal control problem for a nonlinear plankton-fish reaction-diffusion was presented. The model was discussed in the context of biomanipulation of eutrophic lakes. After considering the local dynamics of the system, and proving the global existence and uniqueness of the classical solutions of the state equations, we presented the mathematical analysis of the plankton-fish optimal control problem. Numerical solutions were obtained with the aid of a

semi-implicit Galerkin finite element method with piecewise linear continuous basis functions. The numerical results illustrate the ability of a variable step-size gradient algorithm to drive the plankton dynamics from a chaotic regime to an (arbitrary) ordered state. The time taken to achieve the ordered distribution of phytoplankton in nondimensional units was $t = 100$, which with an assumed maximum growth rate for phytoplankton in eutrophic conditions of $R_0 = 0.5$ per day [33], gives the time taken to achieve this state to be $\tau = 200$ days. The theoretical results in this paper provide the basis for a numerical analysis of the optimal control problem [13]. Furthermore, our results can be generalized in numerous ways to include, for example: convection driven flows, forcing via nutrient inputs, and stochastic influences. We leave these tasks for future work.

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